Arch. Hist. Exact Sci. (2010) 64:301–373
DOI 10.1007/s00407-010-0057-y

# A summary of Euler's work on the pentagonal number theorem

**Jordan Bell**

Received: 22 April 2008 / Published online: 20 May 2010


**Abstract** In this article, we give a summary of Leonhard Euler's work on the pentagonal number theorem. First we discuss related work of earlier authors and Euler himself. We then review Euler's correspondence, papers and notebook entries about the pentagonal number theorem and its applications to divisor sums and integer partitions. In particular, we work out the details of an unpublished proof of the pentagonal number theorem from Euler's notebooks. As we follow Euler's discovery and proofs of the pentagonal number theorem, we pay attention to Euler's ideas about when we can consider a mathematical statement to be true. Finally, we discuss related results in the theory of analytic functions.

## 1 Introduction

The *pentagonal number theorem* is the expansion of the infinite product $(1-x)(1-x^2)(1-x^3)\cdots$ as the following power series

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2}.$$

Communicated by Niccolò Guicciardini.

Part of this work was done at the School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada.

J. Bell (✉)
Department of Mathematics, University of Toronto, Toronto, ON, Canada
e-mail: jordan.bell@gmail.com

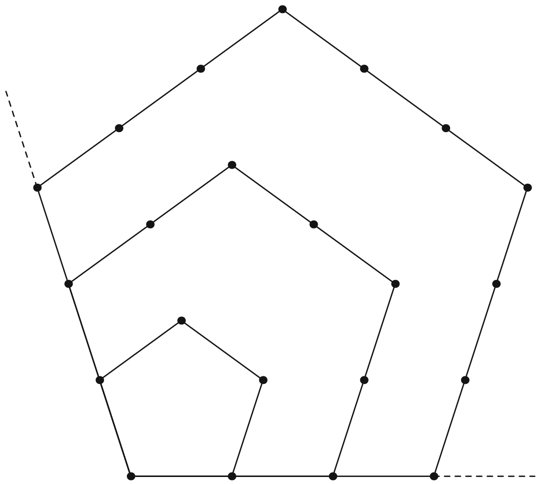

**Fig. 1** The pentagonal numbers $\omega_1 = 1, \omega_2 = 5, \omega_3 = 12, \omega_4 = 22$

This is true as a formal identity; see Rademacher 1973, Chap. 12 for a rigorous explanation of formal power series. This is true analytically for all complex $x$ such that $|x| < 1$.

This result is called the pentagonal number theorem because the numbers $\omega_n = n(3n-1)/2$ are the *pentagonal numbers*, a type of polygonal number. Euler clearly explains polygonal numbers in his *Vollständige Anleitung zur Algebra* (1770, §§402–439); here we give a brief summary. The $n$th $m$-gonal number is the sum of the first $n$ terms of the arithmetic progression starting at 1 with difference $m-2$. For example, the $n$th triangular number (=3-gonal number) is $1+2+3+\cdots+n = \frac{n(n+1)}{2}$. The $n$th square number (=4-gonal number) is $1+3+5+\cdots+(1+2(n-1)) = n+2(1+2+\cdots+(n-1)) = n^2$. The $n$th pentagonal number (=5-gonal number) is $1+4+7+\cdots+(1+3(n-1)) = n+3(1+2+\cdots+(n-1)) = \frac{n(3n-1)}{2}$.

The pentagonal numbers $\omega_n$ are the numbers of points that can be arranged to form homothetic pentagons of side length $n$ with $n$ evenly spaced points on each side, $n = 1, 2, \ldots$. Figure 1 shows this for $n = 1, 2, 3, 4$.

It will be useful to give an equivalent definition of polygonal numbers. A sequence $a_0, a_1, \ldots$ is an arithmetic progression of order $n$ if $\Delta^n a_k$ is the same for all $k$. The $n$th differences are defined by $\Delta a_k = a_{k+1} - a_k$ and $\Delta^{n+1} a_k = \Delta^n a_{k+1} - \Delta^n a_k$. $m$-gonal numbers are second-order arithmetic progressions starting at 1 with a second difference of $m-2$. Thus, the triangular numbers have second difference 1, the squares have second differences 2, the pentagonal numbers have second differences 3, etc.

The purpose of this article is to give a summary of Euler's work on the pentagonal number theorem. First, in §2, we mention related earlier work; in particular we give a sketch of similar work of Euler's on the infinite products for $\sin z$ and $\zeta(s)$. Then, in §3, we outline Euler's discussions in his correspondence about the pentagonal number theorem and explain his published proofs of it. Then, we examine an unpublished proof from Euler's notebooks and fill in the missing details. In §§4 and 5, we review Euler's use of the pentagonal number theorem to prove recurrence relations for the sum of divisors function and partition function, respectively. Finally, in §6, we look at the infinite product $(1-x)(1-x^2)(1-x^3)\cdots$ explicitly as a complex function.

Let us take a moment to explain what this article is useful for. First, it is a more or less exhaustive and reliable reference for use in further study of Euler's work—particularly his extension of the toolbox in analysis of infinite expressions we are familiar with to include infinite products with variable terms—and succeeding work on elliptic functions and analytic number theory (cf. Patterson 2007, p. 511 on determining the argument of Gauss sums). We can sort out the following timeline from the mass of details:

**1740** After being asked about partitions by Philippe Naudé junior in Naudé's August 29, 1740 letter, Euler investigates the products $\prod_{n=1}^{\infty}\frac{1}{1-x^n}$ and $\prod_{n=1}^{\infty}(1-x^n)$. Euler discovers the pentagonal number theorem, and seems to have first mentioned it in a November 30, 1740 letter to Daniel Bernoulli.

**1741** Euler states the pentagonal number theorem without proof at the end of his "Observationes analyticae variae de combinationibus", E158. The paper is presented to the St. Petersburg Academy on April 6, 1741, but is not published until 1751.

**1747** Euler finds that the pentagonal number theorem implies an unexpected recurrence relation for sums of divisors, and mentions this first to Goldbach in his March 21/April 1, 1747 letter, and then in his "Découverte d'une loi tout extraordinaire des nombres, par rapport à la somme de leurs diviseurs", E175, presented to the Berlin Academy on June 22, 1747 and published in 1751.

**1748** The pentagonal number theorem appears without proof in Euler's *Introductio in analysin infinitorum*, E101.

**1750** Euler works out a proof of the pentagonal number theorem, which he sends first to Goldbach on June 9, 1750. This proof is then published in Euler's "Demonstratio theorematis circa ordinem in summis divisorum observatum", E244, in 1760.

**1775** Euler revisits the pentagonal number theorem in two papers, E541 and E542.

Second, this article begins by describing Euler's infinite product for $\sin z$ (which involves Newton's identities for the roots and coefficients of polynomials) and his infinite product for $\zeta(s)$. The article also refers to Euler's work on summing related infinite series (namely in E190) and finding expressions for certain series in terms of more workable functions. The pentagonal number theorem is a significant case of Euler's strategy to find different expressions for the same thing and to exploit these to find out new facts.

Third, this is a detailed case study for Euler's work habits. Following the life of a discovery is an excellent way to appreciate Euler's persistent work solving problems, investigating related problems, and then taking up the topic again later, refining the solutions that he had found and contriving other related problems.

Fourth and finally, this article carefully shows Euler's understanding of the pentagonal number theorem's changing epistemological status (cf. Pólya 1954, Chap. VI). Examples of a statement can either just show that it is true in various cases, or can give insight into how a general proof of the statement would work. The pentagonal number theorem is an excellent case for understanding the relation to Euler between examples of a theorem that seem like they could be generalized and a general proof of the theorem. Having the right notation can let us see more clearly how an argument works and then how to carry it out in general; this article shows the notation that Euler used, and explains these arguments using subscript notation and summation and product notation.

An equivalent statement to the pentagonal number theorem as a formal identity is that if $p_{\mathrm{e}}(n)$ is the number of partitions of $n$ into an even number of distinct positive integers and $p_{\mathrm{o}}(n)$ is the number of partitions of $n$ into an odd number of distinct positive integers, then

$$p_{\mathrm{e}}(n) - p_{\mathrm{o}}(n) = \begin{cases} (-1)^j, & \text{if} \quad n = j(3j-1)/2, \\ 0, & \text{otherwise}, \end{cases}$$

(cf. Legendre 1830, §458). We discuss an entry from Euler's notebooks that says something like this in §5. There is an elementary proof of this version which is explained very clearly by Hardy and Wright (1980, §19.11).

We will now introduce definitions and notation that will be used in this article. The *sum of divisors function* $\sigma(n)$ is defined as the sum of the positive divisors of $n$. The *partition function* $p(n)$ is defined as the number of ways of writing $n$ as a sum of positive integers, disregarding the order. For example, $1 + 1 + 2$ and $1 + 2 + 1$ are considered the same, while $2 + 2$ and $2 + 1 + 1$ are considered different. Euler uses the notation $\int n$ for the sum of divisors function, i.e., $\int n = \sigma(n)$, and $n^{(\infty)}$ for the partition function, that is $n^{(\infty)} = p(n)$; in general he defines $n^{(m)}$ or $(n)^{(m)}$ to be the number of ways of writing $n$ as a sum of positive integers $\leq m$, disregarding the order. Cajori (1929, §§406–420) gives the history of the notations for the sum of divisors function and partition function.

## 2 Background

Pierre de Fermat (1601 or 1607/1608–1665) claims in a note (Tannery and Henry 1891, Deuxième partie, §XVIII) to Question XXXI, Book IV of Bachet's edition of Diophantus's *Arithmetica* that every nonnegative integer is the sum of $m$ or fewer $m$-gonal numbers (cf. Weil 1984, Chap. II, §V). Euler remarks, e.g., in his August 17, 1750 letter to Goldbach (Juškevič and Winter 1965, Brief 147), and in his paper "Considerationes super theoremate Fermatiano de resolutione numerorum in numeros polygonales" (1785), presented to the St. Petersburg Academy on December 12, 1774 (Nevskaja 2000, p. 621) that a very natural way to prove Fermat's claim is to show for

$$\left( \sum_{n=1}^{\infty} x^{b_m(n)} \right)^m = \sum_{n=1}^{\infty} a_n x^n,$$

that $a_n \geq 1$ for all $n \geq 1$, where $b_m(n) = n + (m-2)(1 + 2 + \cdots + (n-1)) = n + \frac{(m-2)(n-1)n}{2}$ is the $n$th $m$-gonal number (cf. Weil 1984, Chap. III, §§IV, XXI). In particular, to show that every nonnegative integer is the sum of four or fewer squares, with

$$s = 1 + x + x^4 + x^9 + x^{16} + x^{25} + x^{36} + \text{etc.}$$

it is equivalent to show that all the coefficients $a_n$ in the power series expansion of $s^4$ are nonzero. A discussion of the history of polygonal numbers is given by Dickson (1919b, Chap. I), who in particular notes that Fermat's polygonal number theorem was first proved by Cauchy.

Earlier work on partitions by Leibniz (1646–1716) is summarized by Knobloch (1974). Eneström (1908) writes about Jacob I Bernoulli's (1654–1705) study of power series whose exponents form a second-order arithmetic progression, like the series expansion in the pentagonal number theorem. For example, Bernoulli looks at the power series

$$m - m^2 + m^4 - m^5 + m^8 - m^9 + m^{13} - m^{14} + m^{19} - \cdots,$$

in his *Ars Conjectandi* (van der Waerden 1975, T 3, Pars Prima, Problema I), in which the exponents are alternately from the sequence $1, 4, 8, 13, 19, \ldots$ and $2, 5, 9, 14, \ldots$, which are both second-order arithmetic progressions, namely with second differences 1. Euler writes to Carl Leonhard Gottlieb Ehler (1685–1753) on February 11/22, 1737 (Smirnov 1963, pp. 369–385), about finding the sum of the series $1 + 1 + d + d^3 + d^6 + \cdots + d^{\frac{(n-1)(n-2)}{2}} + \cdots$; this letter is R. 594 (Résumé number) in the annotated index of Euler's correspondence in the *Opera omnia* (Juškevič et al. 1975). Euler corresponded with Ehler about number theory between 1735 and 1742 (see Juškevič et al. 1975, R. 581–600). Ehler was a Bürgermeister in Danzig in 1740–1753 and presiding Bürgermeister in 1741–1742, 1745–1746 and finally 1750–1751; there were four Bürgermeisters in Danzig, of which one was a president and one was a vice-president. This question is also mentioned in a letter from Daniel Bernoulli (1700–1782) to Euler on April 14, 1742 (Fuss 1843, D. Bernoulli-Euler, Lettre XXIV). In his later paper "De plurimis quantitatibus transcendentibus, quas nullo modo per formulas integrales exprimere licet" (Euler 1784), E565 in the Eneström index (1913), presented to the St. Petersburg Academy on October 16, 1775 and published in the *Acta* in 1784, Euler looks at series which cannot be expressed as a closed formula using integrals of elementary functions. For example, for $|x| < 1$ the power series for the natural logarithm $\log(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, which is equal to the integral of $\frac{1}{1+x}$, and thus the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ can indeed be expressed as the integral of an elementary function. In this article Euler deals with the power series $1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + \text{etc.}$, where the exponents are the triangular (3-gonal) numbers $\frac{n(n+1)}{2}$. As we observed in §1 the triangular numbers are a second-order arithmetic progression, with a constant second difference of 1. He mentions (in §6 of E565) that if we had a finite expression for this power series, then our problem would become resolving the cube of this expression into a power series and showing that all the powers of $x$ occur: "Maximi autem sine dubio momenti foret, si summa huius seriei

$$1 + x + x^3 + x^6 + x^{10} + x^{15} + \text{etc.}$$

finita expressione, quantumvis transcendente, exhiberi posset. Inde enim solidissima demonstratio theorematis Fermatiani peti potest, quod omnes numeri integri sint summae trium trigonalium; tantum enim opus foret cubum illius summae in seriem resolvere atque ostendere tum omnes plane potestates ipsius $x$ occurrere debere." ["It would without doubt be most momentous if the sum of this series

$$1 + x + x^3 + x^6 + x^{10} + x^{15} + \text{etc.}$$

could be exhibited in a finite expression, however transcendental. For from this a very firm demonstration of the theorem of Fermat could be approached, that all integral numbers are the sum of three trigonal numbers; for the problem would just be to resolve the cube of this sum into a series and then to show that all the powers of $x$ must occur."]

Calinger (1996, §II) lists sources that Euler had read. In particular, notes suggest that Euler had examined Jacob I Bernoulli's articles on infinite series and *Ars Conjectandi*, and John Wallis's *Arithmetica infinitorum*, in 1725 when he was a student in Basel. Euler had seen Fermat's *Varia opera mathematica* (in which Fermat states his polygonal number theorem mentioned in §1) in 1729 (Calinger 1996, §III.A), and indeed Euler mentions it in his second letter to Goldbach, on June 4, 1730.

In his paper "De summis serierum reciprocarum" (Euler 1740), E41, presented to the St. Petersburg Academy on December 5, 1735, Euler finds an expression for $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$, namely $\zeta(2) = \frac{\pi^2}{6}$, by representing $\sin s$ as an infinite product. The zeros of $\sin s$ are precisely $\pi n$, $n$ an integer (Euler proves this in 1742 as a consequence of his formula $e^{is} = \cos s + i \sin s$; cf. Weil 1984, Chap. III, §XIX), hence, we have the infinite product expansion

$$\frac{\sin s}{s} = \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1 - \frac{s}{n\pi}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2\pi^2}\right).$$

Therefore, writing $x = s^2$, we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!} = \prod_{n=1}^{\infty} \left(1 - \frac{x}{n^2\pi^2}\right).$$

Euler then applies Newton's identities for the roots and coefficients of a polynomial (given in Isaac Newton's *Arithmetica universalis*; cf. Struik 1969, Chap. II, §9) to the roots of the above infinite product and the coefficients of the above power series to obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} &= \frac{1}{3!} \\
\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} &= \left(\frac{1}{3!}\right)^2 - 2 \cdot \frac{1}{5!} \\
\sum_{n=1}^{\infty} \frac{1}{n^6\pi^6} &= \left(\frac{1}{3!}\right)^3 - 3 \cdot \frac{1}{3!} \cdot \frac{1}{5!} + 3 \cdot \frac{1}{7!} \\
&\text{etc.}
\end{aligned}$$

from which we get $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \pi^4((\frac{1}{6})^2 - 2 \cdot \frac{1}{120}) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, etc. Sandifer (2007b, Chap. 21) thoroughly presents Euler's arguments in E41.

In general, it is often fruitful to expand an infinite product as a power series, or to expand a power series as an infinite product, both as formal identities and functions analytic in some domain.

The Riemann zeta function $\zeta(s)$ is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$. In his paper "Variae observationes circa series infinitas" (Euler 1744, Theorema 8), E72, presented to the St. Petersburg Academy on April 25, 1737 (Nevskaja 2000, p. 192), Euler expresses $\zeta(s)$ as an infinite product over the primes

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}. \tag{1}$$

The Möbius function $\mu(n)$ is defined to be: 0 if $n$ has a repeated prime factor (i.e. $n$ has a square factor), 1 if $n$ is 1, and $(-1)^m$ if $n$ is the product of $m$ distinct prime factors. In his *Introductio in analysin infinitorum* (Euler 1748, Caput XV, §277), E101, Euler notes that

$$\prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \mu(n) n^{-s},$$

so this is the series expansion of the reciprocal of (1). Sandifer (2007b, Chap. 32) explains E72.

Philippe Naudé junior (1684–1745) writes to Euler on August 29, 1740 (Juškevič et al. 1975, R. 1903). Naudé was a professor of mathematics at the Joachimsthal Gymnasium in Berlin. He asks for the details on Euler's summation of $\zeta(2n)$, and asks Euler about the number of ways to write a number $m$ as a sum of $\mu$ positive integers.

Euler responds to Naudé on September 12/23, 1740 (Smirnov 1963, §15, pp. 179–206), R. 1904. He explains that he has been afflicted with weak vision for several weeks. Now his strength has just recently recovered, and he has read Naudé's letter with great pleasure. Euler says that Naudé's problem about the partition of numbers seemed very nice, and that he wanted a complete explanation even more because he had previously thought about this but had not been able to obtain a clear solution, "eoque magis absolutam hac de re explanationem videre cuperem, quod ego quoque jam antehac in eadem problemata incidi, neque vero ad satis concinnam atque expeditam solutionem, qualem desideraveram, pertingere potui" ["and I would like to see a complete explanation on this point, all the more because I too have encountered the same problems, but have not been able to come upon a solution as sufficiently neat and easy as I had wanted"].

Then Euler explains his method of finding $\zeta(2n)$, and looks at expressions for the cotangent series. After this Euler deals with partitions of integers. First, if the infinite product $(1+nx)(1+n^2x)(1+n^3x)(1+n^4x)(1+n^5x)$ etc. is multiplied out ("quodsi enim hi factores actu in se invicem multiplicentur" ["if these factors are actually multiplied into each other"]) then in the resulting series, the coefficient of $x^i n^j$ will be the number of ways in which $j$ can be written as a sum of $i$ distinct positive integers, disregarding order. For example, in the series of terms multiplying $x^2$, the coefficient of $n^j$ indicates the number of ways $j$ can be written as a sum of two distinct positive integers. In the series multiplying $x^3$, the coefficient of $n^j$ shows the number of ways $j$ can be written as a sum of three distinct positive integers, and so on.

> It is also apparent that the series by which $x$ is multiplied arises if $n$ is divided by $1-n$; next the series of $xx$ arises from the division of $n^3$ by $(1-n)(1-nn)$; third, the series of $x^3$ is the quotient arising from the division of the power $n^6$ by $(1-n)(1-n^2)(1-n^3)$, and so on.[1]

Letting $(A)^{(p)}$ denote the number of ways $A$ can be written as a sum of $p$ unequal parts, Euler says "Deinceps etiam ad series sequentes ex antecedentibus eruendas inveni hanc regulam" ["I have also found this rule for working out the following series from the preceding"], namely $(A)^{(p)} = (A-p)^{(p)} + (A-p)^{(p-1)}$, where if $A < \frac{p(p+1)}{2}$ then $(A)^{(p)} = 0$.

Euler then looks at partitions into parts that are not necessarily distinct. The infinite product

$$\frac{1}{(1-nx)(1-n^2x)(1-n^3x)\text{ etc.}}$$

"per continuam divisionem evolvatur" ["expanded by repeated division"] gives the series where the coefficient of $x^i n^j$ indicates the number of ways $j$ can be written as a sum of $i$ positive integers. Letting $(A)^{(p)}$ now be the number of ways to write $A$ as a sum of $p$ positive integers, not necessarily distinct, Euler also observes that $(A)^{(p)} = (A-p)^{(p)} + (A-1)^{(p-1)}$.

For $x = 1$ the second infinite product is the generating function for the partition function,

$$\prod_{k=1}^{\infty} \frac{1}{1-z^k} = \sum_{k=0}^{\infty} p(k) z^k, \tag{2}$$

[1] "Praeterea patet seriem, per quam $x$ multiplicatur, oriri, si $n$ dividatur per $1-n$; deinde seriem ipsius $xx$ oriri ex divisione ipsius $n^3$ per $(1-n)(1-nn)$; tertio seriem ipsius $x^3$ esse quotum ex divisione potestatis $n^6$ per divisorem $(1-n)(1-n^2)(1-n^3)$ ortum, et ita porro." (Smirnov 1963, pp. 179–206).

where we define $p(0) = 1$.

Previously, generating functions had been used by Pierre Rémond de Montmort (1678–1719) in his *Essay d'analyse sur les jeux de hazard* (1708), Abraham de Moivre (1667–1754) in his *The doctrine of chances* (1718, first ed.), and Jacob I Bernoulli in his *Ars Conjectandi* (1713). The publication history of the *Ars Conjectandi* van der Waerden and Kohli (van der Waerden 1975, K. 3, pp. 391–401). Cantor (1901, Kapitel 96) summarizes earlier work on combinatorics and generating functions. However, it seems that Euler came up with the idea of generating functions independently (Scharlau 1983, §4).

## 3 Discovery and Proof

The first time the pentagonal number theorem is mentioned in Euler's published correspondence is in a letter from Daniel Bernoulli to Euler on January 28, 1741, R. 140. In this letter, Bernoulli discusses a number of problems that Euler seems to have proposed in a letter on November 30, 1740 which is not extant (cf. Rudio 1915, p. xviii, footnote 3 and Eneström 1906). In particular, Bernoulli mentions the problem of finding all the partitions of an integer:

> The problem about combinations of numbers making a given sum is quite easy in particular cases: but it turns out that one cannot perceive the general rule, although one can still indicate the general method. I have not done the calculation for your example of dividing the number 50 into 7 parts, but I gave it to my cousin Nicolaus Bernoulli, who found exactly the number that Your Honour worked out. The other problem, to transform the expression $\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{n^3}\right)$ into the series $1-\frac{1}{n}-\frac{1}{n^2}+\frac{1}{n^5}+\frac{1}{n^7}-\frac{1}{n^{12}}-\frac{1}{n^{15}}+$ etc. also comes out easily by induction, when one actually multiplies many factors of the given expression.[2]

It looks like Bernoulli is saying that we cannot hope to find a general formula for partitions because of all the particular cases that arise, but we can at least get a general understanding of how partitions work and maybe find rules to make computations simpler. Here a "regula generalis" ["general rule"] would be something like a formula to give the number of partitions of a number $n$, while a "methodus generalis" ["general method"] would be an algorithm for calculating the number of partitions of $n$. An analogy might be how we would not expect to be able to find a simple formula (namely a rule) for the $n$th prime number (cf. Hardy and Wright 1980, §§2.7, 22.3), although, we can find the $n$th prime just by checking numbers one by one, and indeed

[2] "Das problema de combinandis numeris datam summam efficientibus, ist in casibus particularibus gar leicht: einige Circumstanzen machen, dass man die regulam generalem nicht siehet, doch aber kann man die methodum generalem anzeigen. Den calculum von Ihrem Exempel de numero 50 in 7 partes dividendo habe ich nicht gemacht, solches aber meinem Vetter Nicolao Bernoulli gegeben, welcher eben die Zahl gefunden die Ew. herausgebracht. Das ander problema, transformare expressionem $\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{n^3}\right)$ in seriem $1-\frac{1}{n}-\frac{1}{n^2}+\frac{1}{n^5}+\frac{1}{n^7}-\frac{1}{n^{12}}-\frac{1}{n^{15}}+$ etc. kommt auch leicht per inductionem heraus, wenn man viele factores von der proposita expressione actu multipliciret." (Fuss 1843, D. Bernoulli-Euler, Lettre XX).

there are methods that are better than this, e.g., the sieve of Eratosthenes (Hardy and Wright 1980, §1.4).

Daniel Bernoulli was a close colleague to Euler at the St. Petersburg Academy from 1727 to 1733. Bernoulli came to St. Petersburg in 1725, and Euler came in 1727. Daniel Bernoulli was interested in recurrent series and wrote several papers about them (Gillispie 1980, Volume II, "Daniel Bernoulli", pp. 36–46; cf. Hofmann 1959, §11). Euler's correspondence with Daniel Bernoulli's cousin Nicolaus I Bernoulli on the pentagonal number theorem is summarized later in this section.

Euler states without proof on p. 51 of his Fourth Notebook, "Haec expressio $(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)$ etc. transmutatur in hanc seriem:

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+\text{etc.}$$

exponentes continentur in hac forma $\frac{3xx\pm x}{2}$." ["The expression $(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)$ etc. transforms into this series:

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+\text{etc.}$$

whose exponents are contained in the form $\frac{3xx\pm x}{2}$."]

Euler's Fourth Notebook covers the period 1740–1744, and Euler's Fifth Notebook, titled "Diarium mathematicum", covers 1749–1753 (Mikhajlov 1959, §III). It seems that Euler had one notebook covering the years 1745–1749 that was not kept.

The first paper in which Euler mentions the pentagonal number theorem is his "Observationes analyticae variae de combinationibus", presented to the St. Petersburg Academy on April 6, 1741 (Nevskaja 2000, p. 256) and published in 1751 in the *Commentarii* (Euler 1751b), E158. In this article, Euler first develops some general results about infinite series and products. Generally, Euler is working out a lot of relations between different infinite series and products that are symmetric in the (infinitely many) variables $a, b, c, d, e$, etc.

Then in section §§17–37 of the paper Euler talks about partitions, applying the general results from the first part of the paper with $a = n, b = n^2, c = n^3$, etc. This lets us group lots of terms together as geometric series. For example, instead of $ab + ac + ad + ae + bd +$ etc. we now have $\frac{n^3}{(1-n)(1-n^2)}$. First, Euler proves results about partitions of $m$ into $\mu$ distinct parts and partitions of $m$ into $\mu$ not necessarily distinct parts, which he denotes by $m^{(\mu)i}$ and $m^{(\mu)}$ respectively. In §22, Euler starts to look at how the denominator of $\frac{n^{k(k+1)/2}}{(1-n)(1-n^2)\cdots(1-n^k)}$ expands from a product into a sum, for example $\frac{n^6}{(1-n)(1-n^2)(1-n^3)} = \frac{n^6}{1-n-n^2+n^4+n^5-n^6}$. In §27, he proves the recurrence relation $m^{(\mu-1)i} = (m+\mu)^{(\mu)i} - m^{(\mu)i}$ and in §34, the recurrence relation $m^{(\mu)} = (m-\mu)^{(\mu)} + (m-1)^{(\mu-1)}$, which as we noted in §2 he had already stated in his 1740 letter to Naudé. In §36 of the paper, Euler states that he has found an interesting result that he has not been able to prove with complete rigor:

> There is a noteworthy observation to make at the end of this article, which however I have not been able to prove with geometric rigor. Namely I have observed

that this product of infinitely many factors

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}$$

if expanded by actual multiplication yields this series

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+n^{51}+\text{etc.},$$

where the only powers of $n$ that occur in it are those whose exponents are contained in the form $\frac{3xx\pm x}{2}$; and if $x$ is an odd number the powers of $n$, which are $n^{\frac{3xx\pm x}{2}}$, have the coefficient $-1$, while if $x$ is an even number then the powers $n^{\frac{3xx\pm x}{2}}$ have the coefficient $+1$.[3]

In §37 of the paper, Euler notes as well that the product of the power series on the right-hand side of (2) and the above power series is unity, since they are the series expansions of reciprocal infinite products. We talk about Euler's recurrence relation for $p(n)$ more in §5.

This article was reviewed in the June 1755 *Nova acta eruditorum* (pp. 361–363),

An observation is made at the conclusion of this paper which, though the Author has not yet been able to demonstrate it with geometric rigor, is still noteworthy. For with the terms

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5), \&\text{c.}$$

actually multiplied, a series is produced in which the only powers of the letter $n$ that occur are those whose exponents are contained in the form $\frac{3x^2\pm x}{2}$. Also, if $x$ is an odd number, the powers $n^{\frac{3x^2\pm x}{2}}$ would have the coefficient $-1$, while truly if $x$ is an even number, the coefficient will be $+1$.[4]

---

[3] "Finem huic dissertationi faciat observatio notatu digna, quam quidem rigore geometrico demonstrare mihi nondum licuit. Observavi scilicet hoc infinitorum factorum productum

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}$$

si per multiplicationem actu evolvatur, praebere hanc seriem

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+n^{51}+\text{etc.},$$

ubi eae tantum ipsius $n$ potestates occurrunt, quarum exponentes continentur hac forma $\frac{3xx\pm x}{2}$. Ac si $x$ sit numerus impar, potestates ipsius $n$, quae sunt $n^{\frac{3xx\pm x}{2}}$, coefficientem habent $-1$; si autem $x$ sit numerus par, tum potestates $n^{\frac{3xx\pm x}{2}}$ coefficientem habent $+1$." (Euler 1751b, §36), E158.

[4] "Coronam huic dissertationi imponit observatio, quam quidem rigore geometrico demonstrare Auctori nondum licuit, notatu tamen digna. Terminis enim

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5), \&\text{c.}$$

Sandifer (2007b, Chap. 48) gives a step by step explanation of E158.

E158 was the last paper Euler presented to the St. Petersburg Academy before he went to Berlin. He left St. Petersburg on June 19, 1741, and arrived Berlin on July 25 (Gillispie 1980, Volume IV, “Leonhard Euler”, pp. 467–484).

The next time the expansion of the infinite product $\prod_{n=1}^{\infty}(1-x^n)$ into a series comes up in Euler’s correspondence is in a letter from Euler to Nicolaus I Bernoulli (1687–1759) on September 1, 1742 (Fellmann and Mikhajlov 1998, pp. 510–532). Euler discusses the problem of expressing the power series for $\sin s$, $s - \frac{s^3}{6} + \frac{s^5}{120} -$ etc. as the infinite product $s\left(1-\frac{s^2}{\pi^2}\right)\left(1-\frac{s^2}{4\pi^2}\right)\left(1-\frac{s^2}{9\pi^2}\right)$ etc. in this letter. Euler tells Bernoulli the generating function for the partition function, and that,

> This series also arises from division, if unity is divided by
>
> $$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.};$$
>
> this product, if actually expanded, gives this expression
>
> $$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-\text{etc.}$$
>
> in which I have not been able to observe from the nature of the series how the exponents proceed, although by induction I have concluded that no other exponents occur except those contained in the formula $\frac{3xx\pm x}{2}$; and that a power of $n$ has the $+$ sign if its exponent arises from an even number substituted for $x$.[5]

Then Euler talks about determining the power series expansion in $a$ of $s^2$ for $s = \sum_{k=0}^{\infty}\frac{a^k}{kn+1}$. He says that

$$\frac{s^2}{2} = \sum_{k=0}^{\infty}\frac{a^k}{kn+2}\sum_{j=0}^{k}\frac{1}{jn+1},$$

“cujus veritas quidem per probationem, sed tamen difficulter elucet; demonstrationem vero nonnisi per differentiationem et integrationem adornare possum” [“whose

Footnote 4 continued
actu multiplicatis, prodit series in qua eae tantum occurrunt potestates litterae $n$, quarum exponentes forma $\frac{3x^2\pm x}{2}$ continentur. Ac si $x$ sit numerus impar, potestates $n^{\frac{3x^2\pm x}{2}}$ coefficientem habent $-1$, sin vero $x$ numerus par sit, coefficiens erit $+1$.” June 1755 *Nova acta eruditorum* (pp. 361–363).

[5] “Oritur autem haec series per divisionem, si unitas dividatur per

$$(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5) \text{ etc.}$$

quod productum si actu evolvatur dat hanc expressionem

$$1-n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-\text{etc.}$$

in quam quemadmodum exponentes progrediuntur ex natura seriei perspicere non potui, per inductionem autem conclusi alios exponentes non occurrere, nisi qui in formula $\frac{3xx\pm x}{2}$ contineantur; hocque ita ut potestas ipsius $n$ habeat signum $+$ si ejus exponens ex numero pari pro $x$ substituto nascatur.” (Fellmann and Mikhajlov 1998, pp. 510–532).

truth indeed appears by examination, though with difficulty. In truth I am only able to furnish a demonstration by differentiation and integration"].

Nicolaus I Bernoulli replies to Euler on October 24, 1742 (Fellmann and Mikhajlov 1998, pp. 532–550). Bernoulli says that Euler's discovery of the generating function $\prod_{n=1}^{\infty} \frac{1}{1-x^n}$ for $p(n)$ is an elegant result. Then Bernoulli works out a "novam speciem trianguli arithmetici" ["new type of arithmetic triangle"] for recursively calculating partitions. He works out two triangles. In the first, the $k$th entry in the $r$th row (ending in the Roman numeral $r$) counting from right to left is equal to the number of partitions of $r$ in which the largest part occurs exactly $k$ times; however, the way Bernoulli writes the triangle, this breaks down in the $\left(\lfloor \frac{r}{2} \rfloor + 1\right)$th entry in each row, i.e., the left-most nonzero entry in each row. Bernoulli puts 1 in each of these spots, while there are in fact 0 partitions of $r$ with the largest part occurring exactly $\lfloor \frac{r}{2} \rfloor + 1$ times. However, the way Bernoulli uses the triangle, these 1's could in fact be put at the left end of the row instead of in the $\left(\lfloor \frac{r}{2} \rfloor + 1\right)$th entry of the row. For each partition of $r + 1$ with largest part occurring exactly once, removing 1 from the largest part yields a partition of $r$; and on the other hand, adding 1 to the largest part of a partition of $r$ yields a partition of $r + 1$ with largest part occurring exactly once. Hence the 1st entry in the $r$th row is $p(r - 1)$. Bernoulli spells out what the entries are in row IX, $r = 9$. There are 22 partitions of 9 in which the largest part occurs exactly once, 4 in which it occurs exactly twice, 2 in which it occurs exactly three times, and so on. Bernoulli explains how the entries in the triangle can be recursively calculated, which amounts to there being as many partitions of $r$ with the largest part occurring exactly $k$ times as there are partitions of $r - k$ with the largest part occurring at least $k$ times. It is straightforward to show this using the Ferrers diagram (Andrews 1998, §1.3) for partitions. In each partition of $r$ with the largest part occurring exactly $k$ times, subtract 1 from each of the $k$ largest parts. Then there is a partition of $r - k$ with the largest part occurring at least $k$ times. For example, there are 14 partitions of 13 with largest part occurring exactly twice (here $r = 13$ and $k = 2$), and there are $8 + 3 + 1 + 1 + 1$ partitions of $11 = 13 - 2$ with largest part occurring at least twice.

In the second triangle, the entry in column $r$ (beneath the Roman numeral $r$) and row $j$ counting downwards starting at row $j = 0$ represents the number of partitions of $r$ with largest part equal to $j$. For example, for XII, $r = 12$, 13 is the entry for $j = 5$, in other words there are 13 partitions of 12 with largest part equal to 5. Let $(j, r)$ be the entry in row $j$ and column $r$, so $(5, 12) = 13$. The entries are calculated recursively as follows: the entry in $(j, r)$ is equal to the sum of all the entries $(k, r - j)$, $0 \leq k \leq j$. For $r = 12$, $j = 5$, the entry $(5, 12) = 13$ is equal to the sum of all the entries $(k, 7)$, $0 \leq k \leq 5$, which is $(0, 7)+(1, 7)+(2, 7)+(3, 7)+(4, 7)+(5, 7) = 0+1+3+4+3+2$. For, adding the part $j$ to a partition of $r - j$ with largest part no greater than $j$ yields a partition of $r$ with largest part $j$.

Bernoulli goes on to mention the pentagonal number theorem,

> In this series
>
> $$n^0 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - \text{etc.}$$
>
> which you have found equal to the expanded product $(1-n)(1-nn)(1-n^3)$ etc. the differences of the exponents progress as 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, etc. which

numbers are alternately drawn from the series 1, 3, 5, 7, 9, etc. and from the series 1, 2, 3, 4, 5, etc. This property will perhaps be able to be demonstrated from the nature of the matter and not only by induction; but I do not have time at the moment to look into this matter.[6]

Then Bernoulli talks about the series $s = \sum_{k=0}^{\infty} \frac{a^k}{kn+1}$, writing "Valde mihi placet methodus inveniendi et summandi series per differentiationem et integrationem, eamque ulterius extendi posse existimo" ["I find very pleasant this method for finding and summing series by differentiation and integration, and I believe that it can be further extended"]. Bernoulli works out $s^2$ and $s^3$.

In fact one can easily derive Euler's statement that the general term is of the form $\frac{3x^2 \pm x}{2}$ from Bernoulli's statement about the differences of the terms. From the theory of finite differences (Chrystal 1964, Chap. XXXI, §§1–5), if $a_0, a_1, \ldots$ is an arithmetic progression of order $n$ then

$$a_k = \sum_{j=0}^{n} \binom{k}{j} \Delta^j a_0.$$

Let $c_0 = 0, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 7, c_5 = 12, c_6 = 15, c_7 = 22, c_8 = 26, c_9 = 35$, etc., and let $a_k = c_{2k}$ and $b_k = c_{2k+1}$, so $a_0 = 0, a_1 = 2, a_2 = 7, a_3 = 15, a_4 = 26$, etc. and $b_0 = 1, b_1 = 5, b_2 = 12, b_3 = 22, b_4 = 35$, etc. Bernoulli has observed that $b_k - a_k = 2k + 1$ and $a_k - b_{k-1} = k$. Hence, $b_k - b_{k-1} = 3k + 1$ and $a_{k+1} - a_k = 3k+2$. Here, $\Delta^0 a_0 = 0$, $\Delta a_0 = 2$, $\Delta^2 a_0 = 3$, so $a_k = 0 + 2k + \frac{k(k-1)}{2} \cdot 3 = \frac{3k^2+k}{2}$. On the other hand, $\Delta^0 b_0 = 1$, $\Delta b_0 = 4$, $\Delta^2 b_0 = 3$, so $b_k = 1 + 4k + \frac{3k^2-3k}{2}$, thus $b_{k-1} = \frac{3k^2-k}{2}$.

But it can be useful to formulate a problem in different ways. As we mentioned above, Nicolaus I Bernoulli also had some good ideas about making a "triangle" for finding recurrence relations for partitions.

Euler next writes to Nicolaus I Bernoulli on November 10, 1742 (Fellmann and Mikhajlov 1998, pp. 551–579), the last letter in their correspondence that deals with the pentagonal number theorem. Euler tells Bernoulli how pleased he was with the ideas Bernoulli communicated about partitio numerorum, "Plurimum autem me delectarunt, quae de partitione numerorum (sic enim appellabat hunc problema Clar. Naudaeus, qui id mihi primum jam Petropoli proposuerat) mecum communicare voluisti" ["What you were gracious enough to communicate to me about partitio numerorum (for this is what the problem was called by Mr. Naudé, who first proposed it to me when I was in St. Petersburg) was a great pleasure to me"]. Then Euler explains to

[6] "In hac serie

$$n^0 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - \text{etc.}$$

quam invenisti aequalem producto $(1-n)(1-nn)(1-n^3)$ etc. expanso, differentiae exponentium progrediuntur ita 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, etc. qui numeri alternatim depromti sunt ex serie 1, 3, 5, 7, 9, etc. et ex serie 1, 2, 3, 4, 5, etc. quae proprietas fortassis ex natura rei nec solum per inductionem demonstrari poterit; sed in hanc rem inquirere nunc non vacat." (Fellmann and Mikhajlov 1998, pp. 532–550).

Bernoulli in more detail his approach to finding the number of partitions of a given number $N$ into $n$ distinct parts, and the number of partitions of a given number $N$ into $n$ parts which are not necessarily distinct. For the first problem Euler explains that if the product

$$(1+mz)(1+m^2z)(1+m^3z)(1+m^4z) \text{ etc.}$$

is expanded into a series, "hic ex natura genesis coefficiens numericus cujusque termini indicat, quot variis modis exponens ipsius $m$ in tot partes inaequales dispertiri possit, quot exponens ipsius $z$ contineat unitates" ["here because of how they are formed, the numerical coefficient of each term indicates how many different ways the exponent of $m$ can be distributed into as many unequal parts as the exponent of $z$."] That is, the sum of all the coefficients of the $m^N z^n$ terms is equal to the number of partitions of $N$ into $n$ distinct parts. Letting $(1+mz)(1+m^2z)(1+m^3z)(1+m^4z)$ etc. $= 1+\alpha z+\beta z^2+\gamma z^3+\delta z^4+$ etc., then because $(1+m^2z)(1+m^3z)(1+m^4z)(1+m^5z)$ etc. $=$ $1+\alpha mz+\beta m^2z^2+\gamma m^3z^3+\delta m^4z^4+$ etc., Euler gets

$$\begin{aligned}&(1+mz)(1+m^2z)(1+m^3z)(1+m^4z) \text{ etc.}\\ &\quad = 1+\frac{mz}{1-m}+\frac{m^3z^2}{(1-m)(1-m^2)}+\frac{m^6z^3}{(1-m)(1-m^2)(1-m^3)}+\text{etc.}\end{aligned}$$

Euler explains similarly that when we expand

$$\frac{1}{(1-mz)(1-m^2z)(1-m^3z)(1-m^4z) \text{ etc.}}$$

into a series, "coefficiens cujusque termini indicat quot variis modis exponens ipsius $m$ dispertiri possit in tot partes quot exponens ipsius $z$ continet unitates" ["the coefficient of each term indicates how many ways the exponent of $m$ can be distributed into as many parts as the exponent of $z$."] Euler also states

$$\begin{aligned}&\frac{1}{(1-mz)(1-m^2z)(1-m^3z) \text{ etc.}}\\ &\quad = 1+\frac{mz}{1-m}+\frac{m^2z^2}{(1-m)(1-m^2)}+\frac{m^3z^3}{(1-m)(1-m^2)(1-m^3)}+\text{etc.}\end{aligned}$$

and says that it follows by the same argument as above.

Euler then writes:

> That the expression $(1-n)(1-n^2)(1-n^3)(1-n^4)$ etc. expanded gives the series $1-n-n^2+n^5+n^7-$etc., in which no other exponents occur except those contained in $\frac{3xx\pm x}{2}$, I believe I have concluded by a legitimate induction; but at the same time, I have in no way been able to find a demonstration, even though I expended no small amount of time on it. I have indeed found that the expression $(1-n)(1-n^2)(1-n^3)(1-n^4)$ etc. can be transformed into this series

$$1-\frac{n}{1-n}+\frac{n^3}{(1-n)(1-n^2)}-\frac{n^6}{(1-n)(1-n^2)(1-n^3)}+\text{etc.}$$

whose value would then be equal to the sum of the series $1-n^1-n^2+n^5+n^7-n^{12}-n^{15}+$ etc. Now since the law of the progression of this series is known, then the character of the other series $1+1n+2n^2+3n^3+5n^4+7n^5+$ etc. can be described as recurrent, having this scale of relation

$$1+1+0+0-1+0-1+0+0+0+0+1+0+0+1+0+0+\text{etc.}$$

by means of which it may be easily continued.

What you have written to me, worthy Sir, about the investigation of powers of the series

$$1+\frac{a}{n+1}+\frac{a^2}{2n+1}+\frac{a^3}{3n+1}+\text{etc.}$$

shows quite clearly how much your method, proceeding a priori, surpasses that other method, which I used, proceeding a posteriori. For, from the series which you showed for the cube of this series it would be very difficult to find its sum a posteriori. And so, the more profit I hope that I will reap from this method, the more grateful I am to you.

By the way, on occasion of the series $1-n-n^2+n^5+n^7-n^{12}-n^{15}+$etc. it came to my mind how many truths in mathematics, particularly about the properties of numbers, we take to be accepted just by induction. Of this kind are that every number is a sum of four or fewer squares; that every prime number of the form $4n+1$ is a sum of two squares; and that the sum of two cubes cannot be a cube.[7]

[7] "Quod expressio $(1-n)(1-n^2)(1-n^3)(1-n^4)$ etc. evoluta det seriem $1-n-n^2+n^5+n^7-$ etc., in qua alii exponentes non occurrunt nisi qui contineantur in $\frac{3xx\pm x}{2}$, per legitimam inductionem mihi equidem conclusisse videor; interim tamen demonstrationem nullo pacto invenire potui, etiamsi non parum temporis in id impenderem. Inveni autem expressionem $(1-n)(1-n^2)(1-n^3)(1-n^4)$ etc. quoque in hanc seriem transmutare posse

$$1-\frac{n}{1-n}+\frac{n^3}{(1-n)(1-n^2)}-\frac{n^6}{(1-n)(1-n^2)(1-n^3)}+\text{etc.}$$

cujus adeo valor aequatur summae seriei $1-n^1-n^2+n^5+n^7-n^{12}-n^{15}+$ etc. Quare cum lex progressionis hujus seriei sit cognita, hinc alterius seriei $1+1n+2n^2+3n^3+5n^4+7n^5+$ etc. indoles ita describi poterit, ut sit recurrens, habens scalam relationis hanc

$$1+1+0+0-1+0-1+0+0+0+0+1+0+0+1+0+0+\text{etc.}$$

cujus ope facile continuatur.

Quae mihi scripsisti Vir Amplissime de investigatione potestatum seriei

$$1+\frac{a}{n+1}+\frac{a^2}{2n+1}+\frac{a^3}{3n+1}+\text{etc.}$$

It turns out that Euler does not use the device of differentiation and integration in his proofs of the pentagonal number theorem. However, Euler does use these methods to find his recurrence relation for sums of divisors, which we talk about in §4 of this article. Euler's work on the theorems that every nonnegative integer is a sum of four squares, that every prime of the form $4n + 1$ is a sum of two squares, and that a sum of two cubes cannot be a cube is explained by Weil (1984, Chap. III).

The next time Euler discusses the pentagonal number theorem in his correspondence is in a letter to Christian Goldbach (1690–1764) on October 4/15, 1743 (Juškevič and Winter 1965, Brief 74). Euler states here,

> When these factors $(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)(1-n^6)$ etc. are actually multiplied with each other out to infinity, the following series comes out
>
> $$1 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - n^{40} \\ +n^{51} + n^{57} - \text{etc.},$$
>
> from which it is apparent by induction that all the terms are contained in the form $n^{\frac{3xx \pm x}{2}}$, and that they have the prefixed sign $+$ when $x$ is an even number, but the sign $-$ when $x$ is an odd number. But I have still not been able to find a method by which I could demonstrate the identity of these two expressions. Prof. Nicolaus Bernoulli also has not been able to get anything out of it except by induction.[8]

Goldbach replies to Euler's problem in a letter dated December 1743 (Juškevič and Winter 1965, Brief 75). His copybook has been checked and we now know that the letter was written December 14; Goldbach kept correspondence books in which he entered complete or partial copies of the letters he sent to various correspondents. He does not give any explicit ideas for how to tackle proving the pentagonal number theorem, and instead poses a new related problem. He says,

---

Footnote 7 continued

satis declarant quantopere methodus Tua a priori procedens praestet alteri illi a posteriori, qua usus sum. Ex serie enim, quam pro cubo hujus seriei exhibuisti, difficillimum foret a posteriori ejus summam invenire; eo majores igitur Tibi habeo gratias, quo majorem fructum me ex ea methodo capturum spero.

Caeterum occasione illius seriei $1 - n - n^2 + n^5 + n^7 - n^{12} - n^{15} +$ etc. mihi in mentem venit, quot veritates in mathesi soli inductioni acceptas referamus, praecipue circa proprietates numerorum. Cujusmodi sunt omnem numerum esse summam quatuor pauciorumve quadratorum; item omnem numerum primum formae $4n+1$ esse summam duorum quadratorum; item summam duorum cuborum non posse esse cubum." (Fellmann and Mikhajlov 1998, pp. 551–579).

[8] "Wann diese factores in infinitum würklich mit einander multipliziert werden, $(1-n)(1-n^2)(1-n^3)(1-n^4)(1-n^5)(1-n^6)$ etc., so kommt nachfolgende Series heraus

$$1 - n^1 - n^2 + n^5 + n^7 - n^{12} - n^{15} + n^{22} + n^{26} - n^{35} - n^{40} \\ +n^{51} + n^{57} - \text{etc.},$$

wovon per inductionem leicht erhellet, daß omnes termini in hac forma begriffen sind $n^{\frac{3xx \pm x}{2}}$, und das signum $+$ praefixum haben, wann $x$ ein numerus par, das signum $-$ aber, wann $x$ ein numerus impar ist. Ich habe aber noch keine Methode finden können, wodurch ich die Identität dieser 2 Expressionen demonstrieren könnte. Der H. Prof. Nicolaus Bernoulli hat auch praeter inductionem nichts darüber herausbringen können." (Juškevič and Winter 1965, Brief 74).

> A particular problem has occurred to me concerning the series $(1-n)(1-n^2)(1-n^3)$ etc.: given a series $A$ of infinitely many terms proceeding with the signs $+$ and $-$ varying in a given order, to find a series $B$ of such a nature that in the product $AB$, the signs $+$ and $-$ proceed in the same order as they were proceeding in $A$. This problem can very easily be solved in the case $A = (1-n)(1-n^2)(1-n^3)$ etc., for although Your Honour has noted that the signs $+$, $-$ alternate in a quite unusual way, when I put $B = \left(1-n^{\frac{1}{2}}\right)\left(1-n^{\frac{3}{2}}\right)\left(1-n^{\frac{5}{2}}\right)$ etc., thus $A$ multiplied by $B$ becomes a new series, which itself has the same variation of signs.[9]

Probably what Goldbach means by "die signa +,− auf eine gar ungewöhnliche Art abwechseln" ["the signs +, − alternate in a quite unusual way" ] is the following. When we expand the product $\prod_{n=1}^{\infty}(1-x^n)$ into a power series $\sum_{n=0}^{\infty} a_n x^n$, a rule for the coefficients ("signa", since $a_n = -1, 0$ or 1) $a_n$ is not obvious like a geometric series. It looks like Goldbach did not have any ideas himself how to prove the pentagonal number theorem, but wanted to pose a related question that might inspire Euler.

Euler writes back to Goldbach about this on January 10/21, 1744 (Juškevič and Winter 1965, Brief 76), in which he says,

> Your Honour's reflection on the expression $(1-n)(1-n^2)(1-n^3)$ etc. with respect to the product $\left(1-n^{\frac{1}{2}}\right)\left(1-n^{\frac{3}{2}}\right)\left(1-n^{\frac{5}{2}}\right)$ etc. that this product when expanded gives the same variation of signs, could perhaps be put to some use in other investigations; but so far I haven't been able to make any use of it in the series which I have derived.[10]

On March 25/April 5, 1746, Euler writes another letter to Goldbach about the pentagonal number theorem (Juškevič and Winter 1965, Brief 102). In this letter, he gives some expansions of products. First, he lets

$$s = (1+a)(1+a^2)(1+a^4)\cdots(1+a^{2^n}).$$

But $(1-a)s = 1 - a^{2^{n+1}}$, and hence $s = \frac{a^{2^{n+1}}-1}{a-1}$. However, there is not an obvious simple expression for the product $(1-a)(1-a^2)(1-a^4)\cdots(1-a^{2^n})$, and indeed

9 "Bei der serie $(1-n)(1-n^2)(1-n^3)$ etc. ist mir ein besonderes problema eingefallen: Data serie $A$ infinitorum terminorum signis $+$ et $-$ dato ordine variantibus procedentium invenire seriem $B$ huius naturae, ut in producto $AB$ signa $+$ et $-$ eodem ordine sibi succedant, quo ordine sibi succedebant in $A$. Dieses problema kann sehr leicht solvieret werden in dem casu $A = (1-n)(1-n^2)(1-n^3)$ etc. obgleich darin, wie E. Hochedelgeb. angemerkt haben, die signa +,− auf eine gar ungewöhnliche Art abwechseln, dann wann ich setze $B = \left(1-n^{\frac{1}{2}}\right)\left(1-n^{\frac{3}{2}}\right)\left(1-n^{\frac{5}{2}}\right)$ etc., so wird $A$ multiplicata per $B$ eine neue series, welche dieselbe variationem signorum in sich hält." (Juškevič and Winter 1965, Brief 75).

10 "Ew. Wohlgeb. Reflexion über die Expression $(1-n)(1-n^2)(1-n^3)$ etc. in Ansehung eines factoris $\left(1-n^{\frac{1}{2}}\right)\left(1-n^{\frac{3}{2}}\right)\left(1-n^{\frac{5}{2}}\right)$ etc., also, daß das factum, si evolvatur, eine gleiche Abwechslung der signorum $+$ et $-$ gebe, könnte vielleicht bei andern Untersuchungen einigen Vorteil bringen; allein in der Serie, welche ich daraus hergeleitet, habe ich daraus noch keinen Nutzen ziehen können." (Juškevič and Winter 1965, Brief 76).

Euler remarks that actual multiplication of the first factors of the product gives

$$1 - a - a^2 + a^3 - a^4 + a^5 + a^6 - a^7 - a^8 + a^9 + a^{10} - a^{11} + a^{12} - a^{13} - a^{14} + a^{15} - a^{16} + a^{17} \text{ etc.},$$

"wo die ordo signorum merkwürdig ist" ["where the order of the signs is remarkable"]. Euler then says,

> I believe that I have already also written to Your Honour that when this infinite product $(1-a)(1-a^2)(1-a^3)(1-a^4)(1-a^5)$ etc. is expanded, this series $1 - a - a^2 + a^5 + a^7 - a^{12} - a^{15} + a^{22} + a^{26} - a^{35} - a^{40} +$ etc. results, where the sequence of the exponents is very remarkable, and it can be determined by induction that everything is contained in the formula $\frac{3xx \pm x}{2}$, although I have not been able to derive this observed law from the nature of the matter.[11]

Euler is saying that he has found a pattern when he actually works out particular cases, but that he has not so far been able to understand why this pattern would work in general. However, he has some sense that a proof would be similar to how he verifies the individual cases. For, when the partial products $(1-x)\cdots(1-x^N)$ and $(1-x)\cdots(1-x^{N+1})$ are expanded as sums, they will have the same coefficients at least for all powers $\leq N$, since all the terms in $-x^{N+1}(1-x)\cdots(1-x^N)$ will have exponents $\geq N+1$. Thus to find the coefficients of the powers $\leq N$ in the series expansion of the infinite product $(1-x)(1-x^2)\cdots$, we need to expand at most the first $N$ factors. Still, in effect this means that he has only been able to guess the pattern, and has not found a proof.

In a letter to Euler on April 15, 1747 (Juškevič and Winter 1965, Brief 114), Goldbach responds to Euler's previous letter, which had given a recurrence relation for the divisor function (see §4 of this article). In that letter, Euler had remarked that his proof assumes the pentagonal number theorem, which he had not been able to prove rigorously. Goldbach declares that,

> the observation which Your Honour has communicated to me seems to me by means of the aforementioned induction already shown to the extent that one could bet on its truth one hundred to one. Otherwise, Your Honour already observed awhile ago that $A \ldots (1-x)(1-x^2)(1-x^3)(1-x^4)$ etc. $= B \ldots 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} +$ etc. and I remember that I derived from it the very simple consequence that, when the powers of $x$ are doubled in $B$ and put as $C = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{12} - x^{30} +$ etc., then it must be $\frac{C}{B} = (1-x)(1-x^3)(1-x^5)(1-x^7)$ etc.[12]

[11] "Ew. Hochwohlgeb. glaube ich auch schon geschrieben zu haben, daß wann man dieses productum infinitum $(1-a)(1-a^2)(1-a^3)(1-a^4)(1-a^5)$ etc. evolviert, diese series herauskomme $1 - a - a^2 + a^5 + a^7 - a^{12} - a^{15} + a^{22} + a^{26} - a^{35} - a^{40} +$ etc., wo der ordo exponentium sehr merkwürdig ist und sich per inductionem also bestimmen läßt, daß alle in hac formula $\frac{3xx \pm x}{2}$ enthalten sind, ungeacht ich diese legem observatam noch nicht ex rei natura habe herausbringen können." (Juškevič and Winter 1965, Brief 102).

[12] "die Observation, welch Ew. H. mir kommuniziert haben, scheinet mir bereits durch die angeführte Induktion dermaßen erwiesen, daß man auf deren Wahrheit hundert gegen eins halten könnte. Sonst haben

In his previous letter to Goldbach, supposing the pentagonal number theorem Euler proved a recurrence relation for the sum of divisors function $\sigma(n)$. But Euler said he could not find a rigorous proof of the pentagonal number theorem, which left this recurrence relation hanging. Goldbach seems to say that Euler gave a pretty good argument a few years ago; yet, Euler would like a more rigorous proof. Also, it is in fact $\frac{B}{C} = (1-x)(1-x^3)(1-x^5)(1-x^7)$ etc., not $\frac{C}{B} = (1-x)(1-x^3)(1-x^5)(1-x^7)$ etc. like Goldbach says here.

Euler replies to Goldbach on May 6, 1747 (Juškevič and Winter 1965, Brief 115), saying:

> The remark which Your Honour made about the equality $A \ldots (1-x)(1-x^2)(1-x^3)(1-x^4)$ etc. $= B \ldots 1-x-x^2+x^5+x^7-$ etc., that, if $C = 1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+$ etc. then it would be $\frac{C}{B} = (1-x)(1-x^3)(1-x^5)$ etc., I still remember well. But neither from this nor from other approaches have I been able to properly demonstrate the equality between the formulas $A$ and $B$; for that $A = B$ and that in $B$ the exponents of $x$ just come out according to the series $1, 2, 5, 7, 12, 15, 22, 26, 35, 40$, I have shown only by induction, which I have in fact continued so far that I can hold the theorem as a complete truth; I would however be very eager to see a direct demonstration of it, which certainly would open the way to the discovery of many other marvelous properties of the integers: but so far all my efforts towards this have been in vain.[13]

Then Euler gives some results about representing numbers as sums of squares. As we remarked about the above letter, it is $\frac{B}{C} = (1-x)(1-x^3)(1-x^5)$ etc. not $\frac{C}{B} = (1-x)(1-x^3)(1-x^5)$ etc.; photographs of the original letters have been checked and indeed Goldbach and Euler both make this slip of the pen. Possibly Goldbach made this slip because he uses $A$ and $B$ differently here than in his December 1743 letter (Juškevič and Winter 1965, Brief 75), which has already been quoted above. There $A$ is a product over the even powers of $n^{1/2}$ and $B$ is a product over the odd powers of $n^{1/2}$, and thus $AB$ is a product over all powers of $n^{1/2}$, while here $A = B$ is a product over all powers of $x$ and $C$ is a product over the even powers of $x$, and thus $\frac{B}{C}$ is a product over all the odd powers of $x$.

Footnote 12 continued
E. H. schon längst angemerket, daß $A \ldots (1-x)(1-x^2)(1-x^3)(1-x^4)$ etc. $= B \ldots 1-x-x^2+x^5+x^7-x^{12}-x^{15}+$ etc. und ich erinnere mich, daß ich daraus die an sich selbst sehr leichte consequence gezogen, daß, wann die potestates ipsius $x$ in $B$ verdoppelt werden und $C = 1-x^2-x^4+x^{10}+x^{14}-x^{12}-x^{30}+$etc. gesetzt wird, alsdann $\frac{C}{B} = (1-x)(1-x^3)(1-x^5)(1-x^7)$ etc. sein muß." (Juškevič and Winter 1965, Brief 114).

13 "Die Anmerkung welche Ew. Hochwohlgeb. über die Gleichheit $A \ldots (1-x)(1-x^2)(1-x^3)(1-x^4)$ etc. $= B \ldots 1-x-x^2+x^5+x^7-$etc. gemacht, daß, wann $C = 1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+$etc., alsdann sei $\frac{C}{B} = (1-x)(1-x^3)(1-x^5)$ etc., erinnere ich mich noch wohl. Ich habe aber weder daraus, noch aus andern Betrachtungen die Gleichheit zwischen den Formuln $A$ und $B$ richtig dartun können; dann daß $A = B$ und daß in $B$ die Exponenten von $x$ just nach dieser Serie $1, 2, 5, 7, 12, 15, 22, 26, 35, 40$, etc. fortgehen, habe ich auch nur per inductionem geschlossen, welche ich zwar so weit fortgesetzet, daß ich die Sach für völlig wahr halten kann; allein ich wäre sehr begierig, davon eine demonstrationem directam zu sehen, welche gewiß zu Entdeckung vieler anderer herrlichen Eigenschaften der Zahlen den Weg bahnen würde: bisher ist aber alle meine darauf angewandte Mühe umsonst gewesen." (Juškevič and Winter 1965, Brief 115).

Later, in a letter to Jean le Rond d'Alembert (1717–1783) on December 30, 1747 (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 11), Euler writes that he has learned from Maupertuis (the President of the Berlin Academy) that d'Alembert wants to leave his mathematical research for some time to regain his health. Euler approves this idea, and goes on to say,

> If in your respite you have the desire to do some research that is not too demanding, let me take the liberty of proposing to you this expression $(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., which, when expanded by actual multiplication, gives this series
>
> $$1-x^1-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+x^{57}$$
> $$-x^{70}-x^{77}+\text{etc.}$$
>
> which seems quite remarkable to me because of the pattern that one easily discovers; but I do not see how this law could be deduced from the proposed expression otherwise than by induction.[14]

As a postscript Euler says[15]

> If we let $s=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. then I can demonstrate that we will have
>
> $$s=1-\frac{x}{1-x}+\frac{x^3}{(1-x)(1-x^2)}-\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$
> $$+\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}-\text{etc.}$$

D'Alembert may have been fatigued because of his work on the three-body problem/the motion of the line of apsides of the moon's orbit, and because of his involvement with the *Encyclopédie* (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No.

14 "Si dans vos divertissemens vous avés envie de faire quelque recherche, qui ne demande pas beaucoup d'application, je prendrai la liberté de vous proposer cette expression $(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., laquelle étant developpée par la multiplication actuelle, donne cette serie

$$1-x^1-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+x^{57}-x^{70}-x^{77}+\text{etc.}$$

qui me paroit fort remarquable à cause de la loi qu'on y decouvre aisement; mais je ne voi pas comment cette loi pourroit être deduite sans induction de l'expression proposée même." (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 11).

15 "Si l'on met $s=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. je puis demontrer qu'il y aura

$$s=1-\frac{x}{1-x}+\frac{x^3}{(1-x)(1-x^2)}-\frac{x^6}{(1-x)(1-x^2)(1-x^3)}$$
$$+\frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}-\text{etc."}$$

(Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 11).

11, Note 1). Euler probably thought that this might catch d'Alembert's interest. Euler was looking for people to talk to about number theory in general (see Scriba 1984, §4).

D'Alembert replies to Euler in a letter on January 20, 1748 (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 12). He begins by saying that by taking a break from work his health is presently much better. He says, "à l'egard de la suite dont vous me parlés elle est fort singulière, j'y ay un peu pensé, mais je ne vois que l'induction pour la demontrer" ["with regard to the series which you wrote me about, it is most singular, and I've given it some thought, but I don't see anything other than induction for a proof"].

In §323 (in Chapter XVI, "De partitione numerorum") of the *Introductio in analysin infinitorum* (Euler 1748), E101, published in 1748, Euler states the pentagonal number theorem, and observes the fact that the product $(1-x)(1-x^2)(1-x^3)$ etc. is the reciprocal of the generating function for $p(n)$. Then in §324, he states a recurrence relation for $p(n)$,[16]

Therefore because of the recursion pattern

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \text{etc.},$$

from the expansion of the fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}}$$

this recurrent series will arise

$$\begin{aligned}1 &+ x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8\\ &+30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14}\\ &+176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20}\\ &+792x^{21} + 1002x^{22} + 1255x^{23} + 1575x^{24}\text{ etc.}\end{aligned}$$

This is $\sum p(n)x^n$ for $n$ from 0 to 24.

16 "Cum igitur scala relationis sit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \text{etc.},$$

series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\text{ etc.}}$$

oriunda erit haec

$$\begin{aligned}1 &+ x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8\\ &+30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14}\\ &+176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20}\\ &+792x^{21} + 1002x^{22} + 1255x^{23} + 1575x^{24}\text{ etc.''}\end{aligned}$$

(Euler 1748, §324), E101.

Euler proves several results about series and infinite products in his paper "Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae", E190, read at the June 19, 1749 meeting of the Berlin Academy (Winter 1957, Nr. 134), sent to Johann Daniel Schumacher (1690–1761) on December 18/29, 1749 (Juškevič and Winter 1961, Brief 108), presented to the St. Petersburg Academy on January 26, 1750 (Nevskaja 2000, p. 369), and published in the *Novi Commentarii* in 1753 (Euler 1753a; cf. Knobloch 1984, III, Nr. 168). Schumacher was the administrator of the St. Petersburg Academy.

In this article Euler looks at a series that interpolates $\log_a$. Euler defines the series

$$s = \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^3} + \frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}} + \text{etc.},$$

i.e.,

$$s(x) = \sum_{n=1}^{\infty} \frac{1}{a^{n(n-1)/2} - a^{n(n+1)/2}} \prod_{k=0}^{n-1} (a^k - x),$$

and asks different questions about it. If $x = a^n$ for $n$ a nonnegative integer, then $s = n$. In particular, if $a = 10$ then $s$ interpolates the common logarithm of $x$. For $x = 9$ and $a = 10$ he calculates the first 11 terms of the series to get $s = 0.89705058521067321224$ (the calculation is incorrect, and for $x = 9$ and $a = 10$ we will in fact get $s = 0.897778586588\ldots$). Actually $\log_{10}(9) = 0.9542425094\ldots$. Next, in §6 of the paper, Euler writes $s$ as

$$s = \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \cdots,$$

i.e.,

$$s = \sum_{n=1}^{\infty} \frac{1}{1-a^n} \prod_{k=0}^{n-1} \left(1 - \frac{x}{a^k}\right), \tag{3}$$

and he defines $t(x) = s(ax)$. Subtracting the first series from the second,

$$t - s = x + \frac{x}{a}(1-x) + \frac{x}{a^2}(1-x)\left(1-\frac{x}{a}\right) + \frac{x}{a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \text{etc.}$$

Now, all the terms in the series $1 + s - t$ have a factor of $1 - x$, so

$$1 + s - t = (1-x)\left(1 - \frac{x}{a} - \frac{x}{a^2}\left(1-\frac{x}{a}\right) - \frac{x}{a^3}\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) - \text{etc.}\right).$$

Then, “Hic factor posterior autem porro divisibilis est per $1 - \frac{x}{a}$, unde fit

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{aa} - \frac{x}{a^3}\left(1 - \frac{x}{aa}\right) - \text{etc.}\right).$$

Hic denuo factor deprehenditur $1 - \frac{x}{aa}$ hocque seorsim expresso factor apparebit $1 - \frac{x}{a^3}$ et ita porro, unde tandem reperitur fore

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right)\left(1 - \frac{x}{a^3}\right)\left(1 - \frac{x}{a^4}\right)\left(1 - \frac{x}{a^5}\right) \text{ etc.”}$$

[“Here the second factor is in turn divisible by $1 - \frac{x}{a}$, whence it will be

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{aa} - \frac{x}{a^3}\left(1 - \frac{x}{aa}\right) - \text{etc.}\right).$$

Next, here one sees the factor $1 - \frac{x}{aa}$, and after removing it the factor $1 - \frac{x}{a^3}$ will appear and so on, and thus one finally obtains

$$1 + s - t = (1 - x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right)\left(1 - \frac{x}{a^3}\right)\left(1 - \frac{x}{a^4}\right)\left(1 - \frac{x}{a^5}\right) \text{ etc.”]}$$

Next in §§8–11, Euler takes $x = \frac{1}{a^n}$, and looks for different ways to express $s$ for $n = 1, 2, \ldots$. In particular, he writes $B$ for the series $s$ with $x = \frac{1}{a}$ and says that for $a = 10$,

$$B = -0.109989900001001;$$

of course $\log_{10}(\frac{1}{10}) = -1$, and so $s$ does not interpolate $\log_a$ at negative integer powers of $a$.

In §10 Euler defines $u(x) = s(a^2x)$ and with some simple manipulations gets $u = 2t - s + ax(1 + s - t)$. In §11 he states the following lemma, which is useful when working with infinite products[17]:

For let

$$A, B, C, D, E, F, \text{etc.}$$

[17] “Sit enim

$$A, B, C, D, E, F, \text{etc.}$$

series quantitatum quarumvis sitque

$$(1 - A)(1 - B)(1 - C)(1 - D)(1 - E) \text{ etc.} = S.$$

Atque hinc obtinebitur

$$1 - A - B(1 - A) - C(1 - A)(1 - B) - D(1 - A)(1 - B)(1 - C) - \text{etc.} = S;$$

haec enim formula facillime reducitur ad illam.” (Euler 1753a, §11), E190.

be any series of quantities and let

$$(1-A)(1-B)(1-C)(1-D)(1-E) \text{ etc.} = S.$$

One will then obtain

$$\begin{aligned} &1 - A - B(1-A) - C(1-A)(1-B) \\ &\quad - D(1-A)(1-B)(1-C) - \text{etc.} = S; \end{aligned} \tag{4}$$

for this formula can easily be reduced to the former.

Then in §14 Euler says he wants to express the series $x$ as a power series in $x$: "Revertor autem ad seriem initio assumtam

$$\begin{aligned} s &= \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) \\ &\quad + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \text{etc.} \end{aligned}$$

quam in aliam formam, in qua termini secundum potestates ipsius $x$ procedant, transfundere animus est." ["I shall return now however to the originally presented series

$$\begin{aligned} s &= \frac{1}{1-a}(1-x) + \frac{1}{1-a^2}(1-x)\left(1-\frac{x}{a}\right) \\ &\quad + \frac{1}{1-a^3}(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right) + \text{etc.}, \end{aligned}$$

and the intent is to transform it into another form, in which the terms proceed in powers of $x$"]. Let $s(x) = \sum_{n=0}^{\infty} a_n x^n$. Euler uses the formula $u - 2t + s = ax(1+s-t)$ from §10 to find by comparing coefficients of powers of $x$ that $a_{n-1}a(1-a^{n-1}) = a_n(1-a^n)^2$ and $a_1 = \frac{a}{(1-a)^2}$. This leads in §15 to

$$s(x) = s(0) + \sum_{n=1}^{\infty} \frac{a^n x^n}{(1-a)\cdots(1-a^{n-1})(1-a^n)^2}.$$

In §§16–18, Euler says something like

$$s(ax) - s(x) = -\sum_{n=1}^{\infty} \frac{a^n x^n}{(1-a)\cdots(1-a^n)},$$

which gets rid of the constant term $s(0)$ and the square of the factor $1-a^n$ in the denominator,

Next in §§19–22, Euler looks in particular at the following product $P$ (where $\frac{P}{1-a^m}$ is the $m$th term in the series (3) for $s$),

$$P = (1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\cdots\left(1-\frac{x}{a^{m-1}}\right),$$

as $m \to \infty$. In §26, Euler puts

$$P = 1 - \alpha x + \beta x^2 - \gamma x^3 + \delta x^4 - \epsilon x^5 + \text{etc.}$$

and defines $Q(x) = P(ax)$. Now, $Q = P - axP$, and then comparing the coefficients of powers of $x$ Euler gets

$$\alpha = \frac{a}{a-1}, \quad \beta = \frac{\alpha a}{a^2-1}, \quad \gamma = \frac{\beta a}{a^3-1}, \quad \delta = \frac{\gamma a}{a^4-1} \text{ etc.}$$

and thus

$$P = 1 - \frac{ax}{a-1} + \frac{a^2x^2}{(a-1)(a^2-1)} - \frac{a^3x^3}{(a-1)(a^2-1)(a^3-1)} + \frac{a^4x^4}{(a-1)(a^2-1)(a^3-1)(a^4-1)} - \text{etc.}$$

This article is reviewed in the June 1759 *Nova acta eruditorum* (pp. 315–316).

At last in a letter from Euler to Goldbach on June 9, 1750 (Juškevič and Winter 1965, Brief 144), Euler gives a proof of the pentagonal number theorem. This proof is based on the lemma (4), which expands an infinite product into a series of products. We can write this lemma as

$$\prod_{n=1}^{\infty}(1-a_n) = 1 - \sum_{n=1}^{\infty} a_n(1-a_1)\cdots(1-a_{n-1}). \tag{5}$$

This distributes (trivially) the factor $1-a_1$, then the factor $1-a_2$, then the factor $1-a_3$, etc. Euler describes it as follows,[18]

> But since then I have also found the demonstration of this theorem, which is based on this lemma: $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$ etc. $= 1 - \alpha - \beta(1-\alpha) - \gamma(1-\alpha)(1-\beta) - \delta(1-\alpha)(1-\beta)(1-\gamma) -$ etc., whose demonstration is immediately evident.
>
> Thus according to this lemma it is:
>
> $$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \text{ etc.} = s$$
> $$= 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) - \text{etc.}$$

[18] "Seit der Zeit aber habe ich auch die Demonstration dieses theorematis gefunden, welche sich auf dieses Lemma gründet: $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$ etc. $= 1-\alpha-\beta(1-\alpha)-\gamma(1-\alpha)(1-\beta)-\delta(1-\alpha)(1-\beta)(1-\gamma)-$ etc., dessen Demonstration sogleich in die Augen fällt.

Also ist nach diesem lemmate:

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \text{ etc.} = s$$
$$= 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) - \text{etc.}"$$

(Juškevič and Winter 1965, Brief 144).

We will give an explicit proof by induction. Trivially $1 - a_1 = 1 - a_1$. Assume up to some $N$ that

$$\prod_{n=1}^{N}(1-a_n) = 1 - \sum_{n=1}^{N} a_n(1-a_1)\cdots(1-a_{n-1}).$$

If we subtract $a_{N+1}(1-a_1)\cdots(1-a_N)$ from both sides we get

$$\prod_{n=1}^{N}(1-a_n) - a_{N+1}(1-a_1)\cdots(1-a_N) = 1 - \sum_{n=1}^{N+1} a_n(1-a_1)\cdots(1-a_{n-1}).$$

But the left-hand side is precisely

$$\prod_{n=1}^{N}(1-a_n) - a_{N+1}\prod_{n=1}^{N}(1-a_n) = \prod_{n=1}^{N+1}(1-a_n).$$

Thus the claim holds for $N+1$. Then letting $N \to \infty$ implies (5).

Euler is referring in the above to his March 21/April 1, 1747 letter to Goldbach, where he found the recurrence relation for $\sigma(n)$ assuming the pentagonal number theorem. Now Euler says that since that time he has found a proof of the pentagonal number theorem that satisfies him. Euler then gives a proof by induction of the pentagonal number theorem using (5), taking $a_n = x^n$ for all $n$.

Euler does the following. Let $s = 1 - x - Ax^2$, where $A = 1 - x + x(1-x)(1-x^2) + x^2(1-x)(1-x^2)(1-x^3) +$ etc. Then distributing the factor $1-x$ in each term we get

$$\begin{aligned} A = 1 &- x && - x^2(1-x^2) && - x^3(1-x^2)(1-x^3) && - \text{etc.} \\ &+ x(1-x^2) && + x^2(1-x^2)(1-x^3) && + x^3(1-x^2)(1-x^3)(1-x^4) && + \text{etc.} \end{aligned}$$

Then combining the pairs of terms,

$$A = 1 - x^3 - x^5(1-x^2) - x^7(1-x^2)(1-x^3) - \text{etc.}$$

Euler does not explicitly do the induction step. Instead he works out enough steps to make it clear how we could do the induction step.

Now let $A = 1 - x^3 - Bx^5$, and

$$B = 1 - x^2 + x^2(1-x^2)(1-x^3) + x^4(1-x^2)(1-x^3)(1-x^4) + \text{etc.}$$

Then expanding the factor $1-x^2$,

$$\begin{aligned} B = 1 &- x^2 && - x^4(1-x^3) && - x^6(1-x^3)(1-x^4) && - \text{etc.} \\ &+ x^2(1-x^3) && + x^4(1-x^3)(1-x^4) && + x^6(1-x^3)(1-x^4)(1-x^5) && + \text{etc.} \end{aligned}$$

Again the pairs of terms are combined, which gives

$$B = 1 - x^5 - x^8(1 - x^3) - x^{11}(1 - x^3)(1 - x^4) - \text{etc.}$$

Next let $B = 1 - x^5 - Cx^8$, and so on. Euler explicitly works out this case, then says[19]

It will thus be

$$\begin{array}{rcl} s &=& 1 - x - Ax^2 \\ A &=& 1 - x^3 - Bx^5 \\ B &=& 1 - x^5 - Cx^8 \\ C &=& 1 - x^7 - Dx^{11} \\ D &=& 1 - x^9 - Ex^{14} \\ && \text{etc.} \end{array} \quad \text{or} \quad \begin{array}{rcl} s &=& 1 - x - Ax^2 \\ Ax^2 &=& x^2(1 - x^3) - Bx^7 \\ Bx^7 &=& x^7(1 - x^5) - Cx^{15} \\ Cx^{15} &=& x^{15}(1 - x^7) - Dx^{26} \\ Dx^{26} &=& x^{26}(1 - x^9) - Ex^{40} \\ && \text{etc.} \end{array}$$

from which it follows with no doubt at all that

$$\begin{aligned} s &= 1 - x - x^2(1 - x^3) + x^7(1 - x^5) - x^{15}(1 - x^7) + x^{26}(1 - x^9) - \text{etc., or} \\ s &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{etc.} \end{aligned}$$

Let us now explicitly work out the induction in Euler's above proof of the pentagonal number theorem. Let

$$S_N = \sum_{n=1}^{\infty} x^{N(n-1)}(1 - x^N) \cdots (1 - x^{n+N-1}).$$

Here $S_1 = A$, $S_2 = B$, $S_3 = C$, etc. First, we shall show that

$$S_N = 1 - x^{2N+1} - S_{N+1}x^{3N+2} \tag{6}$$

19 "Also wird sein:

$$\begin{array}{rcl} s &=& 1 - x - Ax^2 \\ A &=& 1 - x^3 - Bx^5 \\ B &=& 1 - x^5 - Cx^8 \\ C &=& 1 - x^7 - Dx^{11} \\ D &=& 1 - x^9 - Ex^{14} \\ && \text{etc.} \end{array} \quad \text{oder} \quad \begin{array}{rcl} s &=& 1 - x - Ax^2 \\ Ax^2 &=& x^2(1 - x^3) - Bx^7 \\ Bx^7 &=& x^7(1 - x^5) - Cx^{15} \\ Cx^{15} &=& x^{15}(1 - x^7) - Dx^{26} \\ Dx^{26} &=& x^{26}(1 - x^9) - Ex^{40} \\ && \text{etc.} \end{array}$$

woraus dann gantz ungezweifelt folget

$$\begin{aligned} s &= 1 - x - x^2(1 - x^3) + x^7(1 - x^5) - x^{15}(1 - x^7) + x^{26}(1 - x^9) - \text{etc., oder} \\ s &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{etc."} \end{aligned}$$

(Juškevič and Winter 1965, Brief 144).

for all $N \geq 1$, like how we showed that $A = 1 - x^3 - Bx^5$, $B = 1 - x^5 - Cx^8$, etc.

$$\begin{aligned}
S_N &= \sum_{n=1}^{\infty} x^{N(n-1)}(1-x^N)\cdots(1-x^{n+N-1}) \\
&= \sum_{n=1}^{\infty} x^{N(n-1)}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&\quad -\sum_{n=1}^{\infty} x^{Nn}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&= 1+\sum_{n=1}^{\infty} x^{Nn}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&\quad -\sum_{n=1}^{\infty} x^{Nn}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&= 1-\sum_{n=1}^{\infty} x^{Nn+N+n}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&= 1-x^{2N+1}-\sum_{n=1}^{\infty} x^{Nn+2N+n+1}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&= 1-x^{2N+1}-x^{3N+2}\sum_{n=1}^{\infty} x^{(N+1)(n-1)}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&= 1-x^{2N+1}-x^{3N+2}S_{N+1},
\end{aligned}$$

thus showing that (6) holds for all $N \geq 1$.

Now, we shall show that

$$s = \sum_{n=0}^{N-1}\left((-1)^n x^{\frac{n(3n+1)}{2}} + (-1)^{n+1} x^{\frac{(n+1)(3n+2)}{2}}\right) + (-1)^N x^{\frac{N(3N+1)}{2}} S_N \tag{7}$$

for all $N \geq 1$. First, $s = 1 - x - x^2 S_1$ since $S_1 = A$. Now assume that (7) holds for some $N \geq 1$. Then using (6),

$$\begin{aligned}
s &= \sum_{n=0}^{N-1}\left((-1)^n x^{\frac{n(3n+1)}{2}} + (-1)^{n+1} x^{\frac{(n+1)(3n+2)}{2}}\right) \\
&\quad +(-1)^N x^{\frac{N(3N+1)}{2}}(1-x^{2N+1}-x^{3N+2}S_{N+1}) \\
&= \sum_{n=0}^{N-1}\left((-1)^n x^{\frac{n(3n+1)}{2}} + (-1)^{n+1} x^{\frac{(n+1)(3n+2)}{2}}\right) \\
&\quad +(-1)^N x^{\frac{N(3N+1)}{2}} + (-1)^{N+1} x^{\frac{N(3N+1)}{2}+2N+1} + (-1)^{N+1} x^{\frac{N(3N+1)}{2}+3N+2} S_{N+1}
\end{aligned}$$

$$= \sum_{n=0}^{N-1} \left( (-1)^n x^{\frac{n(3n+1)}{2}} + (-1)^{n+1} x^{\frac{(n+1)(3n+2)}{2}} \right)$$

$$(-1)^N x^{\frac{N(3N+1)}{2}} + (-1)^{N+1} x^{\frac{(N+1)(3N+2)}{2}} + (-1)^{N+1} x^{\frac{(N+1)(3N+4)}{2}} S_{N+1}$$

$$= \sum_{n=0}^{N} \left( (-1)^n x^{\frac{n(3n+1)}{2}} + (-1)^{n+1} x^{\frac{(n+1)(3n+2)}{2}} \right) + (-1)^{N+1} x^{\frac{(N+1)(3N+4)}{2}} S_{N+1},$$

so (7) holds for all $N \geq 1$, thus proving the pentagonal number theorem.

In his "Demonstratio theorematis circa ordinem in summis divisorum observatum", published in the *Novi Commentarii* in 1760 (Euler 1760a), E244 (cf. Knobloch 1984, I, Nr. 43), Euler gives the above proof of the pentagonal number theorem from his letter to Goldbach with a bit more detail. Schumacher acknowledges in his April 4/15, 1752 letter to Euler (Juškevič and Winter 1961, Brief 189) having received the paper with Euler's March 7/18, 1752 letter (Juškevič and Winter 1961, Brief 185), though the paper is not mentioned in that letter. In this article Euler first recalls that some time ago he had discovered a recurrence relation for the divisor function, for which the differences in the arguments are the pentagonal numbers. However, his proof was not rigorous enough to him, since it relied on the pentagonal number theorem, which previously he had not been able to find a rigorous proof for. But now he declares that he has finally found a demonstration of the pentagonal number theorem. Proposition 1 is our (5), for $a_1 = -\alpha, a_2 = -\beta, a_3 = -\gamma$ and so on. His demonstration is the following,[20]

> For since the first term is $(1+\alpha)$ and the second $= \beta(1+\alpha)$, the sum of the first and second will be $= (1+\alpha)(1+\beta)$; now if the third term $\gamma(1+\alpha)(1+\beta)$ is added it will yield $(1+\alpha)(1+\beta)(1+\gamma)$; let the fourth term, which is

[20] "Cum enim seriei primus terminus sit $(1+\alpha)$ et secundus $= \beta(1+\alpha)$, erit summa primi et secundi $= (1+\alpha)(1+\beta)$; si iam addatur tertius terminus $\gamma(1+\alpha)(1+\beta)$, prodibit $(1+\alpha)(1+\beta)(1+\gamma)$; addatur insuper terminus quartus, qui est $\delta(1+\alpha)(1+\beta)(1+\gamma)$; erit summa

$$= (1+\alpha)(1+\beta)(1+\gamma)(1+\delta).$$

Atque sic in infinitum procedendo summa totius seriei seu omnium eius terminorum perducetur ad hoc productum

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)(1+\zeta) \text{ etc.}$$

Unde manifestum est, si fuerit

$$s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)(1+\zeta) \text{ etc.},$$

fore vicissim

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma) + \text{etc.''}$$

(Euler 1760a, Propositio 1), E244.

$\delta(1+\alpha)(1+\beta)(1+\gamma)$, be added on top; the sum will be

$$= (1+\alpha)(1+\beta)(1+\gamma)(1+\delta).$$

And proceeding thus onto infinity, the whole of the series, or all of its terms, will be led to this product

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)(1+\zeta) \text{ etc.}$$

Whence it is clear that if

$$s = (1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)(1+\zeta) \text{ etc.}$$

it will be in turn

$$s = (1+\alpha) + \beta(1+\alpha) + \gamma(1+\alpha)(1+\beta) + \delta(1+\alpha)(1+\beta)(1+\gamma) + \text{etc.}$$

Euler uses this to prove the pentagonal number theorem. In Proposition 2 he takes $\alpha = -x$, $\beta = -x^2$, $\gamma = -x^3$, $\delta = -x^4$, $\epsilon = -x^5$, etc. Then $s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., and

$$s = 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) - \text{etc.}$$

Then in Proposition 3, Euler proves the pentagonal number theorem. It is the same proof he gave in his above letter to Goldbach, but he gives a bit more detail. He concludes,

Therefore, we will have

$$\begin{aligned} s = 1 - x - x^2(1-x^3) + x^7(1-x^5) - x^{15}(1-x^7) + x^{26}(1-x^9) \\ -x^{40}(1-x^{11}) + \text{etc.} \end{aligned}$$

or that very thing which is to be demonstrated,

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.},$$

from which the law of the exponents indicated above is clearly seen at once.[21]

[21] "Quamobrem habebimus

$$s = 1 - x - x^2(1-x^3) + x^7(1-x^5) - x^{15}(1-x^7) + x^{26}(1-x^9) - x^{40}(1-x^{11}) + \text{etc.}$$

sive id ipsum, quod demonstrari oportet,

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + \text{etc.},$$

unde simul lex exponentium supra indicata per differentias luculenter perspicitur." (Euler 1760a, Propositio 3), E244.

Euler then uses the pentagonal number theorem to finally give a rigorous proof of his recurrence relation for the divisor function, which we discuss in §4.

In the 1752 *Bibliothèque impartiale* (juillet et août), tome VI, première partie, article IX, "Demonstration de la Loi d'une suite de termes de la Quantité composée qui est faite par la multiplication des Binomes $1-x$, $1-x^2$, $1-x^3$, & c.", pp. 111–126, there is a proof of the pentagonal number theorem by an anonymous author.

Euler leaves Berlin on June 9, 1766, spends ten days in Warsaw, and arrives in St. Petersburg on July 28. By 1771, he was completely blind (Gillispie 1980, Volume IV, "Leonhard Euler", pp. 467–484).

Euler gives two proofs of the pentagonal number theorem in his later paper "Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. in seriem simplicem", delivered to the St. Petersburg Academy on August 14, 1775 (Nevskaja 2000, p. 627) and published in 1783 in the *Acta* (Euler 1783b), E541. The first proof is a reworking of the proof in E244, while the second is a little different. We will explicitly work out the inductions in both of them. Euler takes

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4) \text{ etc.}$$

and "facile patet fore" ["it is not difficult to see that"]

$$s = 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) - \text{etc.}$$

and "quaeritur, si singuli eius termini evolvantur, qualis series secundum simplices potestates ipsius $x$ sit proditura" ["we want to find what series will be produced in simple powers of $x$ if all the terms are expanded"].

In the first proof (§§1–10 of E541), let

$$A = \sum_{n=1}^{\infty} x^{n+1}(1-x)\cdots(1-x^n),$$

so $s = 1 - x - A$. We may expand by the factor $1-x$ to get

$$A = \sum_{n=1}^{\infty} x^{n+1}(1-x^2)\cdots(1-x^n) - \sum_{n=1}^{\infty} x^{n+2}(1-x^2)\cdots(1-x^n).$$

Then

$$A = x^2 + \sum_{n=1}^{\infty} x^{n+2}(1-x^2)\cdots(1-x^{n+1}) - \sum_{n=1}^{\infty} x^{n+2}(1-x^2)\cdots(1-x^n),$$

and adding the series together

$$A = x^2 - \sum_{n=1}^{\infty} x^{2n+3}(1-x^2)\cdots(1-x^n).$$

And so

$$A = x^2 - x^5 - \sum_{n=1}^{\infty} x^{2n+5}(1-x^2)\cdots(1-x^{n+1}).$$

Now let

$$B = \sum_{n=1}^{\infty} x^{2n+5}(1-x^2)\cdots(1-x^{n+1}),$$

so $A = x^2 - x^5 - B$. We can expand by the factor $1 - x^2$. Then

$$B = \sum_{n=1}^{\infty} x^{2n+5}(1-x^3)\cdots(1-x^{n+1}) - \sum_{n=1}^{\infty} x^{2n+7}(1-x^3)\cdots(1-x^{n+1})$$

and so

$$B = x^7 + \sum_{n=1}^{\infty} x^{2n+7}(1-x^3)\cdots(1-x^{n+2}) - \sum_{n=1}^{\infty} x^{2n+7}(1-x^3)\cdots(1-x^{n+1}).$$

Then we add the two series together and get

$$B = x^7 - \sum_{n=1}^{\infty} x^{3n+9}(1-x^3)\cdots(1-x^{n+1}),$$

thus

$$B = x^7 - x^{12} - \sum_{n=1}^{\infty} x^{3n+12}(1-x^3)\cdots(1-x^{n+2}).$$

Next let

$$C = \sum_{n=1}^{\infty} x^{3n+12}(1-x^3)\cdots(1-x^{n+2}),$$

so $B = x^7 - x^{12} - C$.

Euler also works through the next cases $C = x^{15} - x^{22} - D$ and $D = x^{26} - x^{35} - E$, where

$$D = \sum_{n=1}^{\infty} x^{4n+22}(1-x^4)\cdots(1-x^{n+3})$$

and

$$E = \sum_{n=1}^{\infty} x^{5n+35}(1-x^5)\cdots(1-x^{n+4}).$$

Euler says in §9,[22]

> From these it is now clear that the differences of the numbers 2, 7, 15, 26, 40, 57, etc. constitute an arithmetic progression, whence the general term of these numbers will be
>
> $$2+5(n-1)+\frac{3(n-1)(n-2)}{1\cdot 2} = \frac{3nn+n}{2}.$$
>
> On the other hand, the exponents of the preceding terms were 1, 5, 12, 22, 35, 51, etc.; these differ from the succeeding exponents by 1, 2, 3, 4, 5, 6, etc. and in general by $n$ itself, so that the exponent preceding the formula $\frac{3nn+n}{2}$ will be
>
> $$\frac{3nn-n}{2}.$$

To do this proof explicitly using induction, let

$$S_N = \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N-1)}{2}}(1-x^N)\cdots(1-x^{n+N-1}).$$

Here, $S_1 = A$, $S_2 = B$, $S_3 = C$, etc. Thus, so far we have done $s = 1-x-S_1$, $S_1 = x^2-x^5-S_2$, $S_2 = x^7-x^{12}-S_3$, $S_3 = x^{15}-x^{22}-S_4$, and $S_4 = x^{26}-x^{35}-S_5$. We shall show that

$$S_N = x^{\frac{N(3N+1)}{2}} - x^{\frac{(N+1)(3N+2)}{2}} - S_{N+1} \tag{8}$$

[22] "Ex his iam manifestum est numerorum 2, 7, 15, 26, 40, 57, etc. differentias progressionem arithmeticam constituere, unde horum numerorum terminus generalis erit

$$2+5(n-1)+\frac{3(n-1)(n-2)}{1\cdot 2} = \frac{3nn+n}{2}.$$

Exponentes autem, qui hos antecedunt, erant 1, 5, 12, 22, 35, 51, etc. ab illis numeris 1, 2, 3, 4, 5, 6, etc. et in genere ipso numero $n$ diversi, ita ut exponens, qui formulam $\frac{3nn+n}{2}$ praecedit, futurus sit

$$\frac{3nn-n}{2}."$$

(Euler 1783b, §9), E541.

for all $N \geq 1$.

$$
\begin{aligned}
S_N &= \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N-1)}{2}}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&\quad - \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N-1)}{2}+N}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&= x^{N+\frac{N(3N-1)}{2}} + \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N-1)}{2}+N}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&\quad - \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N-1)}{2}+N}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&= x^{\frac{N(3N+1)}{2}} - \sum_{n=1}^{\infty} x^{(N+1)n+\frac{N(3N-1)}{2}+2N}(1-x^{N+1})\cdots(1-x^{n+N-1}) \\
&= x^{\frac{N(3N+1)}{2}} - x^{N+1+\frac{N(3N-1)}{2}+2N} \\
&\quad - \sum_{n=1}^{\infty} x^{(N+1)n+N+1+\frac{N(3N-1)}{2}+2N}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&= x^{\frac{N(3N+1)}{2}} - x^{\frac{(N+1)(3N+2)}{2}} - S_{N+1}
\end{aligned}
$$

for all $N \geq 1$.

Next, it is straightforward to show using induction that

$$
s = \sum_{n=0}^{N-1} \left( (-1)^n x^{\frac{n(3n+1)}{2}} + (-1)^{n+1} x^{\frac{(n+1)(3n+2)}{2}} \right) + (-1)^N S_N
$$

for all $N \geq 1$. Since $S_1 = A$ and $s = 1 - x - A$ this is true for $N = 1$. If we assume it is true for some $N \geq 1$, using (8) it follows right away that it is true for $N+1$, which proves the pentagonal number theorem.

This proof is essentially the same as the proof from E244. However, it indeed ends up being simpler to pack everything into $S_N$ and prove (8), since the pentagonal number theorem follows almost immediately from this, while in E244 we have to show (6) and then show (7).

Euler had written down this proof on p. 100 of his Fifth Notebook; his unpublished proof of the pentagonal number theorem is on pp. 97–99 of his Fifth Notebook, and we shall describe that proof in this section after we finish discussing E541. In this proof he takes $s = (1-x)(1-x^2)(1-x^3)(1-x^4)$ etc. and expands this product as $s = 1 - x - x^2(1-x) - x^3(1-x)(1-x^2) - x^4(1-x)(1-x^2)(1-x^3) -$ etc. Euler works out the steps $s = 1 - x - A$, $A = x^2 - x^5 - B$, $B = x^7 - x^{12} - C$, $C = x^{15} - x^{22} - D$ and $D = x^{26} - x^{35} - E$ of the above proof, and then writes down the general step. However, his notation for the general case is unwieldy. Let

$$M = x^m - x^n - x^{n+p}(1-x^p) - x^{n+2p}(1-x^p)(1-x^{p+1})$$
$$-x^{n+3p}(1-x^p)(1-x^{p+1})(1-x^{p+2}) - \text{etc.}$$

Euler puts $M = x^m - x^n - N$, and then finds that

$$N = x^{n+p} - x^{n+3p+1} - x^{n+4p+2}(1-x^{p+1})$$
$$-x^{n+5p+3}(1-x^{p+1})(1-x^{p+2}) - \text{etc.} \tag{9}$$

Using the notation $S_p = \sum_{k=1}^{\infty} x^{pk+\frac{p(3p-1)}{2}}(1-x^p)\cdots(1-x^{k+p-1})$, with $m = (p-1)(3p-2)/2$ and $n = p(3p-1)/2$ we have $M = S_{p-1}$, $N = S_p$. Verifying (9) corresponds to verifying (8). In Euler's published papers he makes clear how the pattern works with the initial cases but he does not explicitly write down the general case as he does here.

Then in the second part of the paper, "Alio investigatio eiusdem seriei" ["Another investigation of the same series"], Euler gives a different proof of the pentagonal number theorem. This proof is more complicated than the first. First,

$$s = 1 - \sum_{n=1}^{\infty} x^n(1-x)\cdots(1-x^{n-1})$$
$$= 1 - x - \sum_{n=1}^{\infty} x^{n+1}(1-x)\cdots(1-x^n),$$

and then

$$s = 1 - x - x^2(1-x) - \sum_{n=1}^{\infty} x^{n+2}(1-x)\cdots(1-x^{n+1}).$$

We take

$$s = 1 - x - x^2 + A,$$

so

$$A = x^3 - \sum_{n=1}^{\infty} x^{n+2}(1-x)\cdots(1-x^{n+1}).$$

We expand out the factor $1-x$ in the series,

$$A = x^3 - \sum_{n=1}^{\infty} x^{n+2}(1-x^2)\cdots(1-x^{n+1}) + \sum_{n=1}^{\infty} x^{n+3}(1-x^2)\cdots(1-x^{n+1}),$$

and then,

$$
\begin{aligned}
A &= x^3 - x^3(1-x^2) - \sum_{n=1}^{\infty} x^{n+3}(1-x^2)\cdots(1-x^{n+2}) \\
&\quad + \sum_{n=1}^{\infty} x^{n+3}(1-x^2)\cdots(1-x^{n+1}) \\
&= x^5 + \sum_{n=1}^{\infty} x^{2n+5}(1-x^2)\cdots(1-x^{n+1}) \\
&= x^5 + x^7(1-x^2) + \sum_{n=1}^{\infty} x^{2n+7}(1-x^2)\cdots(1-x^{n+2}) \\
&= x^5 + x^7 - B,
\end{aligned}
$$

where

$$B = x^9 - \sum_{n=1}^{\infty} x^{2n+7}(1-x^2)\cdots(1-x^{n+2}).$$

Next,

$$
\begin{aligned}
B &= x^9 - \sum_{n=1}^{\infty} x^{2n+7}(1-x^3)\cdots(1-x^{n+2}) \\
&\quad + \sum_{n=1}^{\infty} x^{2n+9}(1-x^3)\cdots(1-x^{n+2}) \\
&= x^9 - x^9(1-x^3) - \sum_{n=1}^{\infty} x^{2n+9}(1-x^3)\cdots(1-x^{n+3}) \\
&\quad + \sum_{n=1}^{\infty} x^{2n+9}(1-x^3)\cdots(1-x^{n+2}) \\
&= x^{12} + \sum_{n=1}^{\infty} x^{3n+12}(1-x^3)\cdots(1-x^{n+2}) \\
&= x^{12} + x^{15}(1-x^3) + \sum_{n=1}^{\infty} x^{3n+15}(1-x^3)\cdots(1-x^{n+3}) \\
&= x^{12} + x^{15} - C,
\end{aligned}
$$

where

$$C = x^{18} - \sum_{n=1}^{\infty} x^{3n+15}(1-x^3)\cdots(1-x^{n+3}).$$

Euler continues this and shows $C = x^{22} + x^{26} - D$ and $D = x^{35} + x^{40} - E$. Using the following definition of $S_N$, we have $A = S_1$, $B = S_2$, $C = S_3$, $D = S_4$ and $E = S_5$. Let

$$S_N = x^{\frac{3N(N+1)}{2}} - \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N+1)}{2}}(1-x^N)\cdots(1-x^{n+N}).$$

Euler writes[23]

> If we now substitute in the values that we've found of the letters $A, B, C, D,$
> $E$, etc., the following series will result
>
> $$1 - x - xx, +x^5 + x^7, -x^{12} - x^{15}, +x^{22} + x^{26}, -x^{35} - x^{40}, + \text{etc.}$$
>
> The order of the exponents here is easily seen. For first, in the values initially given for the letters $A, B, C, D, E$, etc., the initial single terms were $x^3, x^9, x^{18}, x^{30}, x^{45}$, etc. The exponents of these terms are clearly thrice the triangular numbers, and generally the exponent corresponding to the number $n$ will be $\frac{3nn+3n}{2}$. These terms are preceded by two powers of $x$ each less by the same difference $n$; thus by subtracting the number $n$ twice from this formula, two powers in the desired series are produced whose exponents are
>
> $$\frac{3nn+n}{2} \quad \text{and} \quad \frac{3nn-n}{2}.$$

We now work out the induction explicitly. We shall show that

$$S_N = x^{\frac{(N+1)(3N+2)}{2}} + x^{\frac{(N+1)(3N+4)}{2}} - S_{N+1} \tag{10}$$

[23] "Inventis igitur his valoribus litterarum $A, B, C, D, E$, etc. si singuli successive substituantur, resultabit ista series

$$1 - x - xx, +x^5 + x^7, -x^{12} - x^{15}, +x^{22} + x^{26}, -x^{35} - x^{40}, + \text{etc.}$$

Hic autem ordo exponentium facilius perspicitur. Cum enim in valoribus litterarum $A, B, C, D, E$, etc. primo constitutis primi termini simplices essent $x^3, x^9, x^{18}, x^{30}, x^{45}$, etc. exponentes manifesto sunt numeri trigonales triplicati, unde generatim pro numero $n$ erit ipse exponens $\frac{3nn+3n}{2}$. Verum hi termini sequuntur binas potestates ipsius $x$ praecedentes per eandem differentiam $n$, unde numerum $n$ ab hac formula bis subtrahendo orientur binae potestates in seriem quaesitam ingredientes, quarum exponentes consequenter erunt

$$\frac{3nn+n}{2} \quad \text{et} \quad \frac{3nn-n}{2}."$$

(Euler 1783b, §16), E541.

for all $N \geq 1$.

$$\begin{aligned}
S_N &= x^{\frac{3N(N+1)}{2}} - \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N+1)}{2}}(1-x^N)\cdots(1-x^{n+N}) \\
&= x^{\frac{3N(N+1)}{2}} - \sum_{n=1}^{\infty} x^{Nn+\frac{N(3N+1)}{2}}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&\quad + \sum_{n=1}^{\infty} x^{Nn+\frac{3N(N+1)}{2}}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&= x^{\frac{3N(N+1)}{2}} - x^{\frac{3N(N+1)}{2}}(1-x^{N+1}) \\
&\quad - \sum_{n=1}^{\infty} x^{Nn+\frac{3N(N+1)}{2}}(1-x^{N+1})\cdots(1-x^{n+N+1}) \\
&\quad + \sum_{n=1}^{\infty} x^{Nn+\frac{3N(N+1)}{2}}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&= x^{\frac{(N+1)(3N+2)}{2}} + \sum_{n=1}^{\infty} x^{(N+1)n+\frac{(N+1)(3N+2)}{2}}(1-x^{N+1})\cdots(1-x^{n+N}) \\
&= x^{\frac{(N+1)(3N+2)}{2}} + x^{\frac{(N+1)(3N+4)}{2}}(1-x^{N+1}) \\
&\quad + \sum_{n=1}^{\infty} x^{(N+1)n+\frac{(N+1)(3N+4)}{2}}(1-x^{N+1})\cdots(1-x^{n+N+1}) \\
&= x^{\frac{(N+1)(3N+2)}{2}} + x^{\frac{(N+1)(3N+4)}{2}} - S_{N+1}.
\end{aligned}$$

Now, $s = 1 - x - x^2 + S_1$, and one then shows using induction with (10) that

$$s = 1 + \sum_{n=1}^{N}(-1)^n \left(x^{\frac{n(3n-1)}{2}} + x^{\frac{n(3n+1)}{2}}\right) + (-1)^{N+1}S_N$$

for all $N \geq 1$, which yields the pentagonal number theorem.

Andrews (1983) proves the pentagonal number theorem as a particular case of a more general result. Let $f(x,q) = 1-\sum_{n=1}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{n-1})x^{n+1}q^n$. The pentagonal number theorem is the power series expansion of $f(1,q) = \prod_{n=1}^{\infty}(1-q^n)$. One shows that $f(x,q)$ satisfies the functional equation $f(x,q) = 1 - x^2q - x^3q^2 f(xq,q)$, and then that

$$f(x,q) = 1 + \sum_{n=1}^{\infty}(-1)^n(x^{3n-1}q^{n(3n-1)/2} + x^{3n}q^{n(3n+1)/2}).$$

Taking $x = 1$ yields the pentagonal number theorem.

In his Fourth Notebook, p. 174, after noting that (2) is the generating function for $p(n)$, Euler writes, Also, this series emerges from the resolution of the fraction

$$\frac{1}{1-1n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+n^{51}}$$

all of whose exponents are contained in the form $n^{\frac{3xx\pm x}{2}}$.[24]

Euler gives another proof of the pentagonal number theorem on pp. 97–99 of his Fifth Notebook. It is described by Kiselev and Matvievskaja (1965, §6). We will outline Euler's notes, and then fully work out the proof. Let $s=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)$ etc. Then

$$\begin{aligned} s = 1 - \frac{x}{1-x} + \frac{x^3}{(1-x)(1-x^2)} - \frac{x^6}{(1-x)(1-x^2)(1-x^3)} \\ + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} - \text{etc.} \end{aligned}$$

Now, using the fact that $\frac{A}{1-B}=A+\frac{AB}{1-B}$,

$$s = 1 - x - x^2 \underbrace{- \frac{x^3}{1-x} + \frac{x^3}{(1-x)(1-x^2)}}_{+\frac{x^5}{(1-x)(1-x^2)}} - \text{etc.}$$

Then

$$\begin{aligned} s &= 1 - \frac{x}{1-x} + \frac{x^3}{(1-x)(1-x^2)} - \frac{x^6}{(1-x)(1-x^2)(1-x^3)} \\ &\quad + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \\ &\quad - \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} + \text{etc.} \\ &= 1 - P \\ P &= x + x^2 - \frac{x^5}{1-x^2} + \frac{x^9}{(1-x^2)(1-x^3)} - \frac{x^{14}}{(1-x^2)(1-x^3)(1-x^4)} \\ &\quad + \frac{x^{20}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)} - \text{etc.} \\ &= x + x^2 - Q \end{aligned}$$

[24] "Ceterum series haec emergit ex resolutione hujus fractionis

$$\frac{1}{1-1n-n^2+n^5+n^7-n^{12}-n^{15}+n^{22}+n^{26}-n^{35}-n^{40}+n^{51}}$$

qui exponentes omnes continentur in forma $n^{\frac{3xx\pm x}{2}}$." (Euler's Fourth Notebook, p. 174).

$$\begin{aligned}
Q &= x^5 + x^7 - \frac{x^{12}}{1-x^3} + \frac{x^{18}}{(1-x^3)(1-x^4)} - \frac{x^{25}}{(1-x^3)(1-x^4)(1-x^5)} \\
&\quad + \frac{x^{33}}{(1-x^3)(1-x^4)(1-x^5)(1-x^6)} - \text{etc.} \\
&= x^5 + x^7 - R \\
R &= x^{12} + x^{15} - \frac{x^{22}}{1-x^4} + \frac{x^{30}}{(1-x^4)(1-x^5)} - \frac{x^{39}}{(1-x^4)(1-x^5)(1-x^6)} \\
&\quad + \frac{x^{49}}{(1-x^4)(1-x^5)(1-x^6)(1-x^7)} - \text{etc.} \\
&= x^{12} + x^{15} - S \\
S &= x^{22} + x^{26} - \frac{x^{35}}{1-x^5} + \frac{x^{45}}{(1-x^5)(1-x^6)} \\
&\quad - \frac{x^{56}}{(1-x^5)(1-x^6)(1-x^7)} + \frac{x^{68}}{(1-x^5)(1-x^6)(1-x^7)(1-x^8)} - \text{etc.}
\end{aligned}$$

Then, "Generaliter si sit" ["In general if it is"]

$$\begin{aligned}
Y = \frac{x^{\frac{3n^2-n}{2}}}{1-x^n} - \frac{x^{\frac{3n^2+3n}{2}}}{(1-x^n)(1-x^{n+1})} + \frac{x^{\frac{3n^2+7n+2}{2}}}{(1-x^n)(1-x^{n+1})(1-x^{n+2})} \\
- \frac{x^{\frac{3n^2+11n+6}{2}}}{(1-x^n)(1-x^{n+1})(1-x^{n+2})(1-x^{n+3})} + \text{etc.}
\end{aligned}$$

then

$$Y = x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} + \underbrace{\frac{x^{\frac{3n^2+3n}{2}}}{1-x^n} - \frac{x^{\frac{3n^2+3n}{2}}}{(1-x^n)(1-x^{n+1})}}_{-\frac{x^{\frac{3n^2+5n+2}{2}}}{(1-x^n)(1-x^{n+1})}} + \text{etc.}$$

Then putting

$$Y = x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}} - Z,$$

it will be $Z = \frac{x^{\frac{3n^2+5n+2}{2}}}{1-x^{n+1}} - \text{etc.}$, "sicque $Z$ oritur ex $Y$ ponendo $n+1$ loco $n$" ["and $Z$ thus arises from $Y$ by putting $n+1$ in place of $n$"].

This proof is pretty complicated and it is worthwhile to work out all the details. Let

$$s_N = 1 + \sum_{n=1}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2}}}{(1-x)\cdots(1-x^n)},$$

and let

$$s_{M;N} = 1 + \sum_{m=1}^{M-1} (-1)^m \left( x^{\frac{m(3m-1)}{2}} + x^{\frac{m(3m+1)}{2}} \right) + \sum_{n=M}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2}+(M-1)n}}{(1-x^M)\cdots(1-x^n)}.$$

The plan of the proof is to show that $s_\infty = s_{2;\infty}$, and that $s_{M;\infty} = s_{M+1;\infty}$ for all $M$. Since $s_\infty = \prod_{n=1}^{\infty}(1-x^n)$, this will prove the pentagonal number theorem.

First,

$$\begin{aligned} s_2 &= 1 - \frac{x}{1-x} + \frac{x^3}{(1-x)(1-x^2)} \\ &= 1 - x - x^2 - \frac{x^3}{1-x} + \frac{x^3}{(1-x)(1-x^2)} \\ &= 1 - x - x^2 + \frac{x^5}{(1-x)(1-x^2)} \\ &= 1 - x - x^2 + \frac{x^5}{1-x^2} + \frac{x^6}{(1-x)(1-x^2)}. \end{aligned}$$

Now assume that for some $N$

$$s_N = 1 - x - x^2 + \sum_{n=2}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2}+n}}{(1-x^2)\cdots(1-x^n)} + \frac{(-1)^N x^{\frac{(N+1)(N+2)}{2}}}{(1-x)\cdots(1-x^N)}. \tag{11}$$

Then

$$\begin{aligned} s_{N+1} &= 1 - x - x^2 + \sum_{n=2}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2}+n}}{(1-x^2)\cdots(1-x^n)} \\ &\quad + \frac{(-1)^N x^{\frac{(N+1)(N+2)}{2}}}{(1-x)\cdots(1-x^N)} + \frac{(-1)^{N+1} x^{\frac{(N+1)(N+2)}{2}}}{(1-x)\cdots(1-x^{N+1})} \\ &= 1 - x - x^2 + \sum_{n=2}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2}+n}}{(1-x^2)\cdots(1-x^n)} \\ &\quad + \frac{(-1)^{N+1} x^{\frac{(N+1)(N+2)}{2}+N+1}}{(1-x)\cdots(1-x^{N+1})} \\ &= 1 - x - x^2 + \sum_{n=2}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2}+n}}{(1-x^2)\cdots(1-x^n)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^{N+1}x^{\frac{(N+1)(N+2)}{2}+N+1}}{(1-x^2)\cdots(1-x^{N+1})}+\frac{(-1)^{N+1}x^{\frac{(N+1)(N+2)}{2}+N+2}}{(1-x)\cdots(1-x^{N+1})}\\
&=1-x-x^2+\sum_{n=2}^{N+1}\frac{(-1)^n x^{\frac{n(n+1)}{2}+n}}{(1-x^2)\cdots(1-x^n)}\\
&+\frac{(-1)^{N+1}x^{\frac{(N+2)(N+3)}{2}}}{(1-x)\cdots(1-x^{N+1})}.
\end{aligned}
$$

Thus by induction (11) is true for all $N$. That is,

$$
s_N = s_{2;N} + \frac{(-1)^N x^{\frac{(N+1)(N+2)}{2}}}{(1-x)\cdots(1-x^N)}
$$

for all $N$. Therefore, $s_\infty = s_{2;\infty}$.

We will now show that $s_{M;\infty} = s_{M+1;\infty}$ for all $M$.

$$
\begin{aligned}
s_{M;N} &= 1+\sum_{m=1}^{M-1}(-1)^m(x^{\frac{m(3m-1)}{2}}+x^{\frac{m(3m+1)}{2}})\\
&+\frac{(-1)^M x^{\frac{M(M+1)}{2}+(M-1)M}}{1-x^M}\\
&+\sum_{n=M+1}^{N}\frac{(-1)^n x^{\frac{n(n+1)}{2}+(M-1)n}}{(1-x^M)\cdots(1-x^n)}\\
&=1+\sum_{m=1}^{M-1}(-1)^m\left(x^{\frac{m(3m-1)}{2}}+x^{\frac{m(3m+1)}{2}}\right)\\
&+(-1)^M x^{\frac{M(M+1)}{2}+(M-1)M}+(-1)^M x^{\frac{M(M+1)}{2}+(M-1)M+M}\\
&+\frac{(-1)^M x^{\frac{M(M+1)}{2}+(M-1)M+2M}}{1-x^M}\\
&+\sum_{n=M+1}^{N}\frac{(-1)^n x^{\frac{n(n+1)}{2}+(M-1)n}}{(1-x^M)\cdots(1-x^n)}\\
&=1+\sum_{m=1}^{M}(-1)^m\left(x^{\frac{m(3m-1)}{2}}+x^{\frac{m(3m+1)}{2}}\right)\\
&+\frac{(-1)^M x^{\frac{M(M+1)}{2}+(M-1)M+2M}}{1-x^M}\\
&+\sum_{n=M+1}^{N}\frac{(-1)^n x^{\frac{n(n+1)}{2}+(M-1)n}}{(1-x^M)\cdots(1-x^n)}.
\end{aligned}
$$

So for $N = M + 1$ we have

$$
\begin{aligned}
s_{M;M+1} &= 1 + \sum_{m=1}^{M} (-1)^m \left( x^{\frac{m(3m-1)}{2}} + x^{\frac{m(3m+1)}{2}} \right) \\
&\quad + \frac{(-1)^M x^{\frac{M(M+1)}{2} + (M-1)M + 2M}}{1 - x^M} \\
&\quad + \frac{(-1)^{M+1} x^{\frac{(M+1)(M+2)}{2} + (M-1)(M+1)}}{(1 - x^M)(1 - x^{M+1})} \\
&= 1 + \sum_{m=1}^{M} (-1)^m \left( x^{\frac{m(3m-1)}{2}} + x^{\frac{m(3m+1)}{2}} \right) \\
&\quad + \frac{(-1)^{M+1} x^{\frac{M(M+1)}{2} + (M-1)M + 2M + M + 1}}{(1 - x^M)(1 - x^{M+1})} \\
&= 1 + \sum_{m=1}^{M} (-1)^m \left( x^{\frac{m(3m-1)}{2}} + x^{\frac{m(3m+1)}{2}} \right) \\
&\quad + \frac{(-1)^{M+1} x^{\frac{M(M+1)}{2} + (M-1)M + 2M + M + 1}}{1 - x^{M+1}} \\
&\quad + \frac{(-1)^{M+1} x^{\frac{M(M+1)}{2} + (M-1)M + 2M + M + 1 + M}}{(1 - x^M)(1 - x^{M+1})}.
\end{aligned}
$$

Now assume that for some $N \geq M + 1$,

$$
\begin{aligned}
s_{M;N} &= 1 + \sum_{m=1}^{M} (-1)^m \left( x^{\frac{m(3m-1)}{2}} + x^{\frac{m(3m+1)}{2}} \right) \\
&\quad + \sum_{n=M+1}^{N} \frac{(-1)^n x^{\frac{n(n+1)}{2} + Mn}}{(1 - x^{M+1}) \cdots (1 - x^n)} \\
&\quad + \frac{(-1)^N x^{\frac{(N+1)(N+2)}{2} + (M-1)(N+1)}}{(1 - x^M) \cdots (1 - x^N)}.
\end{aligned} \tag{12}
$$

But

$$
\begin{aligned}
&\frac{(-1)^N x^{\frac{(N+1)(N+2)}{2} + (M-1)(N+1)}}{(1 - x^M) \cdots (1 - x^N)} + \frac{(-1)^{N+1} x^{\frac{(N+1)(N+2)}{2} + (M-1)(N+1)}}{(1 - x^M) \cdots (1 - x^{N+1})} \\
&= \frac{(-1)^{N+1} x^{\frac{(N+1)(N+2)}{2} + (M-1)(N+1) + N + 1}}{(1 - x^M) \cdots (1 - x^{N+1})} \\
&= \frac{(-1)^{N+1} x^{\frac{(N+1)(N+2)}{2} + (M-1)(N+1) + N + 1}}{(1 - x^{M+1}) \cdots (1 - x^{N+1})}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(-1)^{N+1}x^{\frac{(N+1)(N+2)}{2}+(M-1)(N+1)+N+1+M}}{(1-x^M)\cdots(1-x^{N+1})}\\
&=\frac{(-1)^{N+1}x^{\frac{(N+1)(N+2)}{2}+M(N+1)}}{(1-x^{M+1})\cdots(1-x^{N+1})}+\frac{(-1)^{N+1}x^{\frac{(N+2)(N+3)}{2}+(M-1)(N+2)}}{(1-x^M)\cdots(1-x^{N+1})}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
s_{M;N+1} &= 1+\sum_{m=1}^{M}(-1)^m\left(x^{\frac{m(3m-1)}{2}}+x^{\frac{m(3m+1)}{2}}\right)\\
&\quad+\sum_{n=M+1}^{N}\frac{(-1)^n x^{\frac{n(n+1)}{2}+Mn}}{(1-x^{M+1})\cdots(1-x^n)}\\
&\quad+\frac{(-1)^N x^{\frac{(N+1)(N+2)}{2}+(M-1)(N+1)}}{(1-x^M)\cdots(1-x^N)}+\frac{(-1)^{N+1}x^{\frac{(N+1)(N+2)}{2}+(M-1)(N+1)}}{(1-x^M)\cdots(1-x^{N+1})}\\
&=1+\sum_{m=1}^{M}(-1)^m\left(x^{\frac{m(3m-1)}{2}}+x^{\frac{m(3m+1)}{2}}\right)\\
&\quad+\sum_{n=M+1}^{N+1}\frac{(-1)^n x^{\frac{n(n+1)}{2}+Mn}}{(1-x^{M+1})\cdots(1-x^n)}\\
&\quad+\frac{(-1)^{N+1}x^{\frac{(N+2)(N+3)}{2}+(M-1)(N+2)}}{(1-x^M)\cdots(1-x^{N+1})}.
\end{aligned}
$$

Thus, (12) is true for all $N$. Therefore $s_{M;\infty}=s_{M+1;\infty}$ for all $M$, thus completing the proof.

In §6, we indicate two other proofs of the pentagonal number theorem, one using theta functions and one using modular forms.

## 4 The Sum of Divisors Function

Two problems involving the sum of divisors function $\sigma(n)$ Euler was interested in are perfect numbers, which are numbers $n$ such that $\sigma(n)=2n$, and amicable numbers, which are pairs of numbers $m$ and $n$ such that $\sigma(m)-m=n$ and $\sigma(n)-n=m$. Fermat looks at a lot of questions about the function $n\mapsto\sigma(n)-n$, the sum of the proper divisors of $n$ (see Weil 1984, Chap. II, §IV). The history of perfect numbers and amicable numbers and in particular Euler's results on them is given by Dickson (1919a, Chap. I).

On p. 445 of his Fourth Notebook, Euler writes that, "Series $1-n-n^2+n^5+n^7-n^{12}-n^{15}+$ etc. $=(1-n)(1-n^2)(1-n^3)$ etc. hanc habet proprietatem, ut, posita series $=s$ sit $-\frac{n\partial s}{s\partial n}=n+3n^2+4n^3+7n^4+6n^5+$ etc. cujus seriei quilibet coefficiens continet summam divisorum exponentis. Ergo erit

$$-ls = n + \frac{3}{2}n^2 + \frac{4}{3}n^3 + \frac{7}{4}n^4 + \frac{6}{5}n^5 + \frac{12}{6}n^6 + \frac{8}{7}n^7 + \text{etc.”}$$

[“The series $1 - n - n^2 + n^5 + n^7 - n^{12} - n^{15} + \text{etc.} = (1-n)(1-n^2)(1-n^3)$ etc. has the property that, with the series put $= s$, it follows that $-\frac{n\partial s}{s\partial n} = n + 3n^2 + 4n^3 + 7n^4 + 6n^5 + \text{etc.}$, of which each coefficient is the sum of the divisors of the exponent. Thus it will be

$$-ls = n + \frac{3}{2}n^2 + \frac{4}{3}n^3 + \frac{7}{4}n^4 + \frac{6}{5}n^5 + \frac{12}{6}n^6 + \frac{8}{7}n^7 + \text{etc.”]}$$

However, Euler does not state his recurrence relation for sums of divisors here.

Euler first discussed his recurrence relation for the divisor function in a letter to Goldbach on March 21/April 1, 1747 (Juškevič and Winter 1965, Brief 113). He says “Letztens habe ich eine sehr wunderbare Ordnung in den Zahlen, welche die summas divisorum der numerorum naturalium darstellen, entdecket, welche mir um so viel merkwürdiger vorkam, da hierin eine große Verknüpfung mit der Ordnung der numerorum primorum zu stecken scheint. Dahero bitte Ew. Hochwohlgeb., diesen Einfall einiger Aufmerksamkeit zu würdigen.” [“I have recently discovered a wonderful pattern among those numbers which represent the sums of the divisors of the natural numbers. This seemed all the more remarkable to me, since there seems to be here a great connection with the sequence of prime numbers. I request Your Honour to give some attention to this idea.”] Euler defines the symbol $\int n$ as the sum of all the divisors of the number $n$ and works it out for $n = 1$ to 16, and then writes

> Taking this meaning for the symbol $\int$, I have thus found that
>
> $$\begin{aligned}\int n = &\int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15)\\ &-\int(n-22) - \int(n-26) + \text{etc.}\end{aligned}$$
>
> where 2 signs + and − constantly follow each other.[25]

Euler says that the recurrence stops at negative numbers, and that $n$ needs to be written in place of $\int(n-n)$.

Euler shows that this rule holds for $n = 1$ to 12, and then says

> The reason for this pattern is so much the less obvious, that one does not see in what way the numbers 1,2,5,7,12,15, etc. would be related with the nature of divisors. Neither can I boast that I have a rigorous demonstration of it. Even

[25] “Diese Bedeutung des Zeichens $\int$ vorausgesetzt, so habe ich gefunden, daß

$$\begin{aligned}\int n = &\int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15)\\ &-\int(n-22) - \int(n-26) + \text{etc.}\end{aligned}$$

wo immer 2 Zeichen + und − aufeinander folgen.” (Juškevič and Winter 1965, Brief 113).

> though I had none, one could still not doubt about the truth, since until over 300 this rule always holds. In the meanwhile, I have nevertheless properly derived this theorem from the following theorem.[26]

Assuming the pentagonal number theorem, Euler gives the following derivation of his recurrence relation for $\int(n)$. Let $s = (1-x)(1-x^2)(1-x^3)$ etc., and so $s = 1 - x - x^2 + x^5 +$ etc. Thus,

$$\frac{\mathrm{d}s}{s} = -\frac{\mathrm{d}x}{1-x} - \frac{2x\mathrm{d}x}{1-x^2} - \frac{3x^2 dx}{1-x^3} - \frac{4x^3\mathrm{d}x}{1-x^4} - \frac{5x^4\mathrm{d}x}{1-x^5} - \text{etc.}$$

and on the other hand

$$\frac{\mathrm{d}s}{s} = \frac{-\mathrm{d}x - 2x\mathrm{d}x + 5x^4\mathrm{d}x + 7x^6\mathrm{d}x - 12x^{11}\mathrm{d}x - 15x^{14}\mathrm{d}x + \text{ etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \text{ etc.}}$$

Therefore,

$$\begin{aligned}&\frac{1 + 2x - 5x^4 - 7x^6 + 12x^{11} + 15x^{14} - \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \text{etc.}}\\ &\quad = \frac{1}{1-x} + \frac{2x}{1-x^2} + \frac{3x^2}{1-x^3} + \frac{4x^3}{1-x^4} + \frac{5x^4}{1-x^5} + \frac{6x^5}{1-x^6} + \text{etc.}\end{aligned}$$

When all the fractions on the right hand side are transformed into geometric series they become

$$\begin{array}{cccccccccccccc}
1 & +x & +x^2 & +x^3 & +x^4 & +x^5 & +x^6 & +x^7 & +x^8 & +x^9 & +x^{10} & +x^{11} & +x^{12} & +\text{etc.}\\
 & +2x & & +2x^3 & & +2x^5 & & +2x^7 & & +2x^9 & & +2x^{11} & & \\
 & & +3x^2 & & & +3x^5 & & & +3x^8 & & & +3x^{11} & & \\
 & & & +4x^3 & & & & +4x^7 & & & & +4x^{11} & & \\
 & & & & +5x^4 & & & & & +5x^9 & & & & \\
 & & & & & +6x^5 & & & & & & +6x^{11} & & \\
 & & & & & & +7x^6 & & & & & & & \\
 & & & & & & & +8x^7 & +9x^8 & +10x^9 & +11x^{10} & +12x^{11} & +13x^{12} & +\text{etc.}
\end{array}$$

and thus

$$\begin{aligned}&1 + \int 2x + \int 3x^2 + \int 4x^3 + \int 5x^4 + \int 6x^5 + \int 7x^6 + \int 8x^7 + \int 9x^8 + \text{etc.}\\ &= \frac{1 + 2x - 5x^4 - 7x^6 + 12x^{11} + 15x^{14} - 22x^{21} - 26x^{25} + 35x^{34} + \text{etc.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \text{etc.}}\end{aligned}$$

[26] "Der Grund dieser Ordnung fällt um so viel weniger in die Augen, da man nicht sieht, was die Zahlen 1, 2, 5, 7, 12, 15, etc. für eine Verwandtschaft mit der natura divisorum haben. Ich kann mich auch nicht rühmen, daß ich davon eine demonstrationem rigorosam hätte. Wann ich aber auch gar keine hätte, so würde man an der Wahrheit doch nicht zweifeln können, weil bis über 300 diese Regel immer eingetroffen. Inzwischen habe ich doch dieses theorema aus folgendem Satz richtig hergeleitet." (Juškevič and Winter 1965, Brief 113).

“woraus das gegebene theorema leicht fließt. Man sieht aber zugleich, daß dasselbe nicht so obvium ist, und daß zweifelsohne darin noch schöne Sachen verborgen liegen müssen.” [“from which the given theorem follows easily. But at the same time one sees that this is still not so obvious, and that without doubt many beautiful things must still lie hidden in it.”]

Let us write this out using summation and product notation. First,

$$\prod_{n=1}^{\infty}(1-x^n)=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}$$

Then taking the logarithm of both sides,

$$\sum_{n=1}^{\infty}\log(1-x^n)=\log\left(\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}\right).$$

Now taking the derivative of both sides,

$$\sum_{n=1}^{\infty}\frac{-nx^{n-1}}{1-x^n}=\frac{\sum_{n=-\infty}^{\infty}(-1)^n\cdot\frac{n(3n-1)}{2}\cdot x^{\frac{n(3n-1)}{2}-1}}{\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}}.$$

Hence,

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}-nx^{n-1}x^{nk}=\frac{\sum_{n=-\infty}^{\infty}(-1)^n\cdot\frac{n(3n-1)}{2}\cdot x^{\frac{n(3n-1)}{2}-1}}{\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}}.$$

Therefore,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}-nx^{kn}=\frac{\sum_{n=-\infty}^{\infty}(-1)^n\cdot\frac{n(3n-1)}{2}\cdot x^{\frac{n(3n-1)}{2}}}{\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}}.$$

The left-hand side is equal to

$$-\sum_{j=1}^{\infty}\sigma(j)x^j.$$

We remark that Jacob I Bernoulli in “Attollere Infinitinomium ad potestatem indefinitam”, in his *Varia Posthuma* (Cramer 1744, Art. I, pp. 993–998), takes two equal expressions

$$\left(\sum_{n=0}^{\infty}a_n x^n\right)^m=\sum_{n=0}^{\infty}b_n x^n$$

takes the logarithm, differentiates and then cross multiplies to express each $b_n$ in terms of $b_0, \ldots, b_{n-1}$ and $a_0, \ldots, a_{n-1}, a_n$. Ferraro (2008, Chap. 10) explains the development of the idea of recurrent series.

The first paper in which Euler gives his recurrence relation for the divisor function is his "Découverte d'une loi tout extraordinaire des nombres, par rapport à la somme de leurs diviseurs", presented to the Berlin Academy on June 22, 1747, and published in the *Bibliothèque impartiale* in 1751 (Euler 1751a), E175 (cf. Winter 1957, Nr. 45 and Knobloch 1984, I, Nr. 5, III, Nr. 84). He begins by saying that it seems that the order of the prime numbers is a mystery the human spirit will never penetrate. Then,

> I too believe myself quite far from this end, but I have discovered a very strange law on the sums of the divisors of the natural numbers, which, at first sight, seem as irregular as the progression of the prime numbers, and which appear even to envelop the latter. This law, which I shall explain, is in my opinion all the more important because it belongs to a type whose truth we can be assured of even if we cannot give a perfect demonstration of it. Nevertheless I shall present such evidence for this that one may consider it as almost equivalent to a rigorous demonstration.[27]

Euler states $\int n$ for $n = 1$ to 100, and then states his recurrence relation for $\int n$ in the following way[28]:

> I remark nevertheless that this progression follows a well-defined law, and can even be counted among the class of progressions which Geometers call recurrent, such that one can always form each term from some of the terms preceding it, according to a constant rule. For if $\int n$ denotes any term of this irregular progression, and $\int(n-1)$, $\int(n-2)$, $\int(n-3)$, $\int(n-4)$, $\int(n-5)$, etc. the preceding terms, I say that the value of $\int n$ is always composed of some of the preceding

[27] "Je me crois aussi bien éloigné de ce but, mais je viens de découvrir une loi fort bizarre parmi les sommes des diviseurs des nombres naturels, qui, au premier coup d'oeil, paroissent aussi irrégulieres que la progression des nombres premiers, et qui semblent même enveloper celle-ci. Cette régle, que je vai expliquer, est à mon avis d'autant plus importante qu'elle appartient à ce genre dont nous pouvons nous assurer de la vérité, sans en donner une démonstration parfaite. Néanmoins, j'en apporterai de telles preuves, qu'on pourra presque les envisager comme équivalentes à une démonstration rigoureuse." (Euler 1751a, §1), E175.

[28] "Néanmoins, j'ai remarqué que cette progression suit une loi bien réglée et qu'elle est même comprise dans l'ordre des progressions que les Geometres nomment recurrentes, de sorte qu'on peut toujours former chacun de ces termes par quelques-uns des précédens, suivant une régle constante. Car si $\int n$ marque un terme quelconque de cette irréguliere progression, et $\int(n-1)$, $\int(n-2)$, $\int(n-3)$, $\int(n-4)$, $\int(n-5)$, etc. des termes précédens, je dis que la valeur de $\int n$ est toujours composée de quelques-uns des précédens suivant cette formule:

$$\begin{aligned}\int n &= \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15)\\ &\quad - \int(n-22) - \int(n-26) + \int(n-35) + \int(n-40) - \int(n-51) - \int(n-57)\\ &\quad + \int(n-70) + \int(n-77) - \int(n-92) - \int(n-100) + \text{etc.''}\end{aligned}$$

(Euler 1751a, §5), E175.

terms according to this formula:

$$\begin{aligned}\int n &= \int(n-1)+\int(n-2)-\int(n-5)-\int(n-7)+\int(n-12)+\int(n-15)\\ &\quad-\int(n-22)-\int(n-26)+\int(n-35)+\int(n-40)-\int(n-51)\\ &\quad-\int(n-57)+\int(n-70)+\int(n-77)-\int(n-92)-\int(n-100)+\text{etc.}\end{aligned}$$

He then states the general rule for the recurrence relation, that the plus and minus signs switch back and forth in pairs, and that the differences 1, 3, 2, 5, 3, 7, 4, 9, 5, etc. of the numbers subtracted from $n$ are alternately from the sequence of all natural numbers and the sequence of the odd numbers, and that $\int(n-n)$ must be replaced by $n$. He verifies the recurrence for $n = 1$ to 20, and also $n = 101$ and $n = 301$.

Euler explains that he found this recurrence relation for sums of divisors in his research on the pentagonal number theorem, which he is still not able to prove:

> I admit that I did not fall upon this discovery purely by chance; rather, another proposition of the same nature, which must be judged to be true even though I am not able to give a demonstration of it, opened the way for me to find this pretty property.[29]

Namely, if

$$\begin{aligned}&(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)\text{ etc.}\\ &= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\text{etc.},\end{aligned}$$

then the recurrence relation for sums of divisors follows by taking the logarithmic derivative of both sides. Euler says,

> Having thus discovered that these two infinite expressions are equal, although the equality cannot be demonstrated, all the conclusions which one can deduce from this equality will be of the same nature, that is to say true without being demonstrated.[30]

Pólya (1954, Chap. VI) looks at Euler's use of "inductive reasoning" to justify the recurrence relation he found for $\sigma(n)$. Pólya translates and gives commentary on E175.

Euler explains his ideas about induction, e.g. in his 1736 paper "Theorematum quorundam ad numeros primos spectantium demonstratio", E54, his 1748 paper "Demonstration sur le nombre des points, où deux lignes des ordres quelconques peuvent se couper", E148, and his 1753 paper "Specimen de usu observationum in Mathesi pura", E256. In the "Summarium" (editorial summary) of E256, Euler writes,

29 "J'avouë aussi que ce n'a pas été par un pur hazard que je suis tombé sur cette découverte; mais une autre proposition d'une pareille nature qui doit être jugée vraie, quoique je n'en puisse donner une démonstration, m'a ouvert la chemin de parvenir à cette belle propriété." (Euler 1751a, §8), E175.

30 "Ayant donc découvert que ces deux expressions infinies sont égales, quoique l'égalité ne puisse être démontrée, toutes les conclusions qu'on pourra déduire de cette égalité seront de même nature, c'est-à-dire vraies sans être démontrées." (Euler 1751a, §10), E175.

> It is clear from this that in the science of numbers, which is still quite imperfect, much can be expected from observations, indeed from which new properties of numbers are constantly recognized, but a great deal of effort would need to be expended subsequently in proving these new properties.[31]

In his published papers on the pentagonal number theorem, Euler does not work out the general case explicitly, but he makes it clear how to do it from the starting cases, i.e. the difference between not having found a proof and having found a proof is that when he found a proof, he finally understood exactly how to do the induction step, while before it was just observation. We saw in §3 that in his unpublished notes he works out the induction explicitly.

Georg Wolfgang Krafft (1701–1754) writes to Euler on July 7, 1747 (Juškevič and Winter 1976, Brief Nr. 169) and says that Euler's recurrence relation for sums of divisors very much impressed him. Krafft says that previously he had considered 1 to be a prime number, but that this article has changed his opinion, since it follows from it that $\sigma(1) = 1$ instead of $\sigma(1) = 2$. Krafft had discussed the sums of divisors of numbers in previous letters to Euler, and also in papers in volume 7 of the *Commentarii* and volume 3 of the *Novi Commentarii*. Krafft was a physicist who had been a colleague of Euler's at the St. Petersburg Academy (Fellmann 1995, Kapitel II, p. 36). Krafft had been in St. Petersburg from 1728 to 1744, and then was a professor of mathematics and natural philosophy at the University of Tübingen.

Euler next writes to d'Alembert about the pentagonal number theorem in a letter on February 15, 1748 (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 13). Euler writes:

> With regard to the series $1-x-x^2+x^5+x^7$ etc. $= (1-x)(1-x^2)(1-x^3)(1-x^4)$ which I have spoken about with you, I have found from it a very singular property of numbers, concerning the sum of the divisors of each number. With $\int n$ indicating the sum of all the divisors of the number $n$, such that $\int 1 = 1$; $\int 2 = 3$; $\int 3 = 4$; $\int 4 = 7$; $\int 5 = 6$; $\int 6 = 12$; $\int 7 = 8$ etc., it seems at first nearly impossible that one could discover any pattern in the series of these numbers 1, 3, 4, 7, 6, 12, 8, [15, ]13, 18, etc., but I have found that each term depends on several of the preceding, according to this formula:
>
> $$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) - \int(n-22) - \text{etc.}$$
>
> where it is remarked 1° that the numbers
>
> $$\begin{array}{ccccccccccccccccccc} 1, & & 2, & & 5, & & 7, & & 12, & & 15, & & 22, & & 26, & & 35, & & 40, \text{ etc.} \\ & 1 & & 3 & & 2 & & 5 & & 3 & & 7 & & 4 & & 9 & & 5 & \end{array}$$
>
> are formed easily by considering the differences alternately.

[31] "Ex quo perspicuum est in scientia numerorum, quae etiam nunc maxime est imperfecta, plurimum ab observationibus esse expectandum, quippe quibus ad novas proprietates numerorum continuo deducimur, in quarum demonstratione deinceps sit elaborandum." (from the "Summarium" of E256).

> 2° In each case one takes only those terms where the numbers after the $\int$ sign are not negative.
> 3° If one comes to the term $\int 0$ or $\int(n-n)$, one takes for its value the number $n$ itself.
> Thus, you can see that
>
> $$\begin{aligned}\int 4 &= \int 3+\int 2=7;\\ \int 9 &= \int 8+\int 7-\int 4-\int 2=15+8-7-3=13;\\ \int 15 &= \int 14+\int 13-\int 10-\int 8+\int 3+\int 0=24+14-18-15+4+15=24;\\ \int 35 &= \int 34+\int 33-\int 30-\int 28+\int 23+\int 20-\int 13-\int 9+\int 0\\ &= 54+48-72-56+24+42-14-13+35=48.\end{aligned}$$
>
> Next, every time that $n$ is a prime number one will find that $\int n=n+1$ and hence, since the nature of the prime numbers enters into this consideration, this law seems to me all the more remarkable.[32]

D'Alembert replies to Euler in a letter on March 30, 1748 (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 14), saying, “Il ne me reste de place que pour vous dire que votre Theoreme sur les suites me paroit tres beau” [“I only have enough space left to say to you that your Theorem on series seems very beautiful to me”]. D'Alembert

[32] “A l'egard de la suite $1-x-x^2+x^5+x^7$ etc. $=(1-x)(1-x^2)(1-x^3)(1-x^4)$ etc. dont je vous ai parlé, j'en ai tiré une proprieté fort singuliere des nombres par rapport à la somme des diviseurs de chaque nombre. Que $\int n$ marque la somme de tous les diviseurs du nombre $n$ de sorte que $\int 1=1$; $\int 2=3$; $\int 3=4$; $\int 4=7$; $\int 5=6$; $\int 6=12$; $\int 7=8$ etc. il paroit d'abord presque impossible de decouvrir aucune loi dans la suite de ces nombres 1, 3, 4, 7, 6, 12, 8, [15, ]13, 18, etc. mais j'ai trouvé que chaque terme depend de quelques uns des precedents selon cette formule:

$$\int n=\int(n-1)+\int(n-2)-\int(n-5)-\int(n-7)+\int(n-12)+\int(n-15)-\int(n-22)-\text{etc.}$$

où il est à remarquer 1° que les nombres

$$\begin{array}{ccccccccccccccccccccc} 1, && 2, && 5, && 7, && 12, && 15, && 22, && 26, && 35, && 40, & \text{etc.}\\ & 1 && 3 && 2 && 5 && 3 && 7 && 4 && 9 && 5 && \end{array}$$

se forment aisement par les differences considerées alternativement.
2° Dans chaque cas on ne prend que les termes, où les nombres apres le signe $\int$ ne sont point negatifs.
3° S'il arrive ce terme $\int 0$ ou $\int(n-n)$, on prendra pour la valeur le nombre $n$ même.
Ainsi vous verrés que

$$\begin{aligned}\int 4 &= \int 3+\int 2=7;\\ \int 9 &= \int 8+\int 7-\int 4-\int 2=15+8-7-3=13;\\ \int 15 &= \int 14+\int 13-\int 10-\int 8+\int 3+\int 0=24+14-18-15+4+15=24;\\ \int 35 &= \int 34+\int 33-\int 30-\int 28+\int 23+\int 20-\int 13-\int 9+\int 0\\ &= 54+48-72-56+24+42-14-13+35=48.\end{aligned}$$

Donc toutes les fois que $n$ est un nombre premier on trouvera que $\int n=n+1$ et partant puisque la nature des nombres premiers entre dans cette consideration cette loi me paroit d'autant plus remarquable.” (Juškevič and Taton 1980, Euler-d'Alembert, Lettre No. 13).

also tells Euler in this letter that his health is now very good, and that he soon hopes to send some memoirs to the Berlin Academy.

On March 7/18, 1752, Euler sends his paper "Observatio de summis divisorum" (Euler 1760b), E243, to Schumacher (Juškevič and Winter 1961, Brief 185). It was presented to the St. Petersburg Academy on April 6, 1752 (Nevskaja 2000, p. 396) and published in 1760 in the *Novi Commentarii*; the paper had been presented to the Berlin Academy on September 9, 1751 (Knobloch 1984, I, Nr. 29; cf. Winter 1957, Nr. 226, Anm. 1). Euler remarks that it is unexpected that the sums of divisors of numbers would have a connection with the pentagonal numbers, writing "Interim tamen non fortuito et quasi divinando ad cognitionem huius veritatis perveni; cui enim in mentem venire potuisset ordinem, qui forte in summis divisorum locum habuerit, ex natura serierum recurrentium ac numerorum pentagonalium per solam coniecturam elicere velle?" ["But in the meantime, I have come to knowledge of this truth not by chance and, as it were, by divination; for to whom could it have occurred to wish to elicit the order that might exist in the sum of divisors from the nature of recurrent series and pentagonal numbers by guesswork alone?"] In this article Euler again derives his recurrence relation for $\int n$ using the pentagonal number theorem:

> I was led to this observation by consideration of this infinite formula
>
> $$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8) \text{ etc.},$$
>
> whose value, if it is expanded by actual multiplication of all the factors and arranged according to powers of $x$, I have found to be converted into the following series
>
> $$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - \text{etc.},$$
>
> where in the exponents of $x$ the same numbers occurs which I described above, namely the pentagonal numbers, both themselves and continued backwards.[33]

This article was reviewed on May 1761 *Nova acta eruditorum* (p. 218), which reviewed volume 5 of the *Novi Commentarii*,

> The illustrious author thus considered in this article the sum of all the divisors of any number, not to the ends that others typically pursue, for the investigation of

[33] "Deductus autem sum ad hanc observationem per considerationem istius formulae infinitae

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8) \text{ etc.},$$

cuius valorem, si multiplicatione singulorum factorum actu instituta evolvatur ac secundum potestates ipsius $x$ disponatur, deprehendi in sequentem seriem converti

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - \text{etc.},$$

ubi in exponentibus ipsius $x$ iidem numeri occurrunt, quos supra descripsi, numeri scilicet pentagonales cum ipsi tum retro continuati." (Euler 1760b), E243.

> perfect or amicable numbers and other questions of this kind, but to consider the order and the law by which the sums of the divisors for all numbers proceed. This should indeed be considered as most deeply hidden, since for prime numbers the sum of divisors exceeds them by one, but for composites it is the greater the more prime factors that comprise them. Therefore since the rule of the progression of the prime numbers is still considered among the mathematical mysteries, in which not even Fermat was permitted to penetrate, could anyone fail to gather from this the difficulty of the problem? In solving this, the Illustrious Euler was so fortunate that he not only uncovered the law but also proved it in his third paper.[34]

The third paper in the mathematics section of volume 5 of the *Novi Commentarii* is Euler's "Demonstratio theorematis circa ordinem in summis divisorum observatum", E244.

Euler's paper "Demonstratio theorematis circa ordinem in summis divisorum observatum" (1760a) was published in the same volume of this journal (cf. Juškevič and Winter 1961, Brief 189). In this article he gives an inductive proof of the pentagonal number theorem, which we discussed in §3. Then in Proposition 4, or "Theorema principale demonstrandum" ["The principal theorem to be demonstrated"] Euler gives a rigorous proof of his recurrence relation for $\int n$. As we remarked previously, Euler had earlier derived the recurrence relation from the pentagonal number theorem, which he hadn't been able to prove. Now he again derives it, and says "sicque habetur plena ac perfecta demonstratio theorematis propositi, quae, cum praeter tractationem serierum infinitarum, per logarithmos et differentialia procedat, minus quidem naturalis, sed ob hoc ipsum multo magis notabilis est aestimanda." ["and thus a complete and perfect demonstration of the proposed theorem is obtained, which, since it goes beyond the treatment of infinite series and proceeds by logarithms and differentials, is indeed not entirely natural, but because of this it should be thought all the more notable."]

Goldbach writes a letter to Euler on May 9, 1752 (Juškevič and Winter 1965, Brief 157; cf. Juškevič and Winter 1961, Brief 185), in which he says

> Your Honour's paper on the sums of divisors, which you sent to our Academy of Science, has been communicated to me by Prof. Grischow. I do not now think myself capable of judging its quality, but your well known insight in these matters permits me no doubts about the correctness of what is contained in the above-mentioned paper; in particular I saw with pleasure that Your Honour noticed such a beautiful pattern in the numbers 1, 2, 5, 7, 12, 15, 22, etc. But it

[34] "Auctor igitur illustris in hac dissertatione summam omnium divisorum cuiuscunque numeri contemplatur, non eo consilio, ut alias in investigatione numerorum perfectorum vel amicabilium, aliarumque huiusmodi quaestionum fieri solet, sed ut ordinem et legem, qua istae summae divisorum singulis numeris convenientes progrediuntur, exploret. [Q]uae profecto maxime abscondita videri debet, cum pro numeris primis summa divisorum ipsos unitate superet, pro compositis vero eo magis, quo plures factores primos in se complectuntur. Quoniam igitur ratio progressionis numerorum primorum inter mysteria mathematica adhuc refertur, in quae ne Fermatio quidem penetrare licuit, ecquis inde difficultatem problematis non colligat? Illustri Eulero in eo solvendo ita felici esse licuit, ut non solum illam legem detegeret, sed IIItia dissertatione etiam demonstraret." May 1761 *Nova acta eruditorum* (p. 218).

occurred to me and I believe completely, that there are series such that the law of the progression, be it ever so short and simple in itself, nevertheless is not to be grasped, even from more than 100 consecutive terms of the series, as for example: Let the series be

$$\begin{array}{lccccccc} & (1) & (2) & (3) & (4) & (5) & (6) & (7) \\ \alpha \ldots & 1, & 1, & 5, & 7, & 1, & 23, & 43, \text{ etc.} \end{array}$$

whose progression is such that given any term $A$ corresponding to the index $n$, let $A \pm \sqrt{(2 \cdot 3^n - 2A^2)}$ = the next following term $B$, taking the sign $+$ or $-$ so that $B$ will not be divisible by 3. This series has a sister series

$$\begin{array}{lccccccc} & (1) & (2) & (3) & (4) & (5) & (6) & (7) \\ \beta \ldots & 1, & 2, & 1, & 4, & 11, & 10, & 13, \text{ etc.} \end{array}$$

made such that twice the square of the term whose exponent is $n$ in the series $\beta$, added to the square of the term whose exponent is the same $n$ in the series $\alpha$, would give $3^n$, whence it is immediately apparent that the law of the progression of this series $\beta$ is $A \pm \sqrt{3^n - 2A^2} = B$, taking $+$ or $-$ so that $B$ will not be divisible by 3.[35]

Augustin Nathanael Grischow (1726–1760) was the conference secretary of the St. Petersburg Academy from 1751 to 1754. In the previous letters in their correspondence Euler and Goldbach were talking about Fermat's polygonal number theorem.

Edward Waring (1736–1798) mentions Euler's recurrence relation for $\sigma(n)$ in the preface of the 1762 edition of his *Miscellanea analytica de aequationibus algebraicis et curvarum proprietatibus*.

[35] "Eurer Hochedelgebornen Dissertation de summis divisorum, welche Sie an die hiesige Akad. der Wiss. übersandt haben, ist mir von dem Prof. Grischow kommunizieret worden. Ich befinde mich jetzo nicht imstande, davon pro dignitate zu urteilen, allein Dero bekannte Einsicht in dergleichen Sachen lässet mich an der Richtigkeit alles dessen, was in bemeldter Dissertation enthalten ist, nicht zweifeln; insonderheit habe ich mit Vergnügen gesehen, daß in den numeris 1, 2, 5, 7, 12, 15, 22, etc. eine so schöne Ordnung von Ew. Hochedelgeb. bemerket worden und glaube gänzlich, daß es series gibt, aus deren mehr als 100 terminis consequentibus die lex progressionis, ob sie gleich an sich selbst kurz und leicht ist, dennoch nicht zu ersehen sein wird, als zum Exempel: Sit series

$$\begin{array}{lccccccc} & (1) & (2) & (3) & (4) & (5) & (6) & (7) \\ \alpha \ldots & 1, & 1, & 5, & 7, & 1, & 23, & 43, \text{ etc.} \end{array}$$

cuius progressio haec est, ut dato termino quocunque $A$ et exponente termini $n$, fiat $A \pm \sqrt{(2 \cdot 3^n - 2A^2)} =$ termino proxime sequenti $B$ sumendo signum $+$ vel $-$ ita ut $B$ non fiat divisibilis per 3, ex quo sequitur seriem habere sororem

$$\begin{array}{lccccccc} & (1) & (2) & (3) & (4) & (5) & (6) & (7) \\ \beta \ldots & 1, & 2, & 1, & 4, & 11, & 10, & 13, \text{ etc.} \end{array}$$

ita comparatam ut duplum quadrati termini cuius exponens est $n$ in serie $\beta$ additum ad quadratum termini cuius exponens est idem $n$ in serie $\alpha$ det $3^n$ unde simul apparet legem progressionis seriei $\beta$ esse $A \pm \sqrt{3^n - 2A^2} = B$, sumendo $+$ vel $-$, ita ut $B$ non fiat divisibilis per 3." (Juškevič and Winter 1965, Brief 157).

Johann Heinrich Lambert (1728–1777) in his 1771 *Anlage zur Architectonic*, volume II, Chap. XXXI, §875, pp. 506–511 deals with the series $\sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$ and $\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}$.

Other recurrence relations and results for the sum of divisors function are given by Dickson (1919a, Chap. X).

## 5 The Partition Function

The partition function $n^{(m)}$ is defined to be the number of ways of expressing $n$ as a sum of positive integers less than or equal to $m$, disregarding order. In particular $p(n) = n^{(\infty)}$. If $\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=0}^{\infty} a_n x^n$, then since $\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n$,

$$p(n)a_0 + p(n-1)a_1 + \cdots = 0$$

for all $n \geq 1$. As we saw in §3, Euler first explicitly notes this in his 1741 "Observationes analyticae variae de combinationibus" (Euler 1751b, §37), E158. We also saw that he explains how the recurrence relation for $p(n)$ follows from the pentagonal number theorem in his November 10, 1742 letter to Nicolaus I Bernoulli. He also states the recurrence in his *Introductio in analysin infinitorum* (Euler 1748, Caput XVI, §324).

The next time the recurrence for $p(n)$ occurs in Euler's work is in the paper "De partitione numerorum", presented to the St. Petersburg Academy on January 26, 1750 (Nevskaja 2000, p. 369) (sent to Schumacher on December 18/29, 1749 (Juškevič and Winter 1961, Brief 108)) and published in the *Novi Commentarii* in 1753 (Euler 1753b), E191. In E158 and the *Introductio*, Euler had only briefly mentioned the recurrence relation for $p(n)$ and he probably wanted to give some detailed examples now. In §§37–39, he looks at recurrence relations for $n^{(\infty)}$. Then in §40, he states the pentagonal number theorem, and that $(1-x)(1-x^2)(1-x^3)$ etc. is the reciprocal of the generating function for $n^{(\infty)}$. In §41, he observes that this provides a recurrence relation for $n^{(\infty)}$.

> This form therefore provides for us a scale of relation of the sought series, by which it is certain that
>
> $$\begin{aligned} n^{(\infty)} &= (n-1)^{(\infty)} + (n-2)^{(\infty)} - (n-5)^{(\infty)} - (n-7)^{(\infty)} + (n-12)^{(\infty)} \\ &\quad +(n-15)^{(\infty)} - (n-22)^{(\infty)} - (n-26)^{(\infty)} + (n-35)^{(\infty)} \\ &\quad +(n-40)^{(\infty)} - (n-51)^{(\infty)} - (n-57)^{(\infty)} + \text{etc.} \end{aligned}$$
>
> That this law of progression holds will be evident to anybody testing it. For let $n = 30$, and it will be found that
>
> $$30^{(\infty)} = 29^{(\infty)} + 28^{(\infty)} - 25^{(\infty)} - 23^{(\infty)} + 18^{(\infty)} + 15^{(\infty)} - 8^{(\infty)} - 4^{(\infty)};$$

taking these numbers from the table it is

$$5604 = 4565 + 3718 - 1958 - 1255 + 385 + 176 - 22 - 5.$$

And the series can thus be continued in this way as far as please.[36]

This article is reviewed in the June 1759 *Nova acta eruditorum* (pp. 316–317).

As we remarked in §1, as a formal identity the pentagonal number theorem is equivalent to

$$p_e(n) - p_o(n) = \begin{cases} (-1)^j, & \text{if} \quad n = j(3j-1)/2, \\ 0, & \text{otherwise}, \end{cases}$$

where $p_e(n)$ is the number of ways to write $n$ as a sum of distinct positive even integers and $p_o(n)$ is the number of ways to write $n$ as a sum of distinct positive odd integers. On p. 97 of his Fifth Notebook, around 1750–1751 (Kiselev and Matvievskaja 1965, §6), Euler puts

$$\begin{aligned} &(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5) \text{ etc.} \\ &\quad = 1 - (1)x - (2)x^2 - (3)x^3 - (4)x^4 - (5)x^5 - (6)x^6 - (7)x^7 - \text{etc.} \end{aligned}$$

where $(n)$ denotes the coefficient of $x^n$ in this power series. He says "et hujus seriei coefficientes ex natura formae ita determinabuntur" ["and the coefficients of this series will be thus determined from the nature of the form"], and works out $(n)$ for $n = 2$ to $n = 22$ in the following way:

$$\begin{aligned} (2) &= 1 \\ (3) &= 1 - 1 \\ (4) &= 1 - 1 \\ (5) &= 1 - 2 \\ (6) &= 1 - 2 + 1 \end{aligned}$$

[36] "Haec igitur forma nobis suppeditat scalam relationis seriei quaesitae, qua constat fore

$$\begin{aligned} n^{(\infty)} = {} & (n-1)^{(\infty)} + (n-2)^{(\infty)} - (n-5)^{(\infty)} - (n-7)^{(\infty)} + (n-12)^{(\infty)} + (n-15)^{(\infty)} \\ & -(n-22)^{(\infty)} - (n-26)^{(\infty)} + (n-35)^{(\infty)} + (n-40)^{(\infty)} - (n-51)^{(\infty)} - (n-57)^{(\infty)} \\ & + \text{etc.} \end{aligned}$$

Hanc autem legem progressionis locum habere tentanti facile patebit. Sit enim $n = 30$; reperietur fore

$$30^{(\infty)} = 29^{(\infty)} + 28^{(\infty)} - 25^{(\infty)} - 23^{(\infty)} + 18^{(\infty)} + 15^{(\infty)} - 8^{(\infty)} - 4^{(\infty)};$$

est enim his numeris ex tabula desumtis

$$5604 = 4565 + 3718 - 1958 - 1255 + 385 + 176 - 22 - 5.$$

Atque hoc modo ista series, quousque libuerit, continuari potest." (Euler 1753b, §41), E191.

$$\begin{aligned}
(7) &= 1 - 3 + 1 \\
(8) &= 1 - 3 + 2 \\
(9) &= 1 - 4 + 3 \\
(10) &= 1 - 4 + 4 - 1 \\
(11) &= 1 - 5 + 5 - 1 \\
(12) &= 1 - 5 + 7 - 2 \\
(13) &= 1 - 6 + 8 - 3 \\
(14) &= 1 - 6 + 10 - 5 \\
(15) &= 1 - 7 + 12 - 6 + 1 \\
(16) &= 1 - 7 + 14 - 9 + 1 \\
(17) &= 1 - 8 + 16 - 11 + 2 \\
(18) &= 1 - 8 + 19 - 15 + 3 \\
(19) &= 1 - 9 + 21 - 18 + 5 \\
(20) &= 1 - 9 + 24 - 23 + 7 \\
(21) &= 1 - 10 + 27 - 27 + 10 - 1 \\
(22) &= 1 - 10 + 30 - 34 + 13 - 1
\end{aligned}$$

This expresses $(n)$ as the number of ways to write $n$ as a sum of one positive integer, minus the number of ways to write $n$ as a sum of two distinct positive integers, plus the number of ways to write $n$ as a sum of three distinct positive integers, minus the number of ways to write $n$ as a sum of four distinct positive integers, etc. For instance, $7 = 7, 6+1, 5+2, 4+3, 3+2+1$, hence $(7) = 1 - 3 + 1 = -1$; while $8 = 8, 7+1, 6+2, 5+3, 5+2+1, 4+3+1$, hence $(8) = 1 - 3 + 2 = 0$. This is the same as saying that $(n) = p_o(n) - p_e(n)$.

Other recurrence relations and results for the partition function are given by Dickson (1919b, Chap. III).

Hardy and Ramanujan (1918, Table IV) uses Euler's recurrence relation for $p(n)$ to determine $p(n)$ for $n \leq 200$.

Let $R(N)$ be the number of steps to compute $p(n)$ for $n = 1, \ldots, N$ using Euler's recurrence relation

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots \tag{13}$$

for $p(n)$. There are approximately $2\sqrt{\frac{2n}{3}}$ terms on the right hand side of (13). Thus to compute $p(n)$ for $n = 1, \ldots, N$, there are approximately

$$2\sqrt{\frac{2}{3}} \sum_{n=1}^{N} \sqrt{n}$$

steps, which is upper bounded by $2\sqrt{\frac{2}{3}} \cdot N\sqrt{N}$. Hence, $R(N) = O(N^{3/2})$.

## 6 Analytic Functions

The *Jacobi theta functions* $\theta_1, \theta_2, \theta_3, \theta$ are functions of two complex variables that are used to define elliptic functions, and which also have interesting applications to number theory. An excellent introduction is Chandrasekharan (1985, Chap. V). In the following we outline two proofs of the pentagonal number thereom using the Jacobi theta functions; the latter also involves the theory of modular forms.

It will be convenient to use $\theta_3$; however, like the trigonometric functions sin, cos, tan, etc., each of the theta functions can be used to define the others. $\theta_3$ is defined by

$$\theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i v}, \tag{14}$$

where $q = e^{\pi i\tau}$, $\Im\tau > 0$ and $v$ is any complex number. For a fixed $\tau$, we can show that $\theta_3$ converges absolutely and uniformly on compact sets, and hence for fixed $\tau$, $\theta_3$ is an entire function of $v$. We can also show that $\theta_3$ has the product expansion

$$\theta_3(v,\tau) = \prod_{n=1}^{\infty}(1-q^{2n}) \cdot \prod_{n=1}^{\infty}(1+q^{2n-1}e^{2\pi i v}) \cdot \prod_{n=1}^{\infty}(1+q^{2n-1}e^{-2\pi i v}). \tag{15}$$

Proofs for these properties of $\theta_3$ are given by Chandrasekharan (1985, Chap. V).

Using the product expansion (15) for $\theta_3$, we find that

$$\theta_3\left(\frac{1}{2}+\frac{z}{2}, 3z\right) = \prod_{n=1}^{\infty}(1-e^{2\pi i n z}).$$

But by the definition of $\theta_3$,

$$\theta_3\left(\frac{1}{2}+\frac{z}{2}, 3z\right) = \sum_{n=-\infty}^{\infty} e^{3\pi i n^2 z + 2\pi i n(\frac{1}{2}+\frac{z}{2})} = \sum_{n=-\infty}^{\infty} (-1)^n e^{\frac{n(3n+1)}{2}\cdot 2\pi i z},$$

which gives the pentagonal number theorem, namely with $x = e^{2\pi i z}$.

Carl Gustav Jacob Jacobi (1804–1851) proved many identities involving series and products. Chandrasekharan (1985, Chap. V) gives notes on this. Jacobi says in a letter (Stäckel and Ahrens 1908, Brief 10) that "Die Eulersche Formel ist ein spezieller Fall einer Formel, welche wohl das wichtigste und fruchtbarste ist, was ich in reiner Mathematik erfunden habe" ["The Eulerian formula is a special case of a formula that is the most important and most fruitful I have found in pure mathematics"], namely the product expansion (15) for $\theta_3$.

The *Dedekind eta function* is defined by $\eta(z) = e^{\pi i z/12} \cdot \prod_{n=1}^{\infty}(1-e^{2\pi i n z})$, for $\Im z > 0$. It is holomorphic in the upper half plane $\Im z > 0$. One can show that the

Dedekind eta function satisfies the functional equation $\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}} \cdot \eta(z)$ for $\Im z > 0$. Clearly $\eta(z+1) = e^{\pi i/12}\eta(z)$.

We will now outline a proof due to Carl Ludwig Siegel (1896–1981) of the pentagonal number theorem using the Dedekind eta function; this proof is given in detail by Chandrasekharan (1985, Chap. VIII). This proof uses the definition of $\theta_3$ by (14) and does not assume the product expansion (15) for $\theta_3$. Let $g(z) = e^{\pi iz/12}\theta_3\left(\frac{1}{2}+\frac{z}{2}, 3z\right)$ for $\Im z > 0$. We can show that $g\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}} \cdot g(z)$. It is clear that $g(z+1) = e^{\pi i/12}g(z)$.

Now define $K(z) = \frac{\eta(z)-g(z)}{\eta(z)}$ for $\Im z > 0$. Then $K(z+1) = K(z)$ and $K(-\frac{1}{z}) = K(z)$ for $\Im z > 0$. Since the full modular group $SL_2(\mathbb{Z})$ is generated by the transformations $z \mapsto z+1$, $z \mapsto -\frac{1}{z}$, it follows that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $K(\frac{az+b}{cz+d}) = K(z)$, for $\Im z > 0$. Also, $\eta$ and $g$ are holomorphic in the half plane $\Im z > 0$, and $\eta$ has no zeros in $\Im z > 0$; hence, $K$ is holomorphic in $\Im z > 0$. This means that $K$ is a modular form of weight 0 for $SL_2(\mathbb{Z})$; therefore, $K(z)$ is constant on $\Im z > 0$, and must be zero because $K(z) \to 0$ as $\Im z \to +\infty$. Therefore, $\eta(z) = e^{\pi iz/12} \cdot \theta_3(\frac{1}{2}+\frac{z}{2}, 3z)$, which yields the pentagonal number theorem.

Chandrasekharan gives historical references for the Jacobi theta functions (1985, Chaps. V and VIII) and the Dedekind eta function (1985, Chap. VIII). Indeed, the pentagonal number theorem is precisely the Fourier expansion of the Dedekind eta function.

The 24th power of $\eta$ is a constant multiple of the *modular discriminant*, $\Delta(z) = (2\pi)^{12}e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}$, $\Im z > 0$. $\Delta$ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$. The Fourier coefficients of modular forms have interesting number theoretic properties. For example, the *Ramanujan tau function* is defined as the Fourier coefficients of $\Delta$. The pentagonal number theorem tells us the Fourier coefficients of $\eta$. Rademacher (1940) talks about the Fourier coefficients of modular forms.

Assuming Euler's pentagonal number theorem, we can prove the functional equation $\eta(-1/z) = \sqrt{-iz} \cdot \eta(z)$ of the eta function using *twisted Poisson summation* (Bump 1998, Chap. 1, Eq. 1.10). Twisted Poisson summation is a way of expressing a series $\sum_{n=-\infty}^{\infty}\chi(n)f(n)$ in terms of $\sum_{n=-\infty}^{\infty}\overline{\chi(n)}\hat{f}(n/N)$; here, $\chi$ is a primitive Dirichlet character modulo $N$ and $\hat{f}$ is the Fourier transform of $f$. Now,

$$\chi(n) = \begin{cases} 1, & n \equiv \pm 1 \pmod{12}, \\ -1, & n \equiv \pm 5 \pmod{12}, \\ 0, & \text{otherwise.} \end{cases}$$

is a primitive Dirichlet character modulo 12. For $q = e^{2\pi iz}$, $\eta(z)$ is equal to

$$q^{\frac{1}{24}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}} = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(6n+1)^2}{24}}$$

$$= \sum_{n=1}^{\infty} \chi(n) q^{\frac{n^2}{24}}.$$

We then apply twisted Poisson summation to this series. This is worked out by Bump (1998, Chap. 1, pp. 29–30).

One can define automorphic forms for $\Gamma = SL_2(\mathbb{Z})$ (or indeed a discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$) for any real weight $k$ using "multiplier systems" (see Iwaniec 1997, Chap. 2). This is to take care of branches of the logarithm. The Dedekind eta function is an automorphic form of weight $k = 1/2$ for $SL_2(\mathbb{Z})$ and a certain multiplier system, which is explained by Iwaniec (1997, §2.8).

We remarked in §1 that Euler thought the most natural way to prove that every nonnegative integer is the sum of four squares is by showing that for

$$\left( \sum_{n=0}^{\infty} x^{n^2} \right)^4 = \sum_{n=0}^{\infty} a_n x^n,$$

$a_n > 0$ for all $n$. Chandrasekharan (1985, Chap. X) gives a proof due to Jacobi using theta functions that

$$\left( \sum_{k=-\infty}^{\infty} q^{k^2} \right)^4 = \sum_{n=0}^{\infty} r(n) q^n,$$

where $r(n) = 8 \sum_{d|n, 4\nmid d} d$ for $n \geq 1$ and $r(0) = 1$, which implies a fortiori that every nonnegative integer is the sum of four squares, since no integer has 4 as its only divisor. Chandrasekharan gives historical references for Jacobi's work on this.

It will be helpful to clarify the relation between formal power series and analytic functions. This will let us understand why showing that the product $\prod_{n=1}^{\infty}(1 - x^n)$ is equal formally to $\sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2}$ suffices to show that they are equal as analytic functions on the unit disk $D$. Let $f_n$ be a sequence of functions analytic on an open set $U$. If $f_n$ converges uniformly to $f$ on every compact subset of $U$, then $f$ is analytic on $U$ (see Titchmarsh 1939, §2.81, iv). Furthermore, the product $\prod(1 + u_n(z))$ converges uniformly in any compact set where the series $\sum |u_n(z)|$ converges uniformly (Titchmarsh 1939, §1.44). Now, let $u_n(z) = -z^n$. The series $\sum_{n=1}^{\infty} |z|^n$ converges uniformly in any compact subset of $|z| < 1$; hence, the product $\prod_{n=1}^{\infty}(1 - z^n)$ converges uniformly in any compact subset of $|z| < 1$. Since, $f_n(z) = (1-z)(1-z^2)\cdots(1-z^n)$ is analytic in $|z| < 1$ for each fixed $n$, $f(z) = \prod_{n=1}^{\infty}(1 - z^n)$ is analytic on $|z| < 1$. Then it follows from Taylor's theorem (Titchmarsh 1939, §2.43) that for some $a_n$, $\prod_{n=1}^{\infty}(1 - z^n) = \sum_{n=1}^{\infty} a_n z^n$ for all $|z| < 1$. Two functions that are analytic in $|z| < R$ are equal in $|z| < R$ if and only if all the coefficients in their power series are equal (Titchmarsh 1939, §2.5).

Let $F$ be the set of all formal power series with complex coefficients. There are two natural topologies for $F$. The first is the product topology on $\mathbb{C}^{\infty}$ where each $\mathbb{C}$ has the discrete topology, and the other is the product topology on $\mathbb{C}^{\infty}$ where each $\mathbb{C}$ has the usual topology, which we will denote by $O_1(F)$ and $O_2(F)$ respectively.

In $O_1(F)$, a sequence $f_n = \sum_{k=0}^{\infty} a_k^{(n)}$ converges to $f = \sum_{k=0}^{\infty} b_k$ if for each fixed $k$, $a_k^{(n)} = b_k$ for all sufficiently large $n$. The idea is that for any given $N$, after some point the terms with superscripts $\leq N$ do not change in the sequence. In $O_2(F)$, a sequence $f_n = \sum_{k=0}^{\infty} a_k^{(n)}$ converges to $f = \sum_{k=0}^{\infty} b_k$ if for each fixed $k$, $a_k^{(n)} \to b_k$ as $n \to \infty$. Also, $O_1(F)$ is finer than $O_2(F)$, i.e., $O_1(F) \subseteq O_2(F)$. So if $f_n \to f$ in $O_1(F)$ then $f_n \to f$ in $O_2(F)$.

On the other hand, the set $H(D)$ of functions that are analytic on the open unit disk can be given the topology of uniform convergence on compact sets, where a sequence $f_n$ converges to $f$ if for every compact subset $K$ of $D$, $f_n$ converges uniformly to $f$ on $K$ (Cartan 1995, Chap. V, §1). Now, the map $H(D) \to H(D)$ that sends $f$ to its $k$th derivative $f^{(k)}$ is continuous. Since $\{0\}$ is a compact subset of $D$, for each fixed $k$ the constant coefficient of $f_n^{(k)}$ approaches the constant coefficient of $f^{(k)}$ as $n \to \infty$. Thus, the map $T_0\colon H(D) \to (F, O_2(F))$ that sends a function analytic in $D$ to its (formal) Taylor series is continuous; this is because $T_0$ preserves the convergence of sequences and $H(D)$ is a metric space (Cartan 1995, Chap. V, Proposition 3.1).

Let $f_n\colon D \to D$ be defined by $z \mapsto \prod_{m=1}^{n}(1 - z^m)$ and $g_n\colon D \to D$ be defined by $z \mapsto \sum_{m=-n}^{n}(-1)^m z^{m(3m-1)/2}$. These are sequences in $H(D)$ that converge in $H(D)$ to some functions $f, g$. Now, the sequence of formal power series $T_0(f_n)$ converges in $O_1(F)$ to $P = \prod_{m=1}^{\infty}(1 - x^m)$, and the sequence of formal power series $T_0(g_n)$ converges in $O_1(F)$ to $Q = \sum_{m=-\infty}^{\infty}(-1)^m x^{m(3m-1)/2}$. Then since $O_1(F)$ is finer than $O_2(F)$, $T_0(f_n)$ converges in $O_2(F)$ to $P$ and $T_0(g_n)$ converges in $O_2(F)$ to $Q$. Now, $P = Q$. But since $T_0\colon H(D) \to (F, O_2(F))$ is continuous, $T_0(f_n) \to T_0(f)$ and $T_0(g_n) \to T_0(g)$ in $O_2(F)$. Indeed $O_2(F)$ is Hausdorff, so it follows that $T_0(f) = P$ and $T_0(g) = Q$; hence, $T_0(f) = T_0(g)$. Therefore, $f = g$ as analytic functions on $D$.

To find recurrence relations and identities for the partition function it suffices to use formal power series. But if we treat the infinite product $\prod_{n=1}^{\infty}(1 - z^n)$ as an analytic function in the unit disk then there are tools from complex analysis that we can use to get more information about the coefficients of

$$F(z) = \frac{1}{\prod_{n=1}^{\infty}(1 - z^n)} = \sum_{n=0}^{\infty} p(n) z^n.$$

Treating the product as an analytic function lets us use information about the function as a whole, instead of just looking at each of the coefficients separately. Using the residue theorem we get

$$p(n) = \frac{1}{2\pi i} \int\limits_{|z|=r} F(z) z^{-n-1} \mathrm{d}z$$

for any $0 < |r| < 1$. In fact one can find an exact formula for $p(n)$ as a certain series. This is worked out by Rademacher (1973, Chap. 14).

Ferraro (2008, Chaps. 17–19) gives some more information on ideas about formal and analytic power series in the eighteenth century.

We saw in §3 that Euler in his 1749 paper "Consideratio quarumdam serierum quae singularibus proprietatibus sunt praeditae" (Euler 1753a), E190, evaluates series related to the infinite product $(1-x)\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a^2}\right)\left(1-\frac{x}{a^3}\right)\left(1-\frac{x}{a^4}\right)$ etc. for $a=10$ and particular values of $x$. In §§28–32 of that paper, Euler is interested in approximating the value of $\sum_{n=1}^{\infty}\frac{1}{a^n-1}$; cf. the last section of Euler's "Methodus generalis summandi progressiones" (Euler 1738, §21), E25, presented to the St. Petersburg Academy on June 20, 1732 (Nevskaja 2000, p. 116). Let $d(n)$ denote the number of positive divisors of $n$, for example $d(1)=1$, $d(6)=4$ and $d(7)=2$. In §32 of E190, Euler remarks that $\sum_{n=1}^{\infty}\frac{1}{a^n-1}=\sum_{n=1}^{\infty}\frac{d(n)}{a^n}$. E190 is discussed in detail by Koelink and Van Assche (2009) and Gautschi (2008).

Let $f(x,m)=1+\sum_{k=1}^{m}(-1)^k(x^{k(3k-1)/2}+x^{k(3k+1)/2})$ and $g(x,m)=\prod_{k=1}^{m}(1-x^k)$. For any $|x|<1$, $f(x,\infty)=g(x,\infty)$. For a given $|x|<1$ and number $r$ of decimal places, we would like to know how many terms $N$ we need to take so that the first $r$ decimal places of $f(x,N)$ or $g(x,N)$ agree with $f(x,\infty)=g(x,\infty)$, in other words so that $|f(x,N)-f(x,\infty)|<10^{-r}$ or $|g(x,N)-g(x,\infty)|<10^{-r}$, respectively.

For instance, $f\left(\frac{1}{2},\infty\right)=g\left(\frac{1}{2},\infty\right)=0.2887880950\ldots$, $f\left(\frac{1}{2},5\right)=0.2887880950\ldots$ and $g\left(\frac{1}{2},5\right)=0.2980041503\ldots$. To get this many digits of accuracy with the partial product, we have to take $m=35$.

Here is another example. $f\left(\frac{1}{5},\infty\right)=g\left(\frac{1}{5},\infty\right)=0.7603327958\ldots$, $f\left(\frac{1}{5},3\right)=0.7603327958\ldots$ and $g\left(\frac{1}{5},3\right)=0.761856$. To get this many digits of accuracy with the partial product we have to take $m=15$.

For an alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ with decreasing terms, $|\sum_{n=N}^{\infty}(-1)^n a_n|\leq a_N$. For real $0\leq x<1$, $f(x,\infty)$ is an alternating series with decreasing terms, which gives us an upper bound for the number of terms of the partial sum we need to take to obtain a given number of accurate digits. Now, let us work out an upper bound for how many terms in the product we need to multiply to get a given number of accurate digits.

$$\begin{aligned}\left|\log\prod_{n=N}^{\infty}(1-x^n)\right| &= \left|\sum_{n=N}^{\infty}\sum_{k=1}^{\infty}\frac{x^{nk}}{k}\right|\\ &\leq \sum_{n=N}^{\infty}\sum_{k=1}^{\infty}|x^{nk}|\\ &= \sum_{k=1}^{\infty}\frac{|x|^{Nk}}{1-|x|^k}.\end{aligned}$$

Now, $\frac{|x|^{Nk}}{1-|x|^k}\leq\frac{1}{1-|x|}|x|^{Nk}$. Hence

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{|x|^{Nk}}{1-|x|^k} &\leq \frac{1}{1-|x|}\sum_{k=1}^{\infty}|x|^{Nk}\\ &= \frac{|x|^N}{1-|x|}\cdot\frac{1}{1-|x|^N}.\end{aligned}$$

Then if $0 \le x < 1$,

$$\prod_{n=N}^{\infty}(1-x^n) \ge \exp\left(-\frac{x^N}{1-x}\cdot\frac{1}{1-x^N}\right).$$

Thus

$$\prod_{n=1}^{N-1}(1-x^n) - \prod_{n=1}^{\infty}(1-x^n) = \prod_{n=1}^{N-1}(1-x^n)\left(1-\prod_{n=N}^{\infty}(1-x^n)\right) \le 1 - \exp\left(-\frac{x^N}{1-x}\cdot\frac{1}{1-x^N}\right).$$

Probably Euler would have been interested in evaluating the product $f(x) = \prod_{n=1}^{\infty}(1-x^n)$ for particular $x$, $|x| < 1$, but it is not simple to find any $x$ aside from $x = 0$ which give a nice result. Thus, Euler mostly used $(1-x)(1-x^2)(1-x^3)$ etc. as a formal power series.

We can get a neat closed expression for $\eta(i)$ using some elliptic function theory. The results about elliptic and modular functions that we use are given by Markushevich (1967, Chap. 5, §§21–24) and Bump (1998, §1.3).

Let $\Lambda$ be the lattice generated by $a$ and $ai$ and $\wp(z) = \wp(z;\Lambda)$ the Weierstrass $\wp$ function for the lattice. Here, $a$ is a positive real number that we will choose appropriately to get a closed expression for $\eta(i)$.

The Weierstrass function $\wp(z)$ satisfies the following differential equation

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3,$$

where $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ are defined by

$$g_2 = 60\sum(ma+nai)^{-4}, \quad g_3 = 140\sum(ma+nai)^{-6},$$

with summation over all $(m,n) \ne (0,0)$.

It is helpful to define the functions

$$G_{2k}(z) = \sum(m+nz)^{-2k}$$

with summation over all $(m,n) \ne (0,0)$; these are called Eisenstein series, and are modular forms for $SL_2(\mathbb{Z})$ of weight $2k$. Then $g_2(\Lambda) = \frac{60}{a^4}G_4(i)$ and $g_3(\Lambda) = \frac{140}{a^6}G_6(i)$.

Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, which sends $z$ to $-1/z$. Since, $G_6$ is a modular form of weight 6,

$$G_6(Si) = i^6 G_6(i).$$

But $Si = i$, so $G_6(i) = 0$.

The *modular discriminant* is defined by $\Delta = (60G_4)^3 - 27(140G_6)^2$. One can show that $\eta^{24} = (2\pi)^{-12}\Delta$. Since, $G_6(i) = 0$, $\Delta(i) = (60G_4(i))^3$. Therefore to find $\eta(i)$ it suffices to find $G_4(i)$.

It follows from the Fourier series expansion of $G_4(z)$ that $G_4(i)$ is a positive real number. Therefore we can choose some (positive real) $a$ to make $g_2 = 4$. Then since $G_4(i) = \frac{a^4}{60}g_2$, to find $G_4(i)$ it suffices to get an expression for $a$. To do this, we will evaluate a certain integral in two different ways. In the first evaluation we will do the substitution $x = \wp(z)$.

Since $g_2 = 4$, $\wp(z)$ satisfies the differential equation

$$\wp'^2(z) = 4(\wp^3(z) - \wp(z)).$$

One can show that $\wp'(z) = 0$ for $z = a/2, (a + ai)/2, ai/2$. Then $\wp(a/2), \wp((a + ai)/2), \wp(ai/2)$ are the three roots of $0 = 4(x^3 - x)$. We know that the three roots of this equation are $0, 1, -1$.

Now, one can show that $\wp$ is one-to-one in the closed rectangle with vertices $0, a/2, (a + ai)/2, ai/2$. Also, $\wp$ is real on the four lines $x = 0, x = a/2, y = 0, y = a/2$. Since $\wp(z) = z^{-2} + o(1)$ at $z = 0$, it follows that as $z$ moves along the rectangle from 0 to $a/2$ to $(a + ai)/2$ to $ai/2$ to 0, $\wp(z)$ goes from $+\infty$ to $-\infty$. Therefore, $\wp(a/2) = 1, \wp((a + ai)/2) = 0, \wp(ai/2) = -1$.

Let $\sqrt{\cdot}$ be the branch of the square root such that $\sqrt{r} > 0$ for positive real $r$.

$$\begin{aligned} I &= \int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^3 - x}} \\ &= \int_0^{a/2} \frac{\wp'(z)\mathrm{d}z}{\sqrt{\wp^3(z) - \wp(z)}} \\ &= 2\int_0^{a/2} \mathrm{d}z \\ &= a. \end{aligned}$$

But on the other hand,

$$\begin{aligned} I &= \int_0^1 \frac{\mathrm{d}y}{\sqrt{y - y^3}} \\ &= \frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{t^{3/4}\sqrt{1-t}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} B(1/4, 1/2) \\
&= \frac{1}{2} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} \\
&= \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}},
\end{aligned}
$$

where $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ is the beta function, $\Re x, \Re y > 0$. The beta function can be expressed using the gamma function as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Therefore, $a = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}}$, and so $G_4(i) = \frac{a^4}{60} g_2 = \frac{\Gamma(1/4)^8}{60 \cdot 16\pi^2}$. But $(\eta(i))^{24} = (2\pi)^{-12} \Delta(i)$ and $\Delta(i) = (60G_4(i))^3$, and putting this together gives $\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}} = 0.7682254223\ldots$. We know by its definition that $\eta(i)$ is a positive real number and we thus took the positive real 24th root of $(2\pi)^{-12}\Delta(i)$.

The Chowla–Selberg formula gives similar evaluations of $\eta(a)$ when $\mathbb{Z}[a]$ is the ring of integers of an imaginary quadratic field with class number one (see Weil 1976, Chap. IX, Eq. 7).

The sum of reciprocals of the triangular numbers is 2, and the sum of reciprocals of the squares is $\frac{\pi^2}{6}$. It is natural to ask what the sum of the reciprocals of the pentagonal numbers is.

Let $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ and let $C$ be Euler's constant,

$$C = \lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Taking the logarithmic derivative of the Weierstrass product for $\Gamma(z)$, namely the logarithmic derivative of $\Gamma(z) = \frac{1}{ze^{Cz}} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$, we obtain

$$\psi(z+1) = -C + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}, \tag{16}$$

Then letting $p_r(n)$ be the $n$th $r$-gonal number, $p_r(n) = \frac{1}{2} n((r-2)n - (r-4))$, and taking $z = -\frac{r-4}{r-2}$ for $r \geq 5$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{p_r(n)} = \frac{-2}{r-4} \left( \psi\left(\frac{2}{r-2}\right) + C \right).$$

It turns out that $\psi\left(\frac{2}{3}\right) = -C + \frac{\pi\sqrt{3}}{6} - \frac{3}{2}\log(3)$. Thus the sum of the reciprocals of the pentagonal numbers (namely $r = 5$) is $3\log(3) - \frac{\pi}{\sqrt{3}}$, which is $= 1.482037501\ldots$.

In particular (16) implies the result of Pietro Mengoli (1625–1686) in his 1650 *Novae quadraturae arithmeticae* that

$$\sum_{k=1}^{\infty} \frac{n}{k(n+k)} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

for positive integral $n$. This is because for positive integral $n$, $\psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - C$.

In §§17-18 of his 1775 paper "Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)$ etc. in seriem simplicem" (Euler 1783b), E541, Euler remarks that because of the pentagonal number theorem, the series $s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} -$ etc. will have the factors $1-x$, $1-x^2$, $1-x^3$, $1-x^4$, $1-x^5$, etc., and thus that every root of unity will be a zero with infinite multiplicity.

The last paper in which Euler discusses the pentagonal number theorem and its applications is his "De mirabilibus proprietatibus numerorum pentagonalium", presented to the St. Petersburg Academy on September 4, 1775 (Nevskaja 2000, p. 628) and published in the *Acta* in 1783 (Euler 1783a), E542. In §2, Euler notes that every pentagonal number is one-third of a triangular number (those numbers in the form $\frac{n(n+1)}{2}$), and in §4 he gives his recurrence relation for the divisor function. Then in §7, Euler states the pentagonal number theorem, remarking that "Haec igitur non minus admirationem nostram meretur quam proprietas ante commemorata, cum nulla certe appareat ratio, unde ullus nexus intelligi possit inter evolutionem illius producti et nostros numeros pentagonales" ["This therefore deserves our admiration no less than the property mentioned above, since there is certainly no obvious reason why there should be any connection that could be understood between the expansion of this product and our pentagonal numbers"].

In the rest of this article, Euler considers the summation of series of the pentagonal numbers. We would say that he sums this series by the $(E, \omega_n)$ and $(A, \omega_n)$ methods of summation; these are defined by Hardy (1949). As

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n+1)}{2}},$$

where every root of unity is a root of the left-hand side, Euler uses Newton's identities to sum series that involve roots of unity and the pentagonal numbers. Writing

$$-s = 1 - 5 + 12 - 22 + 35 - 51 + 70 - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n(3n-1)}{2},$$

$$-t = 2 - 7 + 15 - 26 + 40 - 57 + 77 - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n(3n+1)}{2},$$

Euler obtains

$$-s = \frac{1}{2} - \frac{4}{4} + \frac{3}{8} = -\frac{1}{8}$$
$$-t = \frac{2}{2} - \frac{5}{4} + \frac{3}{8} = \frac{1}{8},$$

and thus

$$-1 - 2 + 5 + 7 - 12 - 15 + 22 + 26 - 35 - 40 + 51 + 57 - 70 - 77 + \cdots$$
$$= \sum_{n=1}^{\infty} (-1)^n (\omega_n + \omega_{-n}) = 0,$$

where $\omega_n = \frac{n(3n-1)}{2}$. This is explained completely by Rademacher (1969).

For $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$, Fabry's gap theorem (Segal 2008, Chap. 6, Theorem 4.3) tells us that if the exponents $\lambda_n$ satisfy

$$\lim_{n \to \infty} \frac{n}{\lambda_n} = 0$$

then there is no analytic continuation of $f$ to a domain that properly contains the unit disc. But $\frac{n}{\omega_n} = \frac{2}{3n-1} \to 0$ and $\frac{n}{\omega_{-n}} = \frac{2}{3n+1} \to 0$ as $n \to \infty$. Thus, $f(z) = 1 + \sum_{n=1}^{\infty} (-1)^n \left( z^{\frac{n(3n-1)}{2}} + z^{\frac{n(3n+1)}{2}} \right)$ cannot be analytically continued outside of the unit disc.

Analytic results on $\sigma(n)$ are stated by Dickson (1919a, Chap. X). Kempner (1923) gives the history of the analytic study of $p(n)$.

## 7 Conclusion

The pentagonal number theorem is important for three reasons:

(1) developing the theory of infinite products,
(2) Euler's theory of partitions, and generating functions,
(3) elliptic and modular functions.

It can be very useful to represent a product or series in different ways. One reason is that it can tell us things about the product or series. For example, when we study the Riemann zeta function, or $L$-functions in general, it can be useful to express the function as an Euler product over the primes or as a product over its zeros. Another reason is that we might be interested in the terms in the power series expansion, and thus if we have two power series expansions we can compare the terms.

In §3, we worked out an unpublished proof of the pentagonal number theorem from Euler's notebooks. It turns out to be much more complicated than Euler's published proof, and this is probably the reason why it wasn't published.

Euler used incomplete induction as an instrument of scientific research. Juškevič (Gillispie 1980, Volume IV, "Leonhard Euler", pp. 467–484) writes the following: "It is frequently said that Euler saw no intrinsic impossibility in the deduction of mathematical laws from a very limited basis in observation; and naturally he employed methods of induction to make empirical use of the results he had arrived at through analysis of concrete numerical material. But he himself warned many times that an incomplete induction serves only as a heuristic device, and he never passed off as finally proved truths the suppositions arrived at by such methods" (also cf. Weil 1984, Chap. II, §III and Cajori 1918).

Euler explains his ideas about truth in letters CXV to CXX of his *Lettres à une princesse d'Allemagne sur divers sujets de physique & de philosophie*, E344, published in 1768 (Euler 1768). He says that all the truths which are within our grasp can be divided into three classes: "toutes les vérités, qui sont à la portée de notre connoissance, se rapportent à trois classes essentiellement distinguées" ["all the truths which are within the reach of our understanding can be divided into three essentially distinguished classes"] (letter CXV, dated March 31, 1761). The first class are truths of experience, and they are established by evidence of the senses. The second class are truths of rational thought, established by correct reasoning. The third are truths of belief, established by credible statements.

Although the three classes of truths are indeed separate, the status of some truth can change with time or be different for different people. As an example, in letter CXV Euler says that the fact that Russians and Austrians (namely soldiers) had been in Berlin is a truth of the third class for the princess, since she did not see them but only was told of them, while for Euler it is a truth of the first class since he saw them and talked with them.

In this letter, Euler also writes that "Dans l'exemple de l'aimant nous ne savons pas, comment l'attraction du fer est un effet nécessaire de la nature, tant de l'aimant, que du fer; mais nous ne sommes pas moins convaincus de la verité du fait" ["With the example of the magnet we do not know how attraction of iron is a necessary effect of the nature of either the magnet or of iron; but we are nonetheless convinced of the truth of the fact"]; earlier in the letter Euler had given the fact that a magnet attracts iron as an example of a truth of the first class. Perhaps if Euler could deduce the behavior of magnets from a more general theory he would have considered this a truth of the second class.

In letter CXX (dated April 18, 1761), Euler talks about how we can learn something about gravity by doing tests dropping stones, even though we have not dropped every possible stone from every possible position at every possible time. But Euler says that, "Cet exemple suffit pour faire voir à V[otre] A[ltesse] comment les experiences, quoiqu'elles ne roulent que sur des objets individuels, ont conduit les hommes à des connoissances très universelles: mais il faut convenir que l'entendement & les autres facultés de l'ame s'y mêlent d'une maniere, qu'il est très difficile de bien développer: & si l'on vouloit être trop scrupuleux sur toutes les circonstances, on n'avanceroit rien dans toutes nos connoissances, & l'on seroit arreté à chaque pas" ["This example suffices to let Your Highness see how experiments, although they are conducted only on individual objects, have led men to very universal knowledge. However, we must also recognize that understanding and the other faculties of the soul are involved in some way that is very difficult to elaborate on; and if we wish to be overly fastidious in all situations, we won't advance at all in our understanding, and we will halt at every step"]. So we do not know a truth in physics just by doing a lot of tests, but rather we also need to use our intellect to know that these tests suffice.

Now, we have seen that when Euler was expanding the product $(1-x)(1-x^2)(1-x^3)$ etc. to check the pentagonal number theorem, he started to get an idea of how to prove the theorem. Working out examples like these gives us insight into why a theorem is true, instead of just showing that the theorem is true for lots of cases.

One reason why Euler would want to talk about a statement being true which he had not been able to demonstrate yet is to focus his work on fruitful areas. If a statement is probably false, then assuming it might only lead to vacuous statements. But certainly Euler still wanted to find a proof of the pentagonal number theorem, even though he was convinced of its truth already by the examples he had checked. Euler mentions that having a proof of the pentagonal number theorem would give us deeper understanding of it and could lead us to new insights. In other words, having a proof would let us know why the theorem is true instead of just knowing it is true.

Euler had a good idea of what a proof was, and would only say he had proved something if he thought he had a really rigorous proof. But he might still say that we can be absolutely certain about something that we cannot yet prove. But to know that something is certain by experience we need more than just a lot of cases, e.g., to know that a proposition $P(n)$ is true for all $n$ we need to know more than just the truth of $P(1), \ldots, P(10000)$, we need some information about $P(n)$ that our insight/experience/understanding can work on.

Subscript notation would make complicated inductive arguments easier. In particular it would make it clearer when it is an actual induction and not just specific cases. Sandifer (2007a) talks about this.

**Acknowledgments** The author was supported by NSERC. The author benefited from many exchanges with Martin Mattmüller. The author would like to thank Gleb K. Mikhajlov for making copies of Euler's notebooks available, and Martin Mattmüller for checking photographs of the original letters between Euler and Goldbach. The author is very grateful to Polina Vinogradova and Stacy Langton for reading several Russian sources. Finally, the author would like to thank John Grant, Rüdiger Thiele and Robin Chapman for answering many questions.

## References

Andrews, G.E. 1983. Euler's pentagonal number theorem. *Mathematics Magazine* 56(5): 279–284.

Andrews, G.E. 1998. *The theory of partitions*. Cambridge University Press.

Bump, D. 1998. *Automorphic forms and representations*, vol. 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press.

Cajori, F. 1918. Origin of the name "mathematical induction". *American Mathematical Monthly* 25(5): 197–201.

Cajori, F. 1929. *A History of Mathematical Notations*, vol. II. Chicago: The Open Court Publishing Company.

Calinger, R. 1996. Leonhard Euler: the first St. Petersburg years (1727–1741). *Historia Mathematica* 23(2): 121–166.

Cantor, M. 1901. *Vorlesungen über Geschichte der Mathematik. Dritter Band. Vom Jahre 1668 biz zum Jahre 1758*. Leipzig: B. G. Teubner, Zweite Auflage. Reprinted by Johnson Reprint Corporation, New York, 1965.

Cartan, H. 1995. *Elementary theory of analytic functions of one or several complex variables*. Mineola, NY: Dover Publications. Translated from the French.

Chandrasekharan, K. 1985. *Elliptic functions*, Die Grundlehren der mathematischen Wissenschaften, vol. 281. Berlin: Springer-Verlag

Chrystal, G. 1964. *Algebra: an elementary text-book, Part II*. New York: Chelsea Publishing Company, seventh edition. Reprinted by AMS Chelsea Publishing, 1999.

Cramer, G., editor. 1744. *Jacobi Bernoulli, Basileensis, Opera. Tomus Secundus*. Sumptibus Haeredum Cramer & Fratrum Philibert=Héritiers Cramer et frères Philibert, Genevae. With the editorial assistance of Nicolaus I Bernoulli.

Dickson, L.E. 1919a. *History of the theory of numbers*, vol. I. Washington, D.C.: Carnegie Institute. Publication Number 256.

Dickson, L.E. 1919b. *History of the theory of numbers*, vol. II. Washington, D.C.: Carnegie Institute. Publication Number 256.

Eneström, G. 1906. Der Briefwechsel zwischen Leonhard Euler und Daniel Bernoulli. *Bibl. Math., 3. Folge* 7: 126–156.

Eneström, G. 1908. Jakob Bernoulli und die Jacobischen Thetafunktionen. *Bibl. Math., 3. Folge* 9: 206–210.

Eneström, G. 1910/1913. *Verzeichnis der Schriften Leonhard Eulers*. B. G. Teubner, Leipzig. Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband IV, Lief. 1, 2.

Euler, L. 1738. Methodus generalis summandi progressiones. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 6: 68–97. E25, Opera omnia I.14, pp. 42–72.

Euler, L. 1740. De summis serierum reciprocarum. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 7: 123–134. E41, Opera omnia I.14, pp. 73–86.

Euler, L. 1744. Variae observationes circa series infinitas. *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, 9: 160–188. E72, Opera omnia I.14, pp. 217–244.

Euler, L. 1748. *Introductio in analysin infinitorum. Tomus primus*. Apud Marcum-Michaelem Bousquet & Socios=Marc-Michel Bousquet & Compagnie, Lausannae. E101, Opera omnia I.8.

Euler, L. 1751a. Découverte d'une loi tout extraordinaire des nombres par rapport à la somme de leurs diviseurs. *Bibliothèque Impartiale*, 3(1): 10–31. E175, Opera omnia I.2, pp. 241–253.

Euler, L. 1751b. Observationes analyticae variae de combinationibus. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 13: 64–93. E158, Opera omnia I.2, pp. 163–193.

Euler, L. 1753a. Consideratio quarumdam serierum, quae singularibus proprietatibus sunt praeditae. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 3: 86–108. E190, Opera omnia I.14, pp. 516–541.

Euler, L. 1753b. De partitione numerorum. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 3: 125–169. E191, Opera omnia I.2, pp. 254–294.

Euler, L. 1760a. Demonstratio theorematis circa ordinem in summis divisorum observatum. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5: 75–83. E244, Opera omnia I.2, pp. 390–398.

Euler, L. 1760b. Observatio de summis divisorum. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5: 59–74. E243, Opera omnia I.2, pp. 373–389.

Euler, L. 1768. *Lettres à une princesse d'Allemagne sur divers sujets de physique & philosophie, Tome second*. Academie Impériale des Sciences, Saint Petersbourg. E344, Opera omnia III.12.

Euler, L. 1770. *Vollständige Anleitung zur Algebra. Erster Theil*. Kays. Acad. der Wissenschaften, St. Petersburg. E387, Opera omnia I.1.

Euler, L. 1783a. De mirabilibus proprietatibus numerorum pentagonalium. *Acta Academiae Scientiarum Imperialis Petropolitanae* 4: 56–75. E542, Opera omnia I.3, pp. 480–496.

Euler, L. 1783b. Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc. in seriem simplicem. *Acta Academiae Scientiarum Imperialis Petropolitanae* 4: 47–55. E541, Opera omnia I.3, pp. 472–479.

Euler, L. 1784. De plurimis quantitatibus transcendentibus, quas nullo modo per formulas integrales exprimere licet. *Acta Academiae Scientiarum Imperialis Petropolitanae* 4: 31–37. E565, Opera omnia I.15, pp. 522–527.

Euler, L. 1785. Considerationes super theoremate Fermatiano de resolutione numerorum in numeros polygonales. *Opuscula Analytica* 2: 3–15. E586, Opera omnia I.4, pp. 125–135.

Fellmann, E.A. 1995. *Leonhard Euler*. Rowohlt Taschenbuch, Reinbek bei Hamburg. Rowohlts Monographien, Band 387.

Fellmann, E.A., and Mikhajlov, G.K., editors. 1998. *Leonhardi Euleri Opera omnia. Series quarta A: Commercium epistolicum. Volumen secundum: Commercium cum Johanne (I) Bernoulli et Nicolao (I) Bernoulli*. Basel: Birkhäuser.

Ferraro, G. 2008. *The rise and development of the theory of series up to the early 1820s*. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag.

Fuss, P.-H., editor. 1843. *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle. Tomes I, II*. L'Académie Impériale des Sciences de Saint-Pétersbourg, St.-Pétersbourg. The Sources of Science, No. 35, Johnson Reprint Corporation, New York and London, 1968.

Gautschi, W. 2008. On Euler's attempt to compute logarithms by interpolation: A commentary to his letter of February 16, 1734 to Daniel Bernoulli. *Journal of Computational and Applied Mathematics* 219(2): 408–415.

Gillispie, C.C., editor. 1970–1980. *Dictionary of Scientific Biography*. New York: Charles Scribner's Sons.

Hardy, G.H. 1949. *Divergent series*. Oxford University Press.

Hardy, G.H., and Ramanujan, S. 1918. Asymptotic formulae in combinatory analysis. *Proceedings of the London Mathematical Society, Series 2* 17: 75–115. Table IV calculated by Percy A. MacMahon.

Hardy, G.H., and Wright, E.M. 1980. *An Introduction to the theory of numbers*. Oxford University Press, fifth edition.

Hofmann, J.E. 1959. Um Eulers erste Reihenstudien. In: Schröder, K. (ed.), *Sammelband der zu Ehren des 250. Geburtstages Leonhard Eulers der Deutschen Akademie der Wissenschaften zu Berlin vorgelegten Abhandlungen*, pp. 139–208. Berlin: Akademie-Verlag.

Iwaniec, H. 1997. *Topics in classical automorphic forms*, vol. 17 of *Graduate Studies in Mathematics*. American Mathematical Society.

Juškevič, A.P., Smirnov, V.I., and Habicht, W., editors. 1975. *Leonhardi Euleri Opera omnia. Series quarta A: Commercium epistolicum. Volumen primum: Descriptio commercii epistolici*. Basel: Birkhäuser.

Juškevič, A.P., and Taton, R., editors. 1980. *Leonhardi Euleri Opera omnia. Series quarta A: Commercium epistolicum. Volumen quintum: Correspondance de Leonhard Euler avec A. C. Clairaut, J. d'Alembert et J. L. Lagrange*. Basel: Birkhäuser.

Juškevič, A.P., and Winter, E. (eds.). 1961. *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers. Teil 2: Der Briefwechsel L. Eulers mit Nartov, Razumovskij, Schumacher, Teplov und der Petersburger Akademie 1730–1763*. Quellen und Studien zur Geschichte Osteuropas. Berlin: Akademie-Verlag.

Juškevič, A.P., and Winter, E. (eds.) 1965. *Leonhard Euler und Christian Goldbach. Briefwechsel 1729–1764*. Berlin: Akademie-Verlag. Abhandlungen der Deutschen Akademie der Wissenschaften zu Berlin. Jahrgang 1965 Nr. 1.

Juškevič, A.P., and Winter, E., editors. 1976. *Die Berliner und die Petersburger Akademie der Wissenschaften im Briefwechsel Leonhard Eulers. Teil 3: Wissenschaftliche und wissenschaftsorganisatorische Korrespondenzen 1726–1774*. Quellen und Studien zur Geschichte Osteuropas. Berlin: Akademie-Verlag.

Kempner, A.J. 1923. The development of "partitio numerorum," with particular reference to the work of Messrs. Hardy, Littlewood and Ramanujan. *American Mathematical Monthly* 30(7): 354–369.

Kiselev, A.A., and Matvievskaja, G.P. 1965. Neopublikovannye zapisi Eilera po partitio numerorum. *Istoriko-Matematicheskie Issledovaniya* 16: 145–180.

Knobloch, E. 1974. The mathematical studies of G. W. Leibniz on combinatorics. *Historia Mathematica* 1(4): 409–430.

Knobloch, W., editor. 1984. *Leonhard Eulers Wirken an der Berliner Akademie der Wissenschaften 1741–1766*. Berlin: Akademie-Verlag. Studien zur Geschichte der Akademie der Wissenschaften der DDR. Band 11.

Koelink, E., and Van Assche, W. 2009. Leonhard Euler and a $q$-analogue of the logarithm. *Proceedings of the American Mathematical Society* 137(5): 1663–1676.

Legendre, A.-M. 1830. *Théorie des nombres. Tome II*. Firmin-Didot, Paris, third edition.

Markushevich, A.I. 1967. *Theory of functions of a complex variable*, vol. III. Prentice-Hall. Translated from the Russian. Revised English edition translated and edited by Richard A. Silverman.

Mikhajlov, G.K. 1959. Notizen über die unveröffentlichten Manuskripte von Leonhard Euler. In Schröder, K. (ed.), *Sammelband der zu Ehren des 250. Geburtstages Leonhard Eulers der Deutschen Akademie der Wissenschaften zu Berlin vorgelegten Abhandlungen*, pages 256–280. Berlin: Akademie-Verlag.

Nevskaja, N.I., editor. 2000. *Letopis' Rossiiskoi Akademii Nauk*. Nauka, St. Petersburg.

Patterson, S.J. 2007. Gauss sums. In Goldstein, C., Schappacher, N., and Schwermer, J., editors. *The shaping of arithmetic after C. F. Gauss's Disquisitiones arithmeticae*, pages 505–528. Berlin: Springer-Verlag.

Pólya, G. 1954. *Mathematics and plausible reasoning. Induction and analogy in mathematics*, vol. I. Princeton University Press.

Rademacher, H. 1940. Fourier expansions of modular forms and problems of partition. *Bulletin of the American Mathematical Society* 46:59–73.

Rademacher, H. (1969). Comments on Euler's "De mirabilibus proprietatibus numerorum pentagonalium". In *Number theory and analysis (papers in Honor of Edmund Landau)*, pages 257–268. New York: Plenum.

Rademacher, H. 1973. *Topics in analytic number theory*. Springer-Verlag, New York. Edited by Grosswald, E., Lehner, J., and Newman, M. Die Grundlehren der mathematischen Wissenschaften, Band 169.

Rudio, F., editor. 1915. *Leonhardi Euleri Opera omnia, series I, volumen II: Leonhardi Euleri Commentationes arithmeticae, volumen primum*. B. G. Teubner, Leipzig and Berlin.

Sandifer, C.E. 2007a. Some facets of Euler's work on series. In Bradley, R.E., and Sandifer, C.E., editors. *Leonhard Euler: Life, Work and Legacy*, pages 279–302. Elsevier. Studies in the History and Philosophy of Mathematics, vol. 5.

Sandifer, C.E. 2007b. *The early mathematics of Leonhard Euler. The MAA Tercentenary Euler Celebration*, vol. 1. Spectrum Series. The Mathematical Association of America.

Scharlau, W. 1983. Eulers Beiträge zur *partitio numerorum* und zur Theorie der erzeugenden Funktionen. In: Burckhardt, J.J., Fellmann, E.A., and Habicht, W., editors. *Leonhard Euler 1707–1783. Beiträge zu Leben und Werk. Gedenkband des Kantons Basel-Stadt*, pages 135–150. Basel: Birkhäuser.

Scriba, C.J. 1984. Eulers zahlentheoretische Studien im Lichte seines wissenschaftlichen Briefwechsels. In: Knobloch, E., Louhivaara, I.S., and Winkler, J., editors. *Zum Werk Leonhard Eulers. Vorträge des Euler-Kolloquiums im Mai 1983 in Berlin*, pages 67–94. Basel: Birkhäuser.

Segal, S.L. 2008. *Nine introductions in complex analysis*, vol. 208 of *North-Holland Mathematics Studies*. Elsevier, revised edition.

Smirnov, V.I., editor. 1963. *Pis'ma k uchenym*. Nauka, Moscow-Leningrad.

Stäckel, P., and Ahrens, W. 1908. *Der Briefwechsel zwischen C. G. J. Jacobi und P. H. von Fuss über die Herausgabe der Werke Leonhard Eulers*. B. G. Teubner, Leipzig.

Struik, D.J., editor. 1969. *A source book in mathematics, 1200-1800*. Source Books in the History of the Sciences. Harvard University Press.

Tannery, P., and Henry, C., editors. 1891. *Oeuvres de Fermat. Tome I*. Gauthier-Villars et fils, Paris.

Titchmarsh, E.C. 1939. *The theory of functions*. Oxford University Press, second edition.

van der Waerden, B.L., editor. 1975. *Die Werke von Jakob Bernoulli. Band 3. Wahrscheinlichkeitsrechnung*. Birkhäuser, Basel. Herausgegeben von der Naturforschenden Gesellschaft in Basel.

Weil, A. 1976. *Elliptic functions according to Eisenstein and Kronecker*, vol. 88 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag.

Weil, A. 1984. *Number theory: an approach through history from Hammurapi to Legendre*. Birkhäuser.

Winter, E., editor. 1957. *Die Registres der Berliner Akademie der Wissenschaften 1746–1766. Dokumente für das Wirken Leonhard Eulers in Berlin*. Berlin: Akademie-Verlag.