Pontryagin Duality and the Fourier Transform

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Contents

1 Introduction

We follow Rudin [3] and Terras [4], and refer to `sp4comm` [2] and Folland [1].

2 $\mathbb{Z}/n\mathbb{Z}$

Let n be a positive integer.

$$\mathbb{Z}/n\mathbb{Z} = \{k + n\mathbb{Z} : k \in \mathbb{Z}\} = \{k + n\mathbb{Z} : 0 \leq k \leq n - 1\}$$

$Z_n = \mathbb{Z}/n\mathbb{Z}$ is a ring. $|Z_n| = n$. We focus on the additive group $(Z_n, +)$.

3 Haar measure

Let

$$\mathbb{C}^{Z_n}$$

be the set of functions $Z_n \to \mathbb{C}$.

We use counting measure on the group Z_n as Haar measure (which is discrete and compact) with total volume n, and Since Z_n has finite Haar measure (being a compact group) and every function $Z_n \to \mathbb{C}$ is continuous (being a discrete group), we have

$$\mathbb{C}^{Z_n} = L^1(Z_n) = L^2(Z_n).$$

We specify function space for emphasis.

For $f, g \in L^2(Z_n)$, define

$$(f, g)_{L^2(Z_n)} = \sum_{x \in Z_n} f(x)\overline{g(x)}$$

Define

$$|f|_{L^2(Z_n)} = \sqrt{(f, f)_{L^2(Z_n)}} = \sqrt{\sum_{x \in Z_n} f(x)\overline{f(x)}}.$$

Define

$$|f|_{L^1(Z_n)} = \sum_{x \in Z_n} |f(x)|.$$

4 δ_a

For $a \in Z_n$, define $\delta_a : Z_n \to \mathbb{C}$ by

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

For $a, b \in Z_n$,

$$\begin{aligned} (\delta_a, \delta_b)_{L^2(Z_n)} &= \sum_{x \in Z_n} \delta_a(x)\overline{\delta_b(x)} \\ &= \sum_{x \in Z_n} \delta_a(x)\delta_b(x) \\ &= \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \end{aligned}$$

For $f \in L^2(Z_n)$,

$$f(x) = \sum_{a \in Z_n} f(a)\delta_a(x) = \sum_{a=0}^{n-1} f(a)\delta_a(x), \qquad x \in Z_n.$$

Indeed, $\{\delta_0, \ldots, \delta_{n-1}\}$ is an orthonormal basis of $L^2(Z_n)$.

5 Convolution

For $f, g \in L^1(Z_n)$, define the **c**onvolution $f * g \in L^1(Z_n)$ by

$$(f * g)(x) = \sum_{y \in Z_n} f(y)g(x - y), \qquad x \in Z_n.$$

$$f * g = g * f, \qquad f, g \in L^1(Z_n).$$

$$f * (g * h) = (f * g) * h, \qquad f, g, h \in L^1(Z_n).$$

For $f \in L^1(Z_n)$ and for $a \in Z_n$,

$$(f * \delta_a)(x) = f(x - a), \qquad x \in Z_n.$$

For $a, b \in Z_n$, and for $x \in Z_n$,

$$(\delta_a * \delta_b)(x) = \delta_a(x - b) = \delta_{a+b}(x).$$

5.1 Example

Take $n = 15$. Define $f = \delta_0 + \delta_1 + \delta_2$. For $x \in Z_{15}$,

$$\begin{aligned}(f * f)(x) &= (\delta_0 + \delta_1 + \delta_2) * (\delta_0 + \delta_1 + \delta_2) \\ &= \delta_0 * \delta_0 + \delta_1 * \delta_1 + \delta_2 * \delta_2 \\ &+ 2\delta_0 * \delta_1 + 2\delta_0 * \delta_2 + 2\delta_1 * \delta_2 \\ &= \delta_0 + \delta_2 + \delta_4 \\ &+ 2\delta_1 + 2\delta_2 + 2\delta_3 \\ &= \delta_0 + 2\delta_1 + 3\delta_2 + 2\delta_3 + \delta_4\end{aligned}$$

6 Dual group

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, which is a multiplicative group.

Let $\widehat{Z_n}$ be the set of group homomorphisms $Z_n \to S^1$.

We use normalized counting measure on the group $\widehat{Z_n}$ as Haar measure (which is discrete and compact) with total volume 1.

For $a \in Z_n$, define $e_a : Z_n \to S^1$ by

$$e_a(x) = \exp\left(2\pi i \frac{ax}{n}\right), \qquad x \in Z_n.$$

e_a is an element of $\widehat{Z_n}$.

Define $\psi : Z_n \to \widehat{Z_n}$ by $\psi(a) = e_a$.

$$\begin{aligned}\widehat{Z_n} &= \{e_a : a \in Z_n\} \\ &= \{\psi(a) : a \in Z_n\} \\ &= \psi(Z_n)\end{aligned}$$

$\psi : Z_n \to \widehat{Z_n}$ is an isomorphism of groups.

7 Fourier transform

Define the **F**ourier transform $\mathscr{F}_n : L^2(Z_n) \to L^2(\widehat{Z_n})$ by

$$(\mathscr{F}_n f)(e_a) = \sum_{x \in Z_n} f(x) e_a(-x), \qquad e_a \in \widehat{Z_n}.$$

8 Pullback

We introduce the operator $F_n : L^2(Z_n) \to L^2(Z_n)$, which is defined via composition with the Fourier transform $\mathscr{F}_n$ and the function ψ as

$$F_n f = (\mathscr{F}_n f) \circ \psi.$$

We pullback $\mathscr{F}_n f : \widehat{Z_n} \to \mathbb{C}$ to a function $Z_n \to \mathbb{C}$.

We remind ourselves (a) that for $a \in Z_n$, the function $e_a : Z_n \to S^1$ is defined by

$$e_a(x) = \exp\left(2\pi i \frac{ax}{n}\right), \qquad x \in Z_n,$$

(b) that $\psi : Z_n \to \widehat{Z_n}$ is defined by $\psi(a) = e_a$,

and (c) that $\psi : Z_n \to \widehat{Z_n}$ is an isomorphism of groups, by

$$\begin{aligned} \widehat{Z_n} &= \{e_a : a \in Z_n\} \\ &= \{\psi(a) : a \in Z_n\} \\ &= \psi(Z_n) \end{aligned}$$

$$\begin{aligned} (\mathscr{F}_n f)(\psi(a)) &= \sum_{x \in Z_n} f(x) e_a(-x) \\ &= \sum_{x \in Z_n} f(x) \overline{e_a(x)} \\ &= (f, e_a)_{L^2(Z_n)} \end{aligned}$$

Thus

$$(F_n f)(x) = (f, e_x)_{L^2(Z_n)}, \qquad x \in Z_n.$$

In the sequel, we use the term Fourier transform to refer both to $\mathscr{F}_n$ and to F_n, but preserve the distinction for calculations.

8.1 Example: $n = 7$ and $f = \delta_3$

By

$$(F_n f)(x) = (f, e_x)_{L^2(Z_n)}, \qquad x \in Z_n$$

we have

$$(F_7 \delta_3)(x) = (\delta_3, e_x)_{L^2(Z_7)}, \qquad x \in Z_7.$$

The inner product $(\delta_3, e_x)_{L^2(Z_7)}$ is given by

$$\begin{aligned}(\delta_3, e_x)_{L^2(Z_7)} &= \sum_{y \in Z_7} \delta_3(y)\overline{e_x(y)} \\ &= \sum_{y=0}^{6} \delta_3(y)\overline{\exp\left(2\pi i \frac{xy}{7}\right)}.\end{aligned}$$

Since $\delta_3(y) = 1$ only when $y = 3$ and 0 otherwise, the sum collapses to a single term:

$$\begin{aligned}(\delta_3, e_x)_{L^2(Z_7)} &= \overline{\exp\left(2\pi i \frac{3x}{7}\right)} \\ &= \exp\left(-2\pi i \frac{3x}{7}\right) \\ &= e_{-3}(x)\end{aligned}$$

Thus,

$$(F_7\delta_3)(x) = e_{-3}(x), \qquad x \in Z_7,$$

namely,

$$F_7\delta_3 = e_{-3}.$$

9 $\widehat{Z_n}$

$\widehat{Z_n}$ is the set of group homomorphisms $Z_n \to S^1$.

$\widehat{Z_n}$ is a group using pointwise multiplication of functions $Z_n \to S^1$, the **P**ontryagin dual group of Z_n.

For $a \in Z_n$, define $e_a \in \widehat{Z_n}$ by

$$e_a(x) = \exp\left(2\pi i \frac{ax}{n}\right), \qquad x \in Z_n,$$

Define $\psi : Z_n \to \widehat{Z_n}$ by

$$\psi(a) = e_a, \qquad a \in Z_n.$$

We have

$$\begin{aligned}\widehat{Z_n} &= \{e_a : a \in Z_n\} \\ &= \{\psi(a) : a \in Z_n\} \\ &= \psi(Z_n)\end{aligned}$$

Thus, $\psi : Z_n \to \widehat{Z_n}$ is an isomorphism of groups.

10 Haar measure

Let G be a locally compact abelian group.

$\widehat{G}$ is the set of continuous group homomorphisms $G \to S^1$. It is a group with operation $(\phi_1\phi_2)(x) = \phi_1(x)\phi_2(x)$, $\phi_1, \phi_2 \in \widehat{G}$, $x \in G$ (namely, pointwise multiplication).

We assign $\widehat{G}$ the coarsest topology such that for each $x \in G$, the map $\phi \mapsto \phi(x)$ is continuous $\widehat{G} \to S^1$ (namely, the final topology on $\widehat{G}$).

One proves that $\widehat{G}$ is a locally compact abelian group.

If G is a discrete LCA group, then $\widehat{G}$ is a compact LCA group.

10.1 Finite LCA groups

Let G be a finite locally compact abelian group. G must have the discrete topology. Hence the Borel σ-algebra of G is equal to the power set of G, denoted $\mathscr{P}(G)$.

Because G has the discrete topology, $\widehat{G}$ is equal to the set of group homomorphisms $G \to S^1$.

Assign G the Haar measure m_G defined by $m_G(A) = |A|$ for $A \in \mathscr{P}(G)$. One checks that m_G indeed is a Haar measure. (Counting measure.)

Assign $\widehat{G}$ the Haar measure $m_{\widehat{G}}$ defined by $m_{\widehat{G}}(A) = \frac{1}{|\widehat{G}|} \cdot |A|$ for $A \in \mathscr{P}(\widehat{G})$. (Normalized counting measure.)

$L^2(G)$ is equal to the set of functions $G \to \mathbb{C}$ and $L^2(\widehat{G})$ is equal to the set of functions $\widehat{G} \to \mathbb{C}$.

11 $L^2(Z_n)$

For $f, g \in L^2(Z_n)$, define

$$(f, g)_{L^2(Z_n)} = \sum_{x \in Z_n} f(x)\overline{g(x)}$$

Define

$$|f|_{L^2(Z_n)} = \sqrt{(f, f)_{L^2(Z_n)}} = \sqrt{\sum_{x \in Z_n} f(x)\overline{f(x)}}.$$

For $a \in Z_n$, define $\delta_a : Z_n \to \mathbb{C}$ by

$$\delta_a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

For $a, b \in Z_n$,

$$
\begin{aligned}
(\delta_a, \delta_b)_{L^2(Z_n)} &= \sum_{x \in Z_n} \delta_a(x)\overline{\delta_b(x)} \\
&= \sum_{x \in Z_n} \delta_a(x)\delta_b(x) \\
&= \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}
\end{aligned}
$$

For $f \in L^2(Z_n)$,

$$
f(x) = \sum_{a \in Z_n} f(a)\delta_a(x) = \sum_{a=0}^{n-1} f(a)\delta_a(x), \qquad x \in Z_n.
$$

12 Fourier transform

Define the **Fourier transform** $\mathscr{F}_n : L^2(Z_n) \to L^2(\widehat{Z_n})$ by

$$
(\mathscr{F}_n f)(e_a) = \sum_{x \in Z_n} f(x)\overline{e_a(x)} = \sum_{x \in Z_n} f(x)e_a(-x), \qquad e_a \in \widehat{Z_n}.
$$

13 Pullback

We introduce the operator $F_n : L^2(Z_n) \to L^2(Z_n)$, which is defined via composition with the Fourier transform $\mathscr{F}_n$ and the function ψ as

$$
F_n f = (\mathscr{F}_n f) \circ \psi.
$$

That is, for $a \in Z_n$,

$$
(F_n f)(a) = (\mathscr{F}_n f)(e_a).
$$

We pullback $\mathscr{F}_n f : \widehat{Z_n} \to \mathbb{C}$ to a function $F_n f : Z_n \to \mathbb{C}$.
We have

$$
\begin{aligned}
(\mathscr{F}_n f)(\psi(a)) &= \sum_{x \in Z_n} f(x)e_a(-x) \\
&= \sum_{x \in Z_n} f(x)\overline{e_a(x)} \\
&= (f, e_a)_{L^2(Z_n)}
\end{aligned}
$$

Thus

$$
(F_n f)(x) = (f, e_x)_{L^2(Z_n)}, \qquad x \in Z_n.
$$

14 Inverse Fourier transform

Define the Haar measure $m_{\widehat{Z_n}}$ on $\widehat{Z_n}$ by $m_{\widehat{Z_n}}(A) = \frac{1}{n} \cdot |A|$ for $A \in \mathscr{P}(\widehat{Z_n})$.

Let $f \in L^2(Z_n)$ and let $x \in Z_n$.

$$\begin{aligned}
\int_{\widehat{Z_n}} (\mathscr{F}_n f)(\gamma)\gamma(x) dm_{\widehat{Z_n}}(\gamma) &= \frac{1}{n} \sum_{\gamma \in \widehat{Z_n}} (\mathscr{F}_n f)(\gamma)\gamma(x) \\
&= \frac{1}{n} \sum_{a \in Z_n} (\mathscr{F}_n f)(e_a) e_a(x) \\
&= \frac{1}{n} \sum_{a \in Z_n} \left(\sum_{y \in Z_n} f(y)\overline{e_a(y)} \right) e_a(x) \\
&= \frac{1}{n} \sum_{a \in Z_n} \sum_{y \in Z_n} f(y)\overline{e_a(y)} e_a(x) \\
&= \frac{1}{n} \sum_{y \in Z_n} f(y) \left(\sum_{a \in Z_n} e_a(x)\overline{e_a(y)} \right)
\end{aligned}$$

We use the orthogonality relations for characters of finite abelian groups. For $a, b \in Z_n$ we have $e_a, e_b \in \widehat{Z_n}$, and

$$\sum_{x \in Z_n} e_a(x)\overline{e_b(x)} = n\delta_{a,b}.$$

Then, as $e_a(x) = e_x(a)$ and $e_a(y) = e_y(a)$,

$$\begin{aligned}
\frac{1}{n} \sum_{y \in Z_n} f(y) \left(\sum_{a \in Z_n} e_a(x)\overline{e_a(y)} \right) &= \frac{1}{n} \sum_{y \in Z_n} f(y) \left(\sum_{a \in Z_n} e_x(a)\overline{e_y(a)} \right) \\
&= \frac{1}{n} \sum_{y \in Z_n} f(y) \cdot n\delta_{x,y} \\
&= \sum_{y \in Z_n} f(y)\delta_{x,y} \\
&= \sum_{y \in Z_n} f(y)\delta_x(y) \\
&= f(x).
\end{aligned}$$

We have established that for $f \in L^2(Z_n)$ and for $x \in Z_n$,

$$\int_{\widehat{Z_n}} (\mathscr{F}_n f)(\gamma)\gamma(x) dm_{\widehat{Z_n}}(\gamma) = \frac{1}{n} \sum_{a \in Z_n} (\mathscr{F}_n f)(e_a) e_a(x) = f(x).$$

Also,

$$\frac{1}{n}\sum_{a\in Z_n}(F_n f)(a)e_a(x) = \frac{1}{n}\sum_{a\in Z_n}(\mathscr{F}_n f)(e_a)e_a(x) = f(x).$$

We have established the **Fourier inversion formula** for $f \in L^2(Z_n)$:

$$f(x) = \frac{1}{n}\sum_{a\in Z_n}(F_n f)(a)e_a(x), \qquad x \in Z_n.$$

15 $\ell^1(\mathbb{Z})$

Let $\mathbb{C}^{\mathbb{Z}}$ be the set of functions $\mathbb{Z} \to \mathbb{C}$.

Let $x \in \ell^1(\mathbb{Z})$ be the set of those $x \in \mathbb{C}^{\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} |x[n]| < \infty$$

and define

$$|x|_{\ell^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |x[n]|.$$

Let $\ell^2(\mathbb{Z})$ be the set of those $x \in \mathbb{C}^{\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} |x[n]|^2 < \infty.$$

16 $\ell^2(\mathbb{Z})$

For $x \in \ell^2(\mathbb{Z})$, define

$$|x|_{\ell^2(\mathbb{Z})} = \sqrt{\sum_{n\in\mathbb{Z}} |x[n]|^2},$$

and for $x, y \in \ell^2(\mathbb{Z})$, define

$$(x, y)_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} x[n]\overline{y[n]}.$$

For $k \in \mathbb{Z}$, define $T_k : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by

$$(T_k x)[n] = x[n-k], \qquad n \in \mathbb{Z}.$$

References

[1] Gerald B. Folland. *A course in abstract harmonic analysis.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2nd updated edition edition, 2015.

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[3] Walter Rudin. *Fourier analysis on groups.* Number 12 in Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers, New York, 1962.

[4] Audrey Terras. *Fourier analysis on finite groups and applications*, volume 43 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1999.