The Wiener-Pitt tauberian theorem

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1 Introduction

For $f \in L^1(\mathbb{R}^d)$, we write

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx, \qquad \xi \in \mathbb{R}^d.$$

The Riemann-Lebesgue lemma tells us that $\hat{f} \in C_0(\mathbb{R}^d)$.

For $f \in C^\infty(\mathbb{R}^d)$ and for multi-indices α, β, write

$$|f|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha (\partial^\beta f)(x)|.$$

We say that f is a **Schwartz function** if for all multi-indices α and β we have $|f|_{\alpha,\beta} < \infty$. We denote by $\mathscr{S}$ the collection of Schwartz functions. It is a fact that $\mathscr{S}$ with this family of seminorms is a Fréchet space.

Let $V_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$, the volume of the unit ball in $\mathbb{R}^d$.

Lemma 1. *For $1 \leq p \leq \infty$, let m be the least integer $\geq \frac{d+1}{p}$. There is some C_d such that for any multi-index β,*

$$\left\|\partial^\beta f\right\|_p \leq V_d^{1/p} |f|_{0,\beta} + C_d V_d^{1/p} \sum_{|\alpha|=m} |f|_{\alpha,\beta}, \qquad f \in \mathscr{S}.$$

Proof. For $p = \infty$, the claim is true with $C_{d,\infty} = 1$. For $1 \leq p < \infty$, let $g = \partial^\beta f$, which satisfies

$$\begin{aligned}
\|g\|_p &= \left(\int_{|x|\le 1} |g(x)|^p dx + \int_{|x|\ge 1} |x|^{d+1} |g(x)|^p |x|^{-(d+1)} dx \right)^{1/p} \\
&\le \left(\|g\|_\infty^p V_d + \sup_{|x|\ge 1} \left(|x|^{d+1} |g(x)|^p \right) \int_{|x|\ge 1} |x|^{-(d+1)} dx \right)^{1/p} \\
&= \left(\|g\|_\infty^p V_d + \sup_{|x|\ge 1} \left(|x|^{d+1} |g(x)|^p \right) \int_1^\infty \left(\int_{S^{d-1}} |r\gamma|^{-(d+1)} d\sigma(\gamma) \right) r^{d-1} dr \right)^{1/p} \\
&= \left(\|g\|_\infty^p V_d + \sup_{|x|\ge 1} \left(|x|^{d+1} |g(x)|^p \right) \cdot V_d \int_1^\infty r^{-2} dr \right)^{1/p} \\
&= V_d^{1/p} \left(\|g\|_\infty^p + \sup_{|x|\ge 1} \left(|x|^{d+1} |g(x)|^p \right) \right)^{1/p} \\
&\le V_d^{1/p} \|g\|_\infty + V_d^{1/p} \sup_{|x|\ge 1} \left(|x|^{\frac{d+1}{p}} |g(x)| \right) \\
&\le V_d^{1/p} \|g\|_\infty + V_d^{1/p} \sup_{|x|\ge 1} \left(|x|^m |g(x)| \right).
\end{aligned}$$

Using that the function $y \mapsto \sum_{|\alpha|=m} |y^\alpha|$ is continuous $S^{d-1} \to \mathbb{R}$, there is some C_d such that

$$|x|^m \le C_d \sum_{|\alpha|=m} |x^\alpha|, \qquad x \in \mathbb{R}^d.$$

This gives us

$$\begin{aligned}
\|g\|_p &\le V_d^{1/p} \|g\|_\infty + V_d^{1/p} \sup_{|x|\ge 1} C_d \sum_{|\alpha|=m} |x^\alpha| |g(x)| \\
&= V_d^{1/p} \left\| \partial^\beta f \right\|_\infty + C_d V_d^{1/p} \sum_{|\alpha|=m} \sup_{|x|\ge 1} |x^\alpha (\partial^\beta f)(x)| \\
&\le V_d^{1/p} |f|_{0,\beta} + C_d V_d^{1/p} \sum_{|\alpha|=m} |f|_{\alpha,\beta}.
\end{aligned}$$

□

The dual space $\mathscr{S}'$ with the weak-* topology is a locally convex space, elements of which are called **tempered distributions**. It is straightforward to check that if $u : \mathscr{S} \to \mathbb{C}$ is linear, then $u \in \mathscr{S}'$ if and only if there is some C and some nonnegative integers m, n such that

$$|u(f)| \le C \sum_{|\alpha|\le m, |\beta|\le n} |f|_{\alpha,\beta}, \qquad f \in \mathscr{S}.$$

For $1 \leq p \leq \infty$ and $g \in L^p(\mathbb{R}^d)$, define $u : \mathscr{S} \to \mathbb{C}$ by

$$u(f) = \int_{\mathbb{R}^d} f(x)g(x)dx, \qquad f \in \mathscr{S}.$$

For $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality tells us

$$|u(f)| \leq \|fg\|_1 \leq \|g\|_p \|f\|_q.$$

By Lemma 1, with m the least integer $\geq \frac{d+1}{q}$,

$$\|f\|_q \leq V_d^{1/q}|f|_{0,0} + C_d V_d^{1/q} \sum_{|\alpha|=m} |f_{\alpha,0}|.$$

Therefore,

$$|u(f)| \leq C_{g,d,q} \sum_{|\alpha|\leq m, |\beta|\leq 0} |f|_{\alpha,\beta},$$

showing that u is continuous. We thus speak of elements of $L^p(\mathbb{R}^d)$ as tempered distributions, and speak about the Fourier transform of an element of $L^p(\mathbb{R}^d)$.

Let $u \in \mathscr{D}'$ be a distribution. For an open set ω, we say that u **vanishes on** ω if $u(\phi) = 0$ for every $\phi \in \mathscr{D}(\omega)$. Let Γ be the collection of open sets ω on which u vanishes, and let $\Omega = \bigcup_{\omega \in \Gamma} \omega$. Γ is an open cover of Ω, and thus there is a locally finite partition of unity ψ_j subordinate to Γ.[1] For $\phi \in \mathscr{D}(\Omega)$, because $\operatorname{supp} \phi$ is compact, there is some open set W, $\operatorname{supp} \phi \subset W \subset \Omega$, and some m such that

$$\psi_1(x) + \cdots + \psi_m(x) = 1, \qquad x \in W.$$

Then

$$u(\phi) = u(\phi(\psi_1 + \cdots + \psi_m)) = u(\psi_1 \phi) + \cdots + u(\psi_m \phi).$$

For each j, $1 \leq j \leq m$, there is some $\omega_j \in \Gamma$ such that $\operatorname{supp} \psi_j \subset \omega_j$, which implies $\operatorname{supp} \psi_j \phi \subset \omega_j$, i.e. $\psi_j \phi \in \mathscr{D}(\omega_j)$. But $\omega_j \in \Gamma$, so $u(\psi_j \phi) = 0$ and hence $u(\phi) = 0$. This shows that $\Omega \in \Gamma$, namely, Ω is the largest open set on which u vanishes. The **support of** u is

$$\operatorname{supp} u = \mathbb{R}^d \setminus \Omega.$$

For $u \in \mathscr{S}'$ we define $\hat{u} : \mathscr{S} \to \mathbb{C}$ by

$$\hat{u}(\phi) = u(\hat{\phi}), \qquad \phi \in \mathscr{S}.$$

It is a fact that $\hat{u} \in \mathscr{S}'$.

For $f : \mathbb{R}^d \to \mathbb{C}$, write $\check{f}(x) = f(-x)$. For $\phi \in \mathscr{S}$,

$$\mathscr{F}(\mathscr{F}(\phi)) = \check{\phi}.$$

[1] Walter Rudin, *Functional Analysis*, second ed., p. 162, Theorem 6.20.

2 Tauberian theory

Lemma 2. *If $f \in L^1(\mathbb{R}^d)$, $\zeta \in \mathbb{R}^d$, and $\epsilon > 0$, then there is some $h \in L^1(\mathbb{R}^d)$ with $\|h\|_1 < \epsilon$ and some $r > 0$ such that*

$$\hat{h}(\xi) = \hat{f}(\zeta) - \hat{f}(\xi), \qquad \xi \in B_r(\zeta).$$

Proof. It is a fact that there is a Schwartz function g such that $\hat{g}(\xi) = 1$ for $|\xi| < 1$. For $\lambda > 0$, let

$$g_\lambda(x) = e^{2\pi i \zeta \cdot x} \lambda^{-d} g(\lambda^{-1} x), \qquad x \in \mathbb{R}^d,$$

which satisfies, for $\xi \in \mathbb{R}^d$,

$$\begin{aligned} \widehat{g_\lambda}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} e^{2\pi i \zeta \cdot x} \lambda^{-d} g(\lambda^{-1} x) dx \\ &= \int_{\mathbb{R}^d} e^{-2\pi i \lambda \xi \cdot y} e^{2\pi i \lambda \zeta \cdot y} g(y) dy \\ &= \hat{g}(\lambda \xi - \lambda \zeta). \end{aligned}$$

In particular, for $\xi \in V_\lambda = B_{\lambda^{-1}}(\zeta)$ we have $\widehat{g_\lambda}(\xi) = 1$. We also define

$$h_\lambda(x) = \hat{f}(\zeta) g_\lambda(x) - (f * g_\lambda)(x), \qquad x \in \mathbb{R}^d,$$

which satisfies, for $\xi \in \mathbb{R}^d$,

$$\widehat{h_\lambda}(\xi) = \hat{f}(\zeta)\widehat{g_\lambda}(\xi) - \widehat{f * g_\lambda}(\xi) = \hat{f}(\zeta)\widehat{g_\lambda}(\xi) - \hat{f}(\xi)\widehat{g_\lambda}(\xi) = \widehat{g_\lambda}(\xi)(\hat{f}(\zeta) - \hat{f}(\xi)).$$

Hence, for $\xi \in V_\lambda$ we have $\widehat{h_\lambda}(\xi) = \hat{f}(\zeta) - \hat{f}(\xi)$.

For $x \in \mathbb{R}^d$,

$$\begin{aligned} h_\lambda(x) &= \int_{\mathbb{R}^d} f(y) e^{-2\pi i \zeta \cdot y} g_\lambda(x) - \int_{\mathbb{R}^d} f(y) g_\lambda(x - y) dy \\ &= \int_{\mathbb{R}^d} f(y) \left(e^{-2\pi i \zeta \cdot y} g_\lambda(x) - g_\lambda(x - y) \right) dy, \end{aligned}$$

for which

$$\begin{aligned} &\left| e^{-2\pi i \zeta \cdot y} g_\lambda(x) - g_\lambda(x - y) \right| \\ =& \left| e^{-2\pi i \zeta \cdot y} e^{2\pi i \zeta \cdot x} \lambda^{-d} g(\lambda^{-1} x) - e^{2\pi i \zeta \cdot (x - y)} \lambda^{-d} g(\lambda^{-1}(x - y)) \right| \\ =& \lambda^{-d} |g(\lambda^{-1} x) - g(\lambda^{-1}(x - y))|. \end{aligned}$$

Then

$$\begin{aligned} \|h_\lambda\|_1 &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)| \lambda^{-d} |g(\lambda^{-1} x) - g(\lambda^{-1}(x - y))| dy \right) dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(y)| |g(u) - g(\lambda^{-1}(\lambda u - y))| dy \right) du \\ &= \int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |g(u) - g(u - \lambda^{-1} y))| du \right) dy. \end{aligned}$$

For each $y \in \mathbb{R}^d$,

$$|f(y)| \left(\int_{\mathbb{R}^d} |g(u) - g(u - \lambda^{-1} y))| du \right) \leq 2 \|g\|_1 |f(y),$$

and hence by the dominated convergence theorem,

$$\int_{\mathbb{R}^d} |f(y)| \left(\int_{\mathbb{R}^d} |g(u) - g(u - \lambda^{-1} y))| du \right) dy \to 0, \qquad \lambda \to \infty.$$

Thus, there is some λ_ϵ such that $\|h_\lambda\|_1 < \epsilon$ when $\lambda \geq \lambda_\epsilon$. For $h = h_{\lambda_\epsilon}$ and $r = \lambda_\epsilon^{-1}$, we have $\hat{h}(\xi) = \hat{f}(\zeta) - \hat{f}(\xi)$ for $\xi \in V_{\lambda_\epsilon} = B_r(\zeta)$ and $\|h\|_1 < \epsilon$, proving the claim. $\square$

We remind ourselves that for $\phi \in L^\infty(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$, the convolution $f * \phi$ belongs to $C_u(\mathbb{R}^d)$, the collection of bounded uniformly continuous functions $\mathbb{R}^d \to \mathbb{C}$. We also remind ourselves that any element of $L^\infty(\mathbb{R}^d)$ is a tempered distribution whose Fourier transform is a tempered distribution.[2]

Theorem 3. *If $\phi \in L^\infty(\mathbb{R}^d)$, Y is a linear subspace of $L^1(\mathbb{R}^d)$, and*

$$f * \phi = 0, \qquad f \in Y,$$

then

$$Z(Y) = \bigcap_{f \in Y} \{\xi \in \mathbb{R}^d : \hat{f}(\xi) = 0\}$$

contains $\operatorname{supp} \hat{\phi}$.

Proof. If $Y = \{0\}$, then $Z(Y) = \mathbb{R}^d$, and the claim is true. If Y has nonzero dimension, let $\zeta \in \mathbb{R}^d \setminus Z(Y)$ and let $f \in Y$ such that $\hat{f}(\zeta) = 1$; that there is such a function f follows from Y being a linear space. Thus by Lemma 2 there is some $h \in L^1(\mathbb{R}^d)$ with $\|h\|_1 < 1$ and some $r > 0$ such that

$$\hat{h}(\xi) = 1 - \hat{f}(\xi), \qquad \xi \in B_r(\zeta);$$

because $Z(Y)$ is closed, we may take r such that $B_r(\zeta) \subset \mathbb{R}^d \setminus Z(Y)$.

Let $\rho \in \mathscr{D}(B_r(\zeta))$, and let $\psi \in \mathscr{S}$ with $\hat{\psi} = \rho$. Define $g_0 = \psi$ and $g_m = h * g_{m-1}$ for $m \geq 1$. By Young's inequality

$$\|g_m\|_1 \leq \|h\|_1^m \|\psi\|_1 ,$$

and because $\|h\|_1 < 1$, this means that the sequence $\sum_{m=0}^M g_m$ is Cauchy in $L^1(\mathbb{R}^d)$ so converges to some G, for which, as $|\hat{h}| \leq \|h\|_1 < 1$,

$$\hat{G} = \sum_{m=0}^\infty \widehat{g_m} = \sum_{m=0}^\infty \hat{\psi} \cdot \hat{h}^m = \hat{\psi} \cdot (1 - \hat{h})^{-1}.$$

[2] Walter Rudin, *Functional Analysis*, second ed., p. 228, Theorem 9.3.

For $\xi \in \operatorname{supp} \hat{\psi} \subset B_r(\zeta)$ we have $\hat{h}(\xi) = 1 - \hat{f}(\xi)$ and so

$$\hat{\psi}(\xi) = \hat{G}(\xi)(1 - \hat{h}(\xi)) = \hat{G}(\xi)\hat{f}(\xi);$$

on the other hand, for $\xi \not\in \operatorname{supp} \hat{\psi}$, $\hat{\psi}(\xi) = 0 = \hat{G}(\xi)\hat{f}(\xi)$, so

$$\hat{\psi} = \hat{G} \cdot \hat{f},$$

which implies that $\psi = G * f$. Then

$$\psi * \phi = G * f * \phi = G * 0 = 0,$$

therefore

$$\hat{\phi}(\rho) = \phi(\hat{\rho}) = \phi(\mathscr{F}^2(\psi)) = \phi(\check{\psi}) = \int_{\mathbb{R}^d} \psi(-x)\phi(x)dx = (\psi * \phi)(0) = 0.$$

This is true for all $\rho \in \mathscr{D}(B_r(\zeta))$, which means that $\hat{\phi}$ vanishes on $B_r(\zeta)$. This is true for any $\zeta \in \mathbb{R}^d \setminus Z(Y)$, so with Ω the union of those open sets on which $\hat{\phi}$ vanishes, $\mathbb{R}^d \setminus Z(Y) \subset \Omega$. Then $Z(Y) \subset \mathbb{R}^d \setminus \Omega = \operatorname{supp} \hat{\phi}$. $\square$

If X is a Banach space and M is a linear subspace of X, we define the **annihilator of** M as

$$M^\perp = \{\gamma \in X^* : \text{if } x \in M \text{ then } \langle x, \gamma \rangle = 0\}.$$

It is immediate that $M^\perp$ is a weak-* closed linear subspace of X^*. If N is a linear subspace of X^*, we define the **annihilator of** N as

$${}^\perp N = \{x \in X : \text{if } \gamma \in N \text{ then } \langle x, \gamma \rangle = 0\}.$$

It is immediate that ${}^\perp N$ is a norm closed linear subspace of the Banach space X. One proves using the Hahn-Banach theorem that ${}^\perp(M^\perp)$ is the norm closure of M in X.[3]

We say that a subspace Y of $L^1(\mathbb{R}^d)$ is **translation-invariant** if $f \in Y$ and $x \in \mathbb{R}^d$ imply that $f_x \in Y$, where $f_x(y) = f(y - x)$. The following theorem gives conditions under which a closed translation-invariant subspace of $L^1(\mathbb{R}^d)$ is equal to the entire space.[4]

Theorem 4. *If Y is a closed translation-invariant subspace of $L^1(\mathbb{R}^d)$ and $Z(Y) = \emptyset$, then $Y = L^1(\mathbb{R}^d)$.*

Proof. Suppose that $\phi \in L^\infty(\mathbb{R}^d)$ and $\int f\check{\phi} = 0$ for each $f \in Y$. Let $f \in Y$ and $x \in \mathbb{R}^d$. As Y is translation-invariant, $f_{-x} \in Y$ so $\int_{\mathbb{R}^d} f(y+x)\phi(-y)dy = 0$, i.e. $(f * \phi)(x) = 0$. This is true for all $x \in \mathbb{R}^d$, which means that $f * \phi = 0$. Theorem 3 then tells us that $\operatorname{supp} \hat{\phi}$ is contained in $Z(Y)$, namely, $\operatorname{supp} \hat{\phi}$ is empty, which means that the tempered distribution $\hat{\phi}$ vanishes on $\mathbb{R}^d$, i.e. $\operatorname{supp} \hat{\phi}$ is the zero

[3] Walter Rudin, *Functional Analysis*, second ed., p. 96, Theorem 4.7.

[4] Walter Rudin, *Functional Analysis*, second ed., p. 228, Theorem 9.4.

element of the locally convex space $\mathscr{S}'$. As the Fourier transform $\mathscr{S}' \to \mathscr{S}'$ is linear and one-to-one, the tempered distribution ϕ is the zero element of $\mathscr{S}'$, which implies that $\phi \in L^\infty(\mathbb{R}^d)$ is zero. As Lebesgue measure on $\mathbb{R}^d$ is σ-finite, for X the Banach space $L^1(\mathbb{R}^d)$ we have $X^* = L^\infty(\mathbb{R}^d)$, with $\langle f, \gamma \rangle = \int f\gamma$. Thus $Y^\perp$ is the zero subspace of $L^\infty(\mathbb{R}^d)$, hence ${}^\perp(Y^\perp) = L^1(\mathbb{R}^d)$. This implies that $L^1(\mathbb{R}^d)$ is equal to the closure of Y in $L^1(\mathbb{R}^d)$, and because Y is closed this means $Y = L^1(\mathbb{R}^d)$, completing the proof. $\square$

Theorem 5. *Suppose that $K \in L^1(\mathbb{R}^d)$ and that Y is the smallest closed translation-invariant subspace of $L^1(\mathbb{R}^d)$ that includes K. $Y = L^1(\mathbb{R}^d)$ if and only if*

$$\hat{K}(\xi) \neq 0, \qquad \xi \in \mathbb{R}^d.$$

Proof. Suppose that $\hat{K}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$. As $K \in Y$, this implies that $Z(Y) = \emptyset$. Thus by Theorem 4 we get $Y = L^1(\mathbb{R}^d)$.

Suppose that $Y = L^1(\mathbb{R}^d)$. Then $f(x) = e^{-\pi|x|^2}$ belongs to Y and $\hat{f}(\xi) = e^{-\pi|\xi|^2}$, which has no zeros, hence $Z(Y) = \emptyset$. For $\xi \in \mathbb{R}^d$, define $\mathrm{ev}_\xi : C_0(\mathbb{R}^d) \to \mathbb{C}$ by $\mathrm{ev}_\xi(g) = g(\xi)$, which is a bounded linear operator. The Fourier transform $\mathscr{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ is a bounded linear operator, hence for each $\xi \in \mathbb{R}^d$, $\mathrm{ev}_\xi \circ \mathscr{F} : L^1(\mathbb{R}^d) \to \mathbb{C}$ is a bounded linear operator. Hence

$$V_\xi = \{f \in L^1(\mathbb{R}^d) : \hat{f}(\xi) = 0\} = \ker(\mathrm{ev}_\xi \circ \mathscr{F})$$

is a closed subspace of $L^1(\mathbb{R}^d)$. If $f \in V$ and $x \in \mathbb{R}^d$, then

$$\widehat{f_x}(\xi) = \int_{\mathbb{R}^d} f(y - x) e^{-2\pi i \xi \cdot y} dy = e^{-2\pi i \xi \cdot x} \hat{f}(\xi) = 0,$$

showing that V_ξ is translation-invariant. Therefore

$$V = \bigcap_{\hat{K}(\xi)=0} V_\xi$$

is a closed translation-invariant subspace of $L^1(\mathbb{R}^d)$, and because Y is the smallest closed translation-invariant subspace of $L^1(\mathbb{R}^d)$, $Y \subset V$. $Y \subset V$ implies $Z(V) \subset Z(Y) = \emptyset$, and applying Theorem 4 we get that $V = L^1(\mathbb{R}^d)$. But there is no ξ for which $V_\xi = L^1(\mathbb{R}^d)$, so $V = L^1(\mathbb{R}^d)$ implies that $\{\xi \in \mathbb{R}^d : \hat{K}(\xi) = 0\} = \emptyset$. $\square$

3 Slowly oscillating functions

Let $B(\mathbb{R}^d)$ be the collection of bounded functions $\mathbb{R}^d \to \mathbb{C}$, which with the supremum norm $\|f\|_u = \sup_{x \in \mathbb{R}^d} |f(x)|$ is a Banach algebra.

A function $\phi \in B(\mathbb{R}^d)$ is said to be **slowly oscillating** if for each $\epsilon > 0$ there is some A and some $\delta > 0$ such that if $|x|, |y| > A$ and $|x - y| < \delta$, then

$|\phi(x)-\phi(y)|<\epsilon$. We now prove the **Wiener-Pitt tauberian theorem**; the statement supposing that a function is slowly oscillating is attributed to Pitt.[5]

Theorem 6 (Wiener-Pitt tauberian theorem)**.** *If $\phi \in B(\mathbb{R}^d)$, $K \in L^1(\mathbb{R}^d)$, $\hat{K}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$, and*

$$\lim_{|x|\to\infty} (K * \phi)(x) = a\hat{K}(0),$$

then for each $f \in L^1(\mathbb{R}^d)$,

$$\lim_{|x|\to\infty} (f * \phi)(x) = a\hat{f}(0). \tag{1}$$

Furthermore, if such ϕ is slowly oscillating then

$$\lim_{|x|\to\infty} \phi(x) = a. \tag{2}$$

Proof. Define $\psi(x) = \phi(x) - a$. Let Y be the set of those $f \in L^1(\mathbb{R}^d)$ for which

$$\lim_{|x|\to\infty} (f * \psi)(x) = 0.$$

It is immediate that Y is a linear subspace of $L^1(\mathbb{R}^d)$. Suppose that $f_i \in Y$ tends to some $f \in L^1(\mathbb{R}^d)$. As $\psi \in B(\mathbb{R}^d)$, $f * \psi$ and $f_i * \psi$ belong to $C_u(\mathbb{R}^d)$. Then

$$\|f * \psi - f_i * \psi\|_u = \|(f - f_i) * \psi\|_u = \|\psi\|_u \|f - f_i\|_1 .$$

There is some i_0 such that $i \geq i_0$ implies $\|f - f_i\|_1 < \epsilon$, and because $f_{i_0} \in Y$ there is some M such that $|x| \geq M$ implies $|(f_{i_0} * \psi)(x)| < \epsilon$. Then for $|x| \geq M$,

$$\begin{aligned} |(f * \psi)(x)| &\leq |(f * \psi)(x) - (f_{i_0} * \psi)(x)| + |(f_{i_0} * \psi)(x)| \\ &\leq \|\psi\|_u \|f - f_i\|_1 + |(f_{i_0} * \psi)(x)| \\ &< \epsilon \cdot (\|\psi\|_u + 1), \end{aligned}$$

showing that $f \in Y$, namely, that Y is closed. Let $f \in Y$ and $x \in \mathbb{R}^d$. $f_x \in L^1(\mathbb{R}^d)$, and for $y \in \mathbb{R}^d$,

$$((\tau_x f) * \psi)(y) = (f * \psi)(y - x),$$

and as $|y| \to \infty$ we have $|y - x| \to \infty$ and thus $(f * \psi)(y - x) \to 0$, hence $\tau_x f \in Y$, i.e. Y is translation-invariant. Therefore Y is a closed translation-invariant subspace of $L^1(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$,

$$(K * \psi)(x) = \int_{\mathbb{R}^d} K(y)(\phi(x - y) - a)dy = (K * \phi)(x) - a\hat{K}(0),$$

[5] Walter Rudin, *Functional Analysis*, second ed., p. 229, Theorem 9.7; Walter Rudin, *Fourier Analysis on Groups*, p. 163, Theorem 7.2.7; Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 116, Theorem 4.72; V. P. Havin and N. K. Nikolski, *Commutative Harmonic Analysis II*, p. 134; Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis II*, p. 511, Theorem 39.37.

and by hypothesis we get $(K * \psi)(x) \to 0$ as $|x| \to \infty$, i.e. $K \in Y$.

Let Y_0 be the smallest closed translation-invariant subspace of $L^1(\mathbb{R}^d)$ that includes K. On the one hand, because Y is a closed translation-invariant subspace of $L^1(\mathbb{R}^d)$ and $K \in Y$ we have $Y_0 \subset Y$. On the other hand, because $\hat{K}(\xi) \neq 0$ for all ξ we have by Theorem 5 that $Y_0 = L^1(\mathbb{R}^d)$. Therefore $Y = L^1(\mathbb{R}^d)$. This means that for each $f \in L^1(\mathbb{R}^d)$, $(f * \psi)(x) \to 0$ as $|x| \to \infty$, i.e. $(f * \phi)(x) \to a\hat{f}(0)$ as $|x| \to \infty$, proving (1)

Assume further now that ϕ is slowly-oscillating and let $\epsilon > 0$. There is some A and some $\delta > 0$ such that if $|x|, |y| > A$ and $|x - y| < \delta$ then

$$|\phi(x) - \phi(y)| < \epsilon.$$

There is a test function h such that $h \geq 0$, $h(x) = 0$ for $|x| \geq \delta$, and $\hat{h}(0) = 1$. By (1),

$$\lim_{|x| \to \infty} (h * \phi)(x) = a\hat{h}(0) = a.$$

On the other hand, for $x \in \mathbb{R}^d$,

$$\begin{aligned} \phi(x) - (h * \phi)(x) &= \hat{h}(0)\phi(x) - (h * \phi)(x) \\ &= \int_{\mathbb{R}^d} (h(y)\phi(x) - \phi(x - y)h(y))dy \\ &= \int_{|y|<\delta} (\phi(x) - \phi(x - y))h(y)dy, \end{aligned}$$

and so for $|x| > A + \delta$,

$$|\phi(x) - (h * \phi)(x)| \leq \int_{|y|<\delta} \epsilon \cdot |h(y)|dy = \epsilon \int_{\mathbb{R}^d} h(y)dy = \epsilon\hat{h}(0) = \epsilon.$$

We have thus established that as $|x| \to \infty$, (i) $(h * \phi)(x) = a + o(1)$ and (ii) $\phi(x) = (h * \phi)(x) + o(1)$, which together yield $\phi(x) = a + o(1)$, i.e. $\phi(x) \to a$ as $|x| \to \infty$, proving (2). □

4 Closed ideals in $L^1(\mathbb{R}^d)$

$L^1(\mathbb{R}^d)$ is a Banach algebra using convolution as the product.[6]

Theorem 7. *Suppose that I is a closed linear subspace of $L^1(\mathbb{R}^d)$. I is translation-invariant if and only if I is an ideal.*

Proof. Assume that I is translation-invariant and let $f \in I$ and $g \in L^1(\mathbb{R}^d)$.

[6] Eberhard Kaniuth, *A Course in Commutative Banach Algebras*, p. 25, Proposition 1.4.7.

For $\phi \in I^\perp \subset L^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\langle g * f, \phi \rangle &= \int_{\mathbb{R}^d} (g*f)(x)\phi(x) dx \\ &= \int_{\mathbb{R}^d} \phi(x) \left(\int_{\mathbb{R}^d} g(x-y) f(y) dy \right) dx \\ &= \int_{\mathbb{R}^d} g(z) \left(\int_{\mathbb{R}^d} \phi(x) f_z(x) dx \right) dz \\ &= \int_{\mathbb{R}^d} g(z) \langle \phi, f_z \rangle dz \\ &= 0,\end{aligned}$$

because $f_z \in I$ for each $z \in \mathbb{R}^d$. This shows that $f * g \in ^\perp (I^\perp)$. But $^\perp(I^\perp)$ is the closure of I in $L^1(\mathbb{R}^d)$,[7] and I is closed so $f * g \in I$, showing that I is an ideal.

Assume that I is an ideal and let $f \in I$ and $x \in \mathbb{R}^d$. Let V be a closed ball centered at 0, and let χ_A be the indicator function of a set A. We have

$$\begin{aligned}\left\| f_x - \frac{1}{\mu(V)} \chi_{x+V} * f \right\|_1 &= \int_{\mathbb{R}^d} \left| f_x(y) - \frac{1}{\mu(V)} (\chi_{x+V} * f)(y) \right| dy \\ &= \int_{\mathbb{R}^d} \left| \frac{1}{\mu(V)} \int_V f_x(y) dz - \frac{1}{\mu(V)} \int_{\mathbb{R}^d} \chi_{x+V}(z) f(y-z) dz \right| dy \\ &= \frac{1}{\mu(V)} \int_{\mathbb{R}^d} \left| \int_V f(y-x) dz - \int_V f(y-z-x) dz \right| dy \\ &= \frac{1}{\mu(V)} \int_{\mathbb{R}^d} \left| \int_V (f(y-x) - f(y-z-x)) dz \right| dy \\ &\leq \frac{1}{\mu(V)} \int_V \left(\int_{\mathbb{R}^d} |f(y-x) - f(y-z-x)| \, dy \right) dz \\ &= \frac{1}{\mu(V)} \int_V \|f_x - f_{z+x}\|_1 \, dz \\ &= \frac{1}{\mu(V)} \int_V \|f - f_z\|_1 \, dz \\ &\leq \sup_{z \in V} \|f - f_z\|_1 .\end{aligned}$$

Let $\epsilon > 0$. The map $z \mapsto f_z$ is continuous $\mathbb{R}^d \to L^1(\mathbb{R}^d)$, so there is some $\delta > 0$ such that if $|z| < \delta$ then $\|f_z - f_0\|_1 < \epsilon$, i.e. $\|f - f_z\|_1 < \epsilon$. Then let V be the closed ball of radius δ, with which

$$\left\| f_x - \frac{1}{\mu(V)} \chi_{x+V} * f \right\|_1 \leq \sup_{z \in V} \|f - f_z\|_1 \leq \epsilon. \tag{3}$$

As I is an ideal and $\frac{1}{\mu(V)} \chi_{x+V} \in L^1(\mathbb{R}^d)$ we have $\frac{1}{\mu(V)} \chi_{x+V} * f \in L^1(\mathbb{R}^d)$, and

[7] Walter Rudin, *Functional Analysis*, second ed., p. 96, Theorem 4.7.

then (3) and the fact that I is closed imply $f_x \in I$. Therefore I is translation-invariant. $\square$