# The Voronoi summation formula

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## 1 Mellin transform

The **Mellin transform** of $f : (0, \infty) \to \mathbb{C}$ is defined by

$$\mathscr{M}(f)(s) = \int_0^\infty x^{s-1} f(x) dx.$$

For example, $s \mapsto \Gamma(s)$ is the Mellin transform of $x \mapsto e^{-x}$.

Suppose that $f$ continuous on $(0, \infty)$, that there is some $\alpha \in \mathbb{R}$ such that $f(x) = O(x^{-\alpha})$ as $x \to 0$, and that for any $n \geq 1$, $\frac{f(x)}{x^n} \to 0$ as $x \to \infty$. Then [4, p. 107, Proposition 9.7.7] $\mathscr{M}(f)(s)$ is holomorphic on $\Re(s) > \alpha$, and for $\sigma > \alpha$ and $x > 0$,

$$f(x) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma} x^{-s} \mathscr{M}(f)(s) ds.$$

(The **Mellin inversion formula**.)

## 2 Generalized Poisson summation formula

Cohen [4, pp. 177–182, §10.2.5] presents a "generalized Poisson summation formula" which yields both the Poisson summation formula and the Voronoi summation formula.

We denote by $\mathscr{S}(\mathbb{R})$ the Fréchet space of Schwartz functions $\mathbb{R} \to \mathbb{C}$.

**Theorem 1.** Let $a$ be arithmetic function and define

$$L(a, s) = \sum_{n=1}^\infty a(n) n^{-s}, \qquad \Re(s) > 1.$$

Suppose that $L(a, s)$ has an analytic continuation to $\mathbb{C}$ whose only possible pole is at $s = 1$. Suppose also that there are $A, a_1, \ldots, a_g > 0$ such that for

$$\gamma(s) = A^s \prod_{j=1}^g \Gamma(a_j s),$$

$L(a,s)$ satisfies the functional equation

$$\gamma(s)L(a,s) = \gamma(1-s)L(a,1-s).$$

Let $f \in \mathscr{S}(\mathbb{R})$ and define for $x > 0$,

$$K(x) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} \frac{\gamma(s)}{\gamma(1-s)} x^{-s} ds, \qquad g(x) = \int_0^\infty f(y)K(xy)dy.$$

Then,

$$\sum_{n=1}^\infty a(n)f(n) = f(0)L(a,0) + \mathrm{Res}_{s=1}\mathscr{M}(f)(s)L(a,s) + \sum_{n=1}^\infty a(n)g(n).$$

*Proof.* Since $f$ is a Schwartz function, $\mathscr{M}(f)$ is holomorphic on $\Re(s) > 0$. Futhermore, for $\Re(s) > 0$, integrating by parts,

$$\mathscr{M}(f)(s) = \int_0^\infty x^{s-1}f(x)dx = f(x)\frac{x^s}{s}\Big|_0^\infty - \int_0^\infty f'(x)\frac{x^s}{s}dx = -\frac{1}{s}\mathscr{M}(f')(s+1).$$

It follows that $\mathscr{M}(f)$ has an analytic continuation to $\mathbb{C}$ possibly with poles at $0, -1, -2, -3, \ldots$. Write $F = \mathscr{M}(f)$. By the Mellin inversion formula we get

$$\begin{aligned}\sum_{n=1}^\infty a(n)f(n) &= \sum_{n=1}^\infty a(n)\frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} n^{-s}F(s)ds\\ &= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} F(s)\sum_{n=1}^\infty a_n n^{-s}ds\\ &= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} F(s)L(a,s)ds.\end{aligned}$$

The only possible pole of $L(a,s)$ is at $s = 1$. From

$$\mathscr{M}(f)(s) = -\frac{1}{s}\mathscr{M}(f')(s+1),$$

the only possible pole of $F(s)$ in the half-plane $\Re(s) > -1$ is at $s = 0$, and the residue of $F(s)L(a,s)$ at $s = 0$ is

$$-\mathscr{M}(f')(1) = -\int_0^\infty f'(x)dx = -(f(\infty) - f(0)) = f(0),$$

so the residue of $F(s)L(a,s)$ at $s = 0$ is

$$f(0)L(a,0).$$

Therefore, by the residue theorem, taking as given that $F(s)L(a,s) \to 0$ uniformly in $-\frac{1}{2} \le \Re(s) \le \frac{3}{2}$ as $|\Im(s)| \to \infty$,

$$\sum_{n=1}^{\infty} a(n)f(n) = f(0)L(a,0) + \mathrm{Res}_{s=1}F(s)L(a,s) + \frac{1}{2\pi i}\int_{\Re(s)=-\frac{1}{2}} F(s)L(a,s)ds.$$

Define

$$G(s) = F(1-s)\frac{\gamma(s)}{\gamma(1-s)}.$$

Using the functional equation for $L(a,s)$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Re(s)=-\frac{1}{2}} F(s)L(a,s)ds &= \frac{1}{2\pi i}\int_{\Re(s)=-\frac{1}{2}} F(s)\frac{\gamma(1-s)}{\gamma(s)}L(a,1-s)ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} F(1-s)\frac{\gamma(s)}{\gamma(1-s)}L(a,s)ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} G(s)L(a,s)ds.
\end{aligned}$$

Furthermore, define

$$J(x) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} \frac{1}{1-s}\frac{\gamma(s)}{\gamma(1-s)}x^{1-s}ds,$$

which satisfies

$$J'(x) = K(x).$$

We have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} x^{-s}G(s)ds &= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} x^{-s}F(1-s)\frac{\gamma(s)}{\gamma(1-s)}ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} x^{-s}\left(-\frac{1}{1-s}\mathscr{M}(f')(2-s)\right)\frac{\gamma(s)}{\gamma(1-s)}ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} x^{-s}\left(-\frac{1}{1-s}\int_0^{\infty} y^{1-s}f'(y)dy\right)\frac{\gamma(s)}{\gamma(1-s)}ds \\
&= -\frac{1}{x}\int_0^{\infty} f'(y)\frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} \frac{1}{1-s}\frac{\gamma(s)}{\gamma(1-s)}(xy)^{1-s}ds \\
&= -\frac{1}{x}\int_0^{\infty} f'(y)J(xy)dy \\
&= -\frac{1}{x}f(y)J(xy)\Big|_0^{\infty} + \frac{1}{x}\int_0^{\infty} f(y)J'(xy)xdy \\
&= 0 + \int_0^{\infty} f(y)J'(xy)dy \\
&= \int_0^{\infty} f(y)K(xy)dy \\
&= g(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} a(n)g(n) &= \sum_{n=1}^{\infty} a(n)\frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} n^{-s}G(s)ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} G(s)\sum_{n=1}^{\infty} a(n)n^{-s}ds \\
&= \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} G(s)L(a,s)ds.
\end{aligned}$$

Thus we have

$$\sum_{n=1}^{\infty} a(n)f(n) = f(0)L(a,0) + \mathrm{Res}_{s=1}F(s)L(a,s) + \sum_{n=1}^{\infty} a(n)g(n)$$

□

Take $a(n) = 1$ for all $n$. Then,

$$L(a,s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s).$$

The Riemann zeta function satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

So with

$$\gamma(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right),$$

we have

$$\gamma(s)\zeta(s) = \gamma(1-s)\zeta(1-s).$$

Using

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$$

and

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z),$$

we have

$$\begin{aligned}
\Gamma\left(\frac{1-s}{2}\right) &= \Gamma\left(1-\frac{s+1}{2}\right) \\
&= \frac{\pi}{\sin\frac{\pi(s+1)}{2}\Gamma\left(\frac{s+1}{2}\right)} \\
&= \frac{\pi}{\sin\frac{\pi(s+1)}{2}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)} \\
&= \frac{\pi\Gamma\left(\frac{s}{2}\right)}{\sin\frac{\pi(s+1)}{2}2^{1-s}\sqrt{\pi}\Gamma(s)},
\end{aligned}$$

and so

$$\frac{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)} = \pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s}{2}\right) \cdot \frac{\sin\frac{\pi(s+1)}{2}2^{1-s}\sqrt{\pi}\Gamma(s)}{\pi\Gamma\left(\frac{s}{2}\right)}$$
$$= \sin\frac{\pi(s+1)}{2} \cdot 2(2\pi)^{-s}\Gamma(s)$$
$$= \cos\frac{\pi s}{2} \cdot 2(2\pi)^{-s}\Gamma(s).$$

Therefore

$$K(x) = \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} \frac{\gamma(s)}{\gamma(1-s)}x^{-s}ds = \frac{1}{2\pi i}\int_{\Re(s)=\frac{3}{2}} \cos\frac{\pi s}{2} \cdot 2(2\pi)^{-s}\Gamma(s)x^{-s}ds$$

But, taking as known

$$\int_0^\infty \cos(2\pi x)x^{s-1}dx = (2\pi)^{-s}\cos\frac{\pi s}{2}\Gamma(s),$$

it follows that

$$K(x) = 2\cos 2\pi x.$$

Thus Theorem 1 tells us that for $f \in \mathscr{S}(\mathbb{R})$,

$$\sum_{n=1}^\infty f(n) = f(0)\zeta(0) + \mathrm{Res}_{s=1}\mathscr{M}(f)(s)\zeta(s) + 2\sum_{n=1}^\infty \int_0^\infty f(y)\cos(2\pi ny)dy,$$

i.e.,

$$\sum_{n=1}^\infty f(n) = -\frac{1}{2}f(0) + \int_0^\infty f(x)dx + 2\sum_{n=1}^\infty \int_0^\infty f(y)\cos(2\pi ny)dy.$$

If $f : \mathbb{R} \to \mathbb{C}$ is even, this is the **Poisson summation formula**.

Take $a(n) = d(n)$ for all $n$. Then,

$$L(d,s) = \sum_{n=1}^\infty d(n)n^{-s} = \zeta^2(s).$$

For

$$\gamma(s) = \pi^{-s}\Gamma\left(\frac{s}{2}\right)^2,$$

it follows from the functional equation for the Riemann zeta function that $L(d,s)$ satisfies the functional equation

$$\gamma(s)L(d,s) = \gamma(1-s)L(d,1-s).$$

We worked out above that

$$\frac{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}{\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)} = \cos\frac{\pi s}{2} \cdot 2(2\pi)^{-s}\Gamma(s),$$

whence

$$\begin{aligned}\frac{\gamma(s)}{\gamma(1-s)} &= (2\pi)^{-2s} 4\cos^2\frac{\pi s}{2}\Gamma(s)^2\\ &= (2\pi)^{-2s}(2+2\cos\pi s)\Gamma(s)^2.\end{aligned}$$

Taking as given two identities for Bessel functions

$$\int_0^\infty x^{s-1}K_0(4\pi x^{1/2})dx = \frac{1}{2}(2\pi)^{-2s}\Gamma(s)^2$$

and

$$\int_0^\infty x^{s-1}Y_0(4\pi x^{1/2})dx = -\frac{1}{\pi}(2\pi)^{-2s}\cos\pi s\Gamma(s)^2,$$

it follows that

$$K(x) = 4K_0(4\pi x^{1/2}) - 2\pi Y_0(4\pi x^{1/2}).$$

Thus Theorem 1 tells us that for $f \in \mathscr{S}(\mathbb{R})$,

$$\begin{aligned}\sum_{n=1}^\infty d(n)f(n) &= f(0)\zeta^2(0) + \mathrm{Res}_{s=1}\mathscr{M}(f)(s)\zeta^2(s)\\ &\quad + \sum_{n=1}^\infty d(n)\int_0^\infty f(y)\left(4K_0(4\pi(ny)^{1/2}) - 2\pi Y_0(4\pi(ny)^{1/2})\right)dy.\end{aligned}$$

Using

$$\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + O(1), \qquad s \to 1,$$

and

$$x^{s-1} = 1 + (s-1)\log x + O(|s-1|^2),$$

we have

$$\mathrm{Res}_{s=1}\mathscr{M}(f)(s)\zeta^2(s) = 2\gamma + \log x,$$

and so

$$\begin{aligned}\sum_{n=1}^\infty d(n)f(n) &= \frac{1}{4}f(0) + \int_0^\infty f(x)(2\gamma+\log x)dx\\ &\quad + \sum_{n=1}^\infty d(n)\int_0^\infty f(y)\left(4K_0(4\pi(ny)^{1/2}) - 2\pi Y_0(4\pi(ny)^{1/2})\right)dy.\end{aligned}$$

# 3 Bernoulli numbers

The **Bernoulli polynomials** are defined by

$$\frac{te^{tx}}{e^t-1} = \sum_{m=0}^\infty B_m(x)\frac{t^m}{m!}.$$

The **Bernoulli numbers** are defined by $B_m = B_m(0)$.

We denote by $[x]$ the greatest integer $\leq x$, and we define $\{x\} = x - [x]$, namely, the fractional part of $x$. We define $P_m(x) = B_m(\{x\})$, the **Bernoulli functions**.

# 4 Wigert

The following result is proved by Wigert [18]. Our proof follows Titchmarsh [13, p. 163, Theorem 7.15]. Cf. Landau [10].

**Theorem 2.** For $\lambda < \frac{1}{2}\pi$ and $N \geq 1$,

$$\sum_{n=1}^{\infty} d(n)e^{-nz} = \frac{\gamma}{z} - \frac{\log z}{z} + \frac{1}{4} - \sum_{n=0}^{N-1} \frac{B_{2n+2}^2}{(2n+2)!(2n+2)} z^{2n+1} + O(|z|^{2N})$$

as $z \to 0$ in any angle $|\arg z| \leq \lambda$.

*Proof.* For $\sigma > 1$, $s = \sigma + it$,

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$

Using this, for $\Re z > 0$ we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\zeta^2(s)z^{-s}ds &= \sum_{n=1}^{\infty} d(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)(nz)^{-s}ds \\ &= \sum_{n=1}^{\infty} d(n)e^{-nz}. \qquad (1)\end{aligned}$$

Define $F(s) = \Gamma(s)\zeta^s(s)z^{-s}$. $F$ has poles at $1, 0$, and the negative odd integers. (At each negative even integer, $\Gamma$ has a first order pole but $\zeta^2$ has a second order zero.) First we determine the residue of $F$ at 1. We use the asymptotic formula

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|), \qquad s \to 1,$$

the asymptotic formula

$$\Gamma(s) = 1 - \gamma(s-1) + O(|s-1|^2), \qquad s \to 1,$$

and the asymptotic formula

$$z^{-s} = \frac{1}{z} - \frac{\log z}{z}(s-1) + O(|s-1|^2), \qquad s \to 1,$$

to obtain

$$\Gamma(s)\zeta^s(s)z^{-s} = (1-\gamma(s-1)+O(|s-1|^2))\cdot\left(\frac{1}{(s-1)^2}+\frac{2\gamma}{s-1}+O(|s-1|^2)\right)$$
$$\cdot\left(\frac{1}{z}-\frac{\log z}{z}(s-1)+O(|s-1|^2)\right)$$
$$=\frac{1}{z(s-1)^2}-\frac{\gamma}{z(s-1)}+\frac{2\gamma}{z(s-1)}-\frac{\log z}{z(s-1)}+O(1)$$
$$=\frac{1}{z(s-1)^2}+\frac{\gamma}{z(s-1)}-\frac{\log z}{z(s-1)}+O(1).$$

Hence the residue of $F$ at 1 is

$$\frac{\gamma}{z}-\frac{\log z}{z}.$$

Now we determine the residue of $F$ at 0. The residue of $\Gamma$ at 0 is 1, and hence the residue of $F$ at 0 is

$$1\cdot\zeta^2(0)\cdot z^0=\zeta^2(0)=\left(-\frac{1}{2}\right)^2=\frac{1}{4}.$$

Finally, for $n\geq 0$ we determine the residue of $F$ at $-(2n+1)$. The residue of $\Gamma$ at $-(2n+1)$ is $\frac{(-1)^{2n+1}}{(2n+1)!}$, hence the residue of $F$ at $-(2n+1)$ is

$$\frac{(-1)^{2n+1}}{(2n+1)!}\cdot\zeta^2(2n+1)\cdot z^{2n+1}=-\frac{B_{2n+2}^2}{(2n+2)!(2n+2)}z^{2n+1}$$

using

$$\zeta(-m)=-\frac{B_{m+1}}{m+1},\qquad m\geq 1.$$

Let $M>0$, and let $C$ be the rectangular path starting at $2-iM$, then going to $2+iM$, then going to $-2N+iM$, then going to $-2N-iM$, and then ending at $2-iM$. By the residue theorem,

$$\int_C F(s)ds=2\pi i\left(\frac{\gamma}{z}-\frac{\log z}{z}+\frac{1}{4}+\sum_{n=0}^{N-1}-\frac{B_{2n+2}^2}{(2n+2)!(2n+2)}z^{2n+1}\right). \qquad (2)$$

Denote the right-hand sideof (2) by $2\pi iR$. We have

$$\int_C F(s)ds=\int_{2-iM}^{2+iM}F(s)ds+\int_{2+iM}^{-2N+iM}F(s)ds+\int_{-2N+iM}^{-2N-iM}F(s)ds+\int_{-2N-iM}^{2-iM}F(s)ds.$$

We shall show that the second and fourth integrals tend to 0 as $M\to\infty$. For $s=\sigma+it$ with $-2N\leq\sigma\leq 2$, Stirling's formula [14, p. 151] tells us that

$$|\Gamma(s)|\sim\sqrt{2\pi}e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}},\qquad |t|\to\infty.$$

As well [13, p. 95], there is some $K > 0$ such that in the half-plane $\sigma \geq -2N$,

$$\zeta(s) = O(|t|^K).$$

Also,

$$\begin{aligned} z^{-s} &= e^{-s\log z} \\ &= e^{-(\sigma+it)(\log|z|+i\arg z)} \\ &= e^{-\sigma\log|z|+t\arg z-i(\sigma\arg z+t\log|z|)}, \end{aligned}$$

and so for $|\arg z| \leq \lambda$,

$$|z^{-s}| = e^{-\sigma\log|z|+t\arg z} \leq e^{-\sigma\log|z|+\lambda|t|} = |z|^{-\sigma}e^{\lambda|t|}.$$

Therefore

$$\left|\int_{2+iM}^{-2N+iM} F(s)ds\right| \leq (2+2N) \sup_{-2N\leq\sigma\leq 2} |F(\sigma+iM)| = O(e^{-\frac{\pi}{2}M}M^{\sigma-\frac{1}{2}}M^{2K}|z|^{-\sigma}e^{\lambda M}),$$

and because $\lambda < \frac{\pi}{2}$ this tends to 0 as $M \to \infty$. Likewise,

$$\left|\int_{-2N-iM}^{2-iM} F(s)ds\right| \to 0$$

as $M \to \infty$. It follows that

$$\int_{2-i\infty}^{2+i\infty} F(s)ds + \int_{-2N+i\infty}^{-2N-i\infty} F(s)ds = 2\pi iR.$$

Hence,

$$\int_{2-i\infty}^{2+i\infty} F(s)ds = 2\pi iR + \int_{-2N-i\infty}^{-2N+i\infty} F(s)ds.$$

We bound the integral on the right-hand side. We have

$$\int_{-2N-i\infty}^{-2N+i\infty} F(s)ds = \int_{\sigma=-2N,|t|\leq 1} F(s)ds + \int_{\sigma=-2N,|t|>1} F(s)ds.$$

The first integral satisfies

$$\left|\int_{\sigma=-2N,|t|\leq 1} F(s)ds\right| \leq \int_{\sigma=-2N,|t|\leq 1} |\Gamma(s)\zeta^2(s)||z|^{-\sigma}e^{\lambda|t|}ds = |z|^{2N}\cdot O(1) = O(|z|^{2N}),$$

because $\Gamma(s)\zeta^2(s)$ is continuous on the path of integration. The second integral satisfies

$$\begin{aligned} \left|\int_{\sigma=-2N,|t|>1} F(s)ds\right| &\leq \int_{\sigma=-2N,|t|>1} e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}|t|^K|z|^{-\sigma}e^{\lambda|t|}ds \\ &= |z|^{2N}\int_{\sigma=-2N,|t|>1} e^{-\frac{\pi}{2}|t|}|t|^{-2N-\frac{1}{2}}|t|^K e^{\lambda|t|}dt \\ &= |z|^{2N}\cdot O(1) \\ &= O(|z|^{2N}), \end{aligned}$$

because $\lambda < \frac{\pi}{2}$. This establishes

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s)ds = R + O(|z|^{2N}).$$

Using (1) and (2), this becomes

$$\sum_{n=1}^{\infty} d(n)e^{-nz} = \frac{\gamma}{z} - \frac{\log z}{z} + \frac{1}{4} - \sum_{n=0}^{N-1} \frac{B_{2n+2}^2}{(2n+2)!(2n+2)} z^{2n+1} + O(|z|^{-2N}),$$

completing the proof. $\square$

For example, as $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}$, the above theorem tells us that

$$\sum_{n=1}^{\infty} d(n)e^{-nz} = \frac{\gamma}{z} - \frac{\log z}{z} + \frac{1}{4} - \frac{z}{144} - \frac{z^3}{86400} - \frac{z^5}{7620480} + O(|z|^6).$$

# 5 Other works on the Voronoi summation formula

Voronoi's papers on the Voronoi summation formula are [15] and [16] and [17].

Iwaniec and Kowalski [9, Chaper 4]

Stein and Shakarchi [12, p. 392, Theorem 8.11].

Ivic [8, pp. 83ff., Chapter 3] and [7]

Miller and Schmid [11]

Hejhal [6]

Flajolet, Gourdon and Dumas [5]

Bettin and Conrey [1]

Chandrasekharan and Narasimhan [3]

Chandrasekharan [2, Chapter VIII]

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