

# Vitali coverings on the real line

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For  $x \in \mathbb{R}$  and  $r > 0$  write

$$B(x, r) = \{y \in \mathbb{R} : |y - x| < r\}.$$

Let  $\lambda$  be Lebesgue measure on the Borel  $\sigma$ -algebra of  $\mathbb{R}$  and let  $\lambda^*$  be Lebesgue outer measure on  $\mathbb{R}$ .

A **Vitali covering** of a set  $E \subset \mathbb{R}$  is a collection  $\mathcal{V}$  of closed intervals such that for  $\epsilon > 0$  and for  $x \in E$  there is some  $I \in \mathcal{V}$  with  $x \in I$  and  $0 < \lambda(I) < \epsilon$ .

The following is the **Vitali covering theorem**.<sup>1</sup>

**Theorem 1** (Vitali covering theorem). *Let  $U$  be an open set in  $\mathbb{R}$  with  $\lambda(U) < \infty$ , let  $E \subset U$ , and let  $\mathcal{V}$  be a Vitali covering of  $E$  each interval of which is contained in  $U$ . Then for any  $\epsilon > 0$ , there are disjoint  $I_1, \dots, I_n \in \mathcal{V}$  such that*

$$\lambda^* \left( E \setminus \bigcup_{j=1}^n I_j \right) < \epsilon.$$

*Proof.* Suppose that  $I_1, \dots, I_n \in \mathcal{V}$  are pairwise disjoint. If  $E \subset \bigcup_{j=1}^n I_j$  then  $I_1, \dots, I_n$  satisfy the claim, and otherwise, let

$$U_n = U \setminus \bigcup_{j=1}^n I_j,$$

and there exists some  $x \in E \cap U_n$ . As  $x \in U_n$  and  $U_n$  is open, there is some  $\eta > 0$  such that  $B(x, \eta) \subset U_n$  and then as  $\mathcal{V}$  is a Vitali covering of  $E$  there is some  $I \in \mathcal{V}$  with  $x \in I \subset B(x, \eta) \subset U_n$ . Thus  $\delta_n > 0$  for

$$\delta_n = \sup \{ \lambda(I) : I \in \mathcal{V}, I \subset U_n \},$$

and there is some  $I_{n+1} \in \mathcal{V}$  with  $I_{n+1} \subset U_n$  and  $\lambda(I_{n+1}) > \frac{\delta_n}{2}$ .

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<sup>1</sup>Klaus Bichteler, *Integration – A Functional Approach*, p. 161, Lemma 10.5; John J. Benedetto and Wojciech Czaja, *Integration and Modern Analysis*, p. 179, Theorem 4.3.1; Russell A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, p. 52, Lemma 4.6.

For  $j \geq 1$  write  $I_j = [x_j - r_j, x_j + r_j]$  and let  $J_j = [x_j - 5r_j, x_j + 5r_j]$ , namely  $J_j$  is concentric with  $I_j$  and  $\lambda(J_j) = 5\lambda(I_j)$ . Then, as the intervals  $I_1, I_2, \dots$  are pairwise disjoint Borel sets each contained in  $U$ ,

$$\sum_{j=1}^{\infty} \lambda(J_j) = 5 \sum_{j=1}^{\infty} \lambda(I_j) = 5\lambda\left(\bigcup_{j=1}^{\infty} I_j\right) \leq 5\lambda(U) < \infty$$

and it follows from  $\sum_{j=1}^{\infty} \lambda(J_j) < \infty$  that  $\sum_{j=M}^{\infty} \lambda(J_j) \rightarrow 0$  as  $M \rightarrow \infty$ , which with

$$\lambda\left(\bigcup_{j=M}^{\infty} J_j\right) \leq \sum_{j=M}^{\infty} \lambda(J_j)$$

yields  $\lambda\left(\bigcup_{j=M}^{\infty} J_j\right) \rightarrow 0$  as  $M \rightarrow \infty$ .

Let  $M \geq 1$ . If  $x \in E \setminus \bigcup_{j=1}^{\infty} I_j$  then  $x \in E \setminus \bigcup_{j=1}^M I_j$  and so  $x \in U_M$ , and as  $U_M$  is open there is some  $\eta > 0$  with  $B(x, \eta) \subset U_M$ . But  $x \in E$  and  $\mathcal{V}$  is a Vitali covering of  $E$ , so there is some  $I \in \mathcal{V}$  with  $x \in I$  and  $I \subset B(x, \eta) \subset U_M$ . Now,  $\lambda(I_{j+1}) > \frac{\delta_j}{2}$  and  $\sum_{j=1}^{\infty} \lambda(I_j) < \infty$  together imply  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , so there is some  $n$  for which  $\delta_n < \lambda(I)$ . By the definition of  $\delta_n$  as a supremum, this means that  $I \not\subset U_n$  and so it makes sense to define  $N$  to be a minimal positive integer such that  $I \not\subset U_N$ .  $M < N$ : if  $M \geq N$  then  $I \subset U_M \subset U_N$ , contradicting  $I \not\subset U_N$ . (We shall merely use that  $M \leq N$ .) The fact that  $I \not\subset U_N$  and  $I \subset U_{N-1}$  means that  $I \cap I_N \neq \emptyset$  and also, by the definition of  $\delta_{N-1}$ ,  $\lambda(I) \leq \delta_{N-1} < 2\lambda(I_N)$ . Write  $I = [y - r, y + r]$ .  $I \cap I_N \neq \emptyset$  tells us  $y - r \leq x_N + r_N$  and  $y + r \geq x_N - r_N$ , and  $\lambda(I) < 2\lambda(I_N)$  tells us  $2r < 4r_N$ , hence

$$y + r \leq x_N + r_N + 2r \leq x_N + 5r_N, \quad y - r \geq x_N - r_N - 2r \geq x_N - 5r_N,$$

showing that

$$x \in I = [y - r, y + r] \subset J_N \subset \bigcup_{j=M}^{\infty} J_j.$$

This is true for each  $x \in E \setminus \bigcup_{j=1}^{\infty} I_j$ , which means that

$$E \setminus \bigcup_{j=1}^{\infty} I_j \subset \bigcup_{j=M}^{\infty} J_j.$$

Because  $\lambda(\bigcup_{j=M}^{\infty} J_j) \rightarrow 0$  as  $M \rightarrow \infty$ , this yields

$$\lambda^*\left(E \setminus \bigcup_{j=1}^{\infty} I_j\right) = 0.$$

But  $E \setminus \bigcup_{j=1}^n I_j$  is an increasing sequence of sets tending to  $E \setminus \bigcup_{j=1}^{\infty} I_j$ , therefore

$$\lambda^*\left(E \setminus \bigcup_{j=1}^n I_j\right) \rightarrow \lambda^*\left(E \setminus \bigcup_{j=1}^{\infty} I_j\right) = 0, \quad n \rightarrow \infty,$$

so there is some  $n$  such that  $\lambda^* \left( E \setminus \bigcup_{j=1}^n I_j \right) < \epsilon$  and then  $I_1, \dots, I_n$  satisfy the claim.  $\square$