The C^∞ Urysohn lemma

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Define $\eta : \mathbb{R} \to \mathbb{R}$ by
$$\eta(t) = e^{-1/t} 1_{(0,\infty)}(t).$$
It is a fact that η is C^∞. This is proved by showing that for each $k \geq 1$ there is a polynomial P_k of degree $2k$ such that $\eta^{(k)}(t) = P_k(t^{-1})e^{-1/t}$ for $t > 0$, and that $\eta^{(k)}(0) = 0$, which together imply that $\eta \in C^k$.

Define $\psi : \mathbb{R}^d \to \mathbb{R}$ by
$$\psi(x) = \eta(1 - |x|^2) = \begin{cases} e^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$
Because $x \mapsto 1 - |x|^2$ is $C^\infty : \mathbb{R}^d \to \mathbb{R}$, the chain rule tells us that ψ is C^∞.

For a function ϕ on $\mathbb{R}^d$ and for $t > 0$, we define
$$\phi_t(x) = t^{-d}\phi(t^{-1}x).$$

We now construct bump functions.[1]

Theorem 1 (C^∞ Urysohn lemma)**.** *If K is a compact subset of $\mathbb{R}^d$ and U is an open set containing K, then there exists $\phi \in C^\infty(\mathbb{R}^d)$ with $0 \leq \phi \leq 1$, $\phi = 1$ on K, and $\operatorname{supp} \phi \subset U$. Moreover, if K is invariant under $SO(d)$ then the function ϕ constructed here is radial.*

Proof. Let
$$\delta = d(K, U^c),$$
which is positive because K is compact and U^c is closed. Let
$$V = \left\{ x \in \mathbb{R}^d : d(x, K) < \frac{\delta}{3} \right\} = K + B_{\delta/3},$$
and define f on $\mathbb{R}^d$ by
$$f = \left(\int_{\mathbb{R}^d} \psi(x)dx \right)^{-1} \psi_{\delta/3},$$

[1] The following construction of a bump function follows Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 245, Lemma 8.18.

whose support is

$$\operatorname{supp} f = \operatorname{supp} \psi_{\delta/3} = \overline{B_{\delta/3}}.$$

Finally define ϕ on $\mathbb{R}^d$ by

$$\phi = 1_V * f.$$

Because V is bounded and f is C^∞, the function ϕ is C^∞. The support of ϕ is

$$\operatorname{supp} \phi = \operatorname{supp}(1_V * f) \subset \overline{\operatorname{supp} 1_V + \operatorname{supp} f} = \overline{V + \overline{B_{\delta/3}}} = K + \overline{B_{2\delta/3}} \subset U.$$

Because 1_V and f are nonnegative, so is their convolution ϕ. For any x,

$$\phi(x) = \int_{\mathbb{R}^d} 1_V(x-y) f(y) dy \leq \int_{\mathbb{R}^d} f(y) dy = 1,$$

so $0 \leq \phi \leq 1$. For $x \in K$, if $y \in V^c$ then $|x-y| \geq \delta/3$. But $f(u) = 0$ for $|u| \geq \delta/3$, so in this case $f(x-y) = 0$. This implies that for $x \in K$ the functions $y \mapsto 1_V(y) f(x-y)$ and $y \mapsto f(x-y)$ are equal, hence

$$\phi(x) = \int_{\mathbb{R}^d} 1_V(y) f(x-y) dy = \int_{\mathbb{R}^d} f(x-y) dy = \int_{\mathbb{R}^d} f(y) dy = 1.$$

This shows that $\phi = 1$ on K, verifying all the assertions made about ϕ.

The function ψ is radial and so f is too. If V is invariant under $SO(d)$, then the indicator function 1_V is radial. Thus, if K is invariant under $SO(d)$ then 1_V is radial, and the convolution of two radial functions is also radial, which means that ϕ is radial in this case. □

For example, take $d = 1$, take K to be the closed ball of radius 1, and take U to be the open ball of radius 2. Then $\delta = d(K, U^c) = 1$ and $V = B_{4/3}$. In Figure 1 we plot the bump function ϕ constructed in the above theorem.

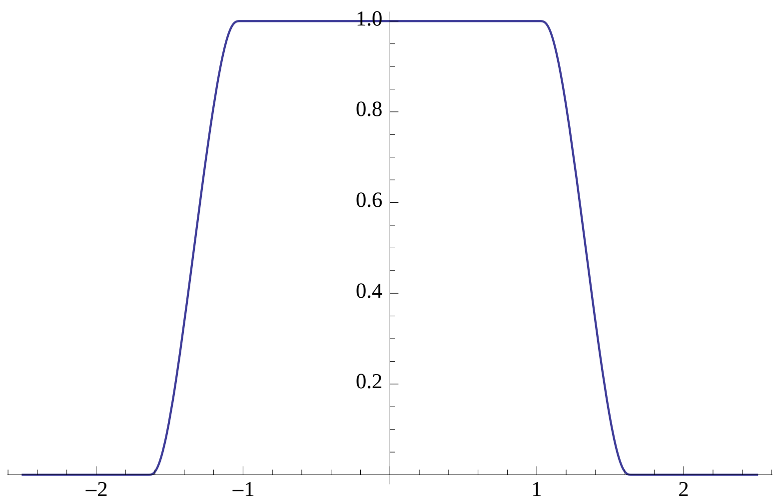

Figure 1: The bump function ϕ, for $d = 1$, $K = [-1, 1]$, $U = (-2, 2)$; $\delta = 1$ and $V = (-4/3, 4/3)$