Unordered sums in Hilbert spaces

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1 Preliminaries

Let $\mathbb{N}$ be the set of positive integers. We say that a set is countable if it is bijective with a subset of $\mathbb{N}$; thus a finite set is countable. In this note I do not presume unless I say so that any set is countable or that any topological space is separable. A neighborhood of a point in a topological space is a set that contains an open set that contains the point; one reason why it can be handy to speak about neighborhoods of a point rather than just open sets that contain the point is that the set of all neighborhoods of a point is a filter, whereas it is unlikely that the set of all open sets that contain a point is a filter.

2 Unordered sums in normed spaces

A *partially ordered set* is a set J and a binary relation $\leq$ on J that is reflexive ($\alpha \leq \alpha$), antisymmetric (if both $\alpha \leq \beta$ and $\beta \leq \alpha$ then $\alpha = \beta$), and transitive (if both $\alpha \leq \beta$ and $\beta \leq \gamma$ then $\alpha \leq \gamma$).[1] A *directed set* is a partially ordered set $(J, \leq)$ such that if $\alpha, \beta \in J$ then there is some $\gamma \in J$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. If X is a topological space, a *net* in X is a function from some directed set to X. If $z : J \to X$ is a net in X and N is a subset of X, we say that z is *eventually in* N if there is some $\alpha \in J$ such that $\alpha \leq \beta$ implies $z(\beta) \in N$. We say that the net z *converges* to $x \in X$ if for every neighborhood of x the net is eventually in that neighborhood. The importance of the notion of a net is that if X and Y are topological spaces and f is a function $X \to Y$ then f is continuous if and only if for every $x \in X$ and for every net $z : J \to X$ that converges to x, the net $f \circ z : J \to Y$ converges to $f(x)$.[2]

Let X be a normed space, let I be a set, and let $\mathscr{F}$ be the set of all finite subsets of I. $\mathscr{F}$ is a directed set ordered by set inclusion. Define $S : \mathscr{F} \to X$ by

$$S(F) = \sum_{i \in F} f(i) \in X, \qquad F \in \mathscr{F}.$$

S is a net in X, and if the net S converges to $x \in X$, we say that the sum $\sum_{i \in I} f(i)$ *converges to* x, and write $\sum_{i \in I} f(i) = x$.

[1] Paul R. Halmos, *Naive Set Theory*, §14.

[2] James R. Munkres, *Topology*, second ed., p. 188.

Theorem 1. *If X is a normed space, $f : I \to X$ is a function, $x \in X$, and I_0 is a subset of I such that if $i \in I \setminus I_0$ then $f(i) = 0$, then $\sum_{i\in I} f(i)$ converges to x if and only if $\sum_{i\in I_0} f(i)$ converges to x.*

Proof. Let $\mathscr{F}$ be the set of all finite subsets of I, let $\mathscr{F}_0$ be the set of all finite subsets of I_0, define $S : \mathscr{F} \to X$ by $S(F) = \sum_{i\in F} f(i)$, and let S_0 be the restriction of S to $\mathscr{F}_0$. Suppose that $\sum_{i\in I} f(i)$ converges to x, and let $\epsilon > 0$. There is some $F_\epsilon \in \mathscr{F}$ such that if $F_\epsilon \subseteq F \in \mathscr{F}$ then $\|S(F) - x\| < \epsilon$. Let $G_\epsilon = F_\epsilon \cap I_0$. If $G_\epsilon \subseteq G \in \mathscr{F}_0$, then

$$S_0(G) - x = \sum_{i\in G} f(i) - x = \sum_{i\in F} f(i) - x = S(F) - x,$$

giving $\|S_0(G) - x\| = \|S(F) - x\|$. Hence $G_\epsilon \subseteq G \in \mathscr{F}_0$ implies that $\|S_0(G) - x\| < \epsilon$, showing that the net S_0 converges to x, i.e. that $\sum_{i\in I_0} f(i)$ converges to x.

Suppose that $\sum_{i\in I_0} f(i)$ converges to x, and let $\epsilon > 0$. There is some $G_\epsilon \in \mathscr{F}_0$ such that if $G_\epsilon \subseteq G \in \mathscr{F}_0$ then $\|S_0(G) - x\| < \epsilon$. If $G_\epsilon \subseteq F \in \mathscr{F}$, then, with $G = F \cap I_0$,

$$S(F) - x = \sum_{i\in F} f(i) - x = \sum_{i\in G} f(i) - x = S_0(G) - x,$$

so $G_\epsilon \subseteq F \in \mathscr{F}$ implies that $\|S(F) - x\| < \epsilon$. This shows that S converges to x, that is, that $\sum_{i\in I} f(i)$ converges to x. □

Theorem 2. *If X is a normed space, $f : I \to X$ is a function, and $\sum_{i\in I} f(i)$ converges, then $\{i \in I : f(i) \neq 0\}$ is countable.*

Proof. Suppose that $\sum_{i\in I} f(i)$ converges to x, let $\mathscr{F}$ be the set of all finite subsets of I, and let $S(F) = \sum_{i\in I} f(i)$, $F \in \mathscr{F}$. For each $n \in \mathbb{N}$, let $F_n \in \mathscr{F}$ be such that if $F_n \subseteq F \in \mathscr{F}$ then

$$\|S(F) - x\| < \frac{1}{n}.$$

If $G \in \mathscr{F}$ and $G \cap F_n = \emptyset$, then

$$\|S(G)\| = \|S(G \cup F_n) - S(F_n)\| \leq \|S(G \cup F_n) - x\| + \|S(F_n) - x\| < \frac{2}{n}.$$

Let $J = \bigcup_{n\in\mathbb{N}} F_n$. If $i \in I \setminus J$, then for each $n \in \mathbb{N}$, we have $\{i\} \cap F_n = \emptyset$, whence $\|S(\{i\})\| < \frac{2}{n}$. That is, if $i \in I \setminus J$ then for each $n \in \mathbb{N}$ we have $\|f(i)\| < \frac{2}{n}$, which implies that if $i \in I \setminus J$ then $f(i) = 0$. Therefore $\{i \in I : f(i) \neq 0\} \subseteq J$, and as J is countable, the set $\{i \in I : f(i) \neq 0\}$ is countable. □

However, we already have a notion of infinite sums: a series is the limit of a sequence of partial sums.

Theorem 3. *If X is a normed space, $x_n \in X$, and $\sum_{n\in\mathbb{N}} x_n$ converges to x, then $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$.*

Proof. Let $\epsilon > 0$, let $\mathscr{F}$ be the set of all finite subsets of $\mathbb{N}$, and let $S : \mathscr{F} \to X$ be $S(F) = \sum_{n \in F} x_n$. The net S converges to x, so there is some $F_\epsilon \in \mathscr{F}$ such that if $F_\epsilon \subseteq F$ then $\|S(F) - x\| < \epsilon$. Let $N_\epsilon = \max F_\epsilon$. If $N \geq N_\epsilon$, then for $F = \{1, \ldots, N\}$ we have $F_\epsilon \subseteq F$ and so

$$\left\| \sum_{n=1}^{N} x_n - x \right\| = \|S(F) - x\| < \epsilon,$$

showing that $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$. □

When we talk about the sum $\sum_{i \in I} f(i)$, the set of all finite subsets of I is ordered by set inclusion, but we don't care about any ordering of the set I itself. If the sum $\sum_{n \in \mathbb{N}} x_n$ converges then for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n \in \mathbb{N}} x_n$. If x_n is a sequence in a normed space and for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges, we say that the sequence x_n is *unconditionally summable*. If an unordered sum converges, then it is unconditionally summable, and if a countable unordered sum is unconditionally summable the unordered sum converges.

Theorem 4. *If X is a Banach space, $x_n \in X$, and $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n \in \mathbb{N}} x_n$ converges.*

Proof. For each $k \in \mathbb{N}$ there is some $K(k)$ such that

$$\sum_{n=K(k)+1}^{\infty} \|x_n\| < \frac{1}{k};$$

suppose that if $j < k$ then $K(j) < K(k)$. Define

$$v_k = \sum_{n=1}^{K(k)} x_n.$$

For $\epsilon > 0$, let $N > \frac{1}{\epsilon}$. If $k > j \geq N$, then

$$\|v_k - v_j\| = \left\| \sum_{n=1}^{K(k)} x_n - \sum_{n=1}^{K(j)} x_n \right\| = \left\| \sum_{n=K(j)+1}^{K(k)} x_n \right\| \leq \sum_{n=K(j)+1}^{K(k)} \|x_n\| \leq \sum_{n=K(j)+1}^{\infty} \|x_n\|,$$

hence if $k > j \geq N$, then $\|v_k - v_j\| < \frac{1}{j} \leq \frac{1}{N}$. This shows that v_k is a Cauchy sequence, and hence v_k converges to some $x \in X$.

Let $\mathscr{F}$ be the set of all finite subsets of $\mathbb{N}$ and define $S : \mathscr{F} \to X$ by $S(F) = \sum_{n \in F} x_n$. Let $\epsilon > 0$, and as $v_k \to x$ there is some N_1 such that if $k \geq N_1$ then $\|v_k - x\| < \epsilon$. Let $N_2 > \frac{1}{\epsilon}$, put $N = \max\{N_1, N_2\}$, and put

$F_\epsilon = \{1, \dots, K(N)\}$. If $F_\epsilon \subseteq F \in \mathscr{F}$, then

$$\begin{aligned}
\|S(F) - x\| &= \left\| \sum_{n\in F} x_n - x \right\| \\
&\leq \left\| \sum_{n\in F} x_n - \sum_{n\in F_\epsilon} x_n \right\| + \left\| \sum_{n\in F_\epsilon} x_n - x \right\| \\
&= \left\| \sum_{n\in F\setminus F_\epsilon} x_n \right\| + \|v_N - x\| \\
&< \sum_{n\in F\setminus F_\epsilon} \|x_n\| + \epsilon \\
&\leq \sum_{n=K(N)+1}^{\infty} \|x_n\| + \epsilon \\
&< \frac{1}{N} + \epsilon \\
&< 2\epsilon.
\end{aligned}$$

Therefore the net S converges to x, i.e. $\sum_{n\in\mathbb{N}} x_n$ converges to x. □

The following theorem shows us in particular that the converse of Theorem 3 is false. One direction of the following theorem is Theorem 4 with $X = \mathbb{C}$. The other direction follows from the Riemann rearrangement theorem.[3]

Theorem 5. *If $\alpha_n \in \mathbb{C}$, then $\sum_{n\in\mathbb{N}} \alpha_n$ converges if and only if $\sum_{n=1}^{\infty} |\alpha_n| < \infty$.*

Let X be a normed space and $z : J \to X$ a net. We say that z is *Cauchy* if for every $\epsilon > 0$ there is some $\alpha \in J$ such that $\alpha \leq \beta$ and $\alpha \leq \gamma$ together imply that $\|z(\beta) - z(\gamma)\| < \epsilon$.[4]

Theorem 6. *If X is a Banach space and $z : J \to X$ is a Cauchy net, then there is some $x \in X$ such that z converges to x.*

Proof. Let $\alpha_1 \in J$ such that if $\alpha_1 \leq \alpha$ then $\|z(\alpha) - z(\alpha_1)\| < 1$, and for $n > 1$ let $\alpha_n \in J$ be such that if $\alpha_n \leq \alpha$ then $\|z(\alpha) - z(\alpha_n)\| < \frac{1}{n}$ and such that $\alpha_{n-1} \leq \alpha_n$. Define $x_n = z(\alpha_n)$. For $\epsilon > 0$, let $N > \frac{1}{\epsilon}$. If $n \geq m \geq N$, then, as $\alpha_n \geq \alpha_m$,

$$\|x_n - x_m\| = \|z(\alpha_n) - z(\alpha_m)\| < \frac{1}{m} \leq \frac{1}{N},$$

showing that x_n is a Cauchy sequence in X. Hence there is some $x \in X$ such that $x_n \to x$.

[3] Walter Rudin, *Principles of Mathematical Analysis*, third ed., p. 76, Theorem 3.54.

[4] Ronald G. Douglas, *Banach Algebra Techniques in Operator Theory*, second ed., p. 3, Proposition 1.7.

Let $\epsilon > 0$, let $N_1 > \frac{1}{\epsilon}$, let N_2 be such that if $n \geq N_2$ then $\|x_{N_2} - x\| < \epsilon$, and set $N = \max\{N_1, N_2\}$. If $\alpha_N \leq \alpha$, then, by construction of the sequence α_n,

$$\begin{aligned}
\|z(\alpha) - x\| &\leq \|z(\alpha) - z(\alpha_N)\| + \|z(\alpha_N) - x\| \\
&= \|z(\alpha) - z(\alpha_N)\| + \|x_N - x\| \\
&< \frac{1}{N} + \epsilon \\
&< 2\epsilon,
\end{aligned}$$

showing that the net z converges to x. □

Theorem 7. *If H is an infinite dimensional Hilbert space and $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set in H, then $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges.*

Proof. Let $\mathscr{F}$ be the set of finite subsets of $\mathbb{N}$ and let $S(F) = \sum_{n\in F} \frac{1}{n} e_n$, $F \in \mathscr{F}$. Define $v_N = \sum_{n=1}^{N} \frac{1}{n} e_n$. If $N_1 > N_2 \geq N$, then, as e_n are orthonormal,

$$\|v_{N_1} - v_{N_2}\|^2 = \left\| \sum_{n=N_2+1}^{N_1} \frac{1}{n} e_n \right\|^2 = \sum_{n=N_2+1}^{N_1} \frac{1}{n^2} < \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \sum_{n=N}^{\infty} \frac{1}{n(n+1)} = \frac{1}{N},$$

so v_N is a Cauchy sequence in H and hence converges to some $h \in H$. For $\epsilon > 0$, let $N_1 > \frac{1}{\epsilon}$, let $\|v_{N_2} - h\|^2 < \epsilon$, put $N = \max\{N_1, N_2\}$, and put $F_\epsilon = \{1, \ldots, N\}$. If $F_\epsilon \subseteq F \in \mathscr{F}$, then, using that e_n are orthonormal and $0 \leq (a-b)^2 = a^2 - 2ab + b^2$,

$$\begin{aligned}
\|S(F) - h\|^2 &\leq (\|S(F) - S(F_\epsilon)\| + \|S(F_\epsilon) - h\|)^2 \\
&\leq 2\|S(F) - S(F_\epsilon)\|^2 + 2\|S(F_\epsilon) - h\|^2 \\
&= 2\left\| \sum_{n\in F\setminus F_\epsilon} \frac{1}{n} e_n \right\|^2 + 2\|v_N - h\|^2 \\
&= 2 \sum_{n\in F\setminus F_\epsilon} \frac{1}{n^2} + 2\|v_N - h\|^2 \\
&< 4\epsilon.
\end{aligned}$$

This shows that the net S converges to h, that is, that $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges to h. □

We have proved that if H is an infinite dimensional Hilbert space and $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set in H, then $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges, although $\sum_{n=1}^{\infty} \left\| \frac{1}{n} e_n \right\| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. This shows that the converse of Theorem 4 is false. In fact, the Dvoretsky-Rogers theorem states that if X is an infinite dimensional Banach space then there is some countable subset $\{x_n : n \in \mathbb{N}\}$ of X such that $\sum_{n\in\mathbb{N}} x_n$ converges but $\sum_{n\in\mathbb{N}} \|x_n\| = \infty$.[5]

[5] Joseph Diestel, *Sequences and Series in Banach Spaces*, p. 59, chapter VI.

3 Orthogonal projections

If $S_i, i \in I$, are subsets of a Hilbert space H, we define $\bigvee_{i\in I} S_i$ to be the closure of the span of $\bigcup_{i\in I} S_i$. If $i \neq j$ implies that $S_i \perp S_j$, we say that the sets S_i are *mutually orthogonal*. To say that $\{e_i : i \in I\}$ is an orthonormal basis for H is to say that $\{e_i : i \in I\}$ is an orthonormal set and that $H = \bigvee_{i\in I}\{e_i\}$.

If $M_n, n \in \mathbb{N}$, are mutually orthogonal closed subspaces of M, we denote

$$\bigoplus_{n\in\mathbb{N}} M_n = \bigvee_{n\in\mathbb{N}} M_n,$$

which we call an *orthogonal direct sum*.

If H is a Hilbert space and M is a closed subspace of H, then for every $h \in H$ there is a unique $v_h \in M$ such that

$$\|h - v_h\| = \inf_{v\in M} \|h - v\|,$$

and $h - v_h \in M^\perp$.[6] This gives

$$H = M \oplus M^\perp.$$

The *orthogonal projection of H onto M* is the map $P : H \to H$ defined by

$$P(h_1 + h_2) = h_1, \qquad h_1 \in M, h_2 \in M^\perp.$$

It is straightforward to check that P is linear, $\|P\| \leq 1$ ($\|P\| = 1$ if and only if M is nonzero), $P^2 = P$, and $\ker P = M^\perp$ and $P(H) = M$.[7] Rather than specifying a closed subspace of H and talking about the orthogonal projection onto M, we can talk about an orthogonal projection in H, which is the orthogonal projection onto its image.

Bessel's inequality[8] states that if $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set in a Hilbert space H and $h \in H$, then

$$\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \|h\|^2. \tag{1}$$

Theorem 8. *If H is a Hilbert space, $\mathscr{E}$ is an orthonormal set in H, and $h \in H$, then there are only countably many $e \in \mathscr{E}$ such that $\langle h, e\rangle \neq 0$.*

Proof. Let

$$\mathscr{E}_n = \left\{ e \in \mathscr{E} : |\langle h, e\rangle| \geq \frac{1}{n} \right\}.$$

If $\mathscr{E}_n$ were infinite, let $\{e_j : j \in \mathbb{N}\}$ be a subset of it, and this gives us a contradiction by (1). Therefore each $\mathscr{E}_n$ is finite. But if $\langle h, e\rangle \neq 0$ then there is

[6] John B. Conway, *A Course in Functional Analysis*, second ed., p. 9, Theorem 2.6.
[7] John B. Conway, *A Course in Functional Analysis*, second ed., p. 10, Theorem 2.7.
[8] John B. Conway, *A Course in Functional Analysis*, second ed., p. 15, Theorem 4.8.

some n such that $|\langle h, e\rangle| \geq \frac{1}{n}$, so

$$\mathscr{E} = \bigcup_{n=1}^{\infty} \mathscr{E}_n.$$

Therefore $\mathscr{E}$ is countable. □

Bessel's inequality makes sense for an orthonormal set of any cardinality in a Hilbert space, rather than just for a countable orthonormal set.

Theorem 9 (Bessel's inequality)**.** *If H is a Hilbert space, $\mathscr{E}$ is an orthonormal set in H, and $h \in H$, then*

$$\sum_{e \in \mathscr{E}} |\langle h, e\rangle|^2 \leq \|h\|^2 .$$

Proof. By Theorem 8, there are only countably many $e \in \mathscr{E}$ such that $\langle h, e\rangle \neq 0$; let them be $\{e_n : n \in \mathbb{N}\}$. $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set, so by (1) we have

$$\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \|h\|^2 .$$

Theorem 4 states that if X is a Banach space, $x_n \in X, n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then the unordered sum $\sum_{n \in \mathbb{N}} x_n$ converges. Thus, with $X = \mathbb{C}$ and $x_n = |\langle h, e_n\rangle|^2$, the unordered sum $\sum_{n \in \mathbb{N}} |\langle h, e_n\rangle|^2$ converges, say to S. Because $\sum_{n \in \mathbb{N}} |\langle h, e_n\rangle|^2$ converges to S, by Theorem 3 the series $\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2$ converges to S. But we already know that this series is $\leq \|h\|^2$, so

$$\sum_{n \in \mathbb{N}} |\langle h, e_n\rangle|^2 \leq \|h\|^2 .$$

By Theorem 1, the unordered sum $\sum_{e \in \mathscr{E}} |\langle h, e\rangle|^2$ converges if and only if the unordered sum $\sum_{n \in \mathbb{N}} |\langle h, e_n\rangle|^2$ converges, and if they converge they have the same value. Therefore, the unordered sum $\sum_{e \in \mathscr{E}} |\langle h, e\rangle|^2$ indeed converges, and it is $\leq \|h\|^2$. □

4 Convergence of unordered sums in the strong operator topology

Let H be a Hilbert space and let $\mathscr{B}(H)$ be the set of bounded linear maps $H \to H$. It is straightforward to check that $\mathscr{B}(H)$ is a normed space with the operator norm $\|T\| = \sup_{\|h\| \leq 1} \|Th\|$. (In fact it is a Banach space, actually a Banach algebra, actually a C^*-algebra; each of these statements implies the previous one.) The *strong operator topology* on $\mathscr{B}(H)$ can be characterized in the following way: a net $f : I \to \mathscr{B}(H)$ converges to $T \in \mathscr{B}(H)$ in the strong operator topology if for all $h \in H$ the net $f(i)h$ converges to Th in H.[9]

If I is a set, $\mathscr{F}$ is the set of all finite subsets of I, and $f : I \to \mathscr{B}(H)$ is a function, define $S : \mathscr{F} \to \mathscr{B}(H)$ by

$$S(F) = \sum_{i \in I} f(i) \in \mathscr{B}(H).$$

S is a net in $\mathscr{B}(H)$, and if the net converges to $T \in \mathscr{B}(H)$ in the strong operator topology we say that the unordered sum $\sum_{i \in I} f(i)$ *converges strongly* to T. To say that the net S converges to T in the strong operator topology is to say that if $h \in H$ then $\sum_{i \in I} f(i)h$ converges to Th in H.

If $f, g \in H$, we define $f \otimes g : H \to H$ by

$$f \otimes g(h) = \langle h, g \rangle f.$$

It is apparent that $f \otimes g$ is linear, and

$$\|f \otimes g(h)\| = \|\langle h, g \rangle f\| = |\langle h, g \rangle| \, \|f\| \leq \|h\| \, \|g\| \, \|f\| \, ,$$

so $\|f \otimes g\| \leq \|f\| \, \|g\|$, giving $f \otimes g \in \mathscr{B}(H)$. Additionally,

$$\langle f \otimes g(h_1), h_2 \rangle = \langle \langle h_1, g \rangle f, h_2 \rangle = \langle h_1, g \rangle \langle f, h_2 \rangle = \langle h_1, \langle h_2, f \rangle g \rangle = \langle h_1, g \otimes f(h_2) \rangle,$$

showing that $(f \otimes g)^* = g \otimes f$.

Theorem 10. *If H is a Hilbert space, $\mathscr{E}$ is an orthonormal set in H, and P is the orthogonal projection onto $\bigvee \mathscr{E}$, then $\sum_{e \in \mathscr{E}} e \otimes e$ converges strongly to P.*

Proof. Let $h \in H$. By Theorem 8 there are only countably many $e \in \mathscr{E}$ such that $\langle h, e \rangle \neq 0$, and we denote these by $\{e_n : n \in \mathbb{N}\}$. By Bessel's inequality,

$$\sum_{e \in \mathscr{E}} |\langle h, e \rangle|^2 = \sum_{n \in \mathbb{N}} |\langle h, e_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 \leq \|h\|^2 \, . \tag{2}$$

Let $\mathscr{F}$ be the set of all finite subsets of $\mathbb{N}$ and for $F \in \mathscr{F}$ let

$$S(F) = \sum_{n \in F} \langle h, e_n \rangle e_n \in H.$$

If $\epsilon > 0$, then by (2) there is some N such that $\sum_{n=N+1}^{\infty} |\langle h, e_n \rangle|^2 < \epsilon^2$. If $F_\epsilon = \{1, \ldots, N\}$ and $F, G \in \mathscr{F}$ both contain F_ϵ, then, because the e_n are

[9] For the strong operator topology see John B. Conway, *A Course in Functional Analysis*, second ed., p. 256.

orthonormal,

$$\begin{aligned}\|S(F)-S(G)\|^2 &= \left\|\sum_{n\in F}\langle h,e_n\rangle e_n - \sum_{n\in G}\langle h,e_n\rangle e_n\right\|^2\\ &= \sum_{n\in(F\cup G)\setminus(F\cap G)} \|\langle h,e_n\rangle e_n\|^2\\ &= \sum_{n\in(F\cup G)\setminus(F\cap G)} |\langle h,e_n\rangle|^2\\ &\leq \sum_{n=N+1}^{\infty} |\langle h,e_n\rangle|^2\\ &< \epsilon^2.\end{aligned}$$

Therefore, if $F, G \in \mathscr{F}$ both contain F_ϵ then $\|S(F) - S(G)\| < \epsilon$. This means that S is a Cauchy net, and hence, by Theorem 6, has a limit $v \in H$. That is, the unordered sum $\sum_{n\in\mathbb{N}}\langle h, e_n\rangle e_n$ converges to v.

As the unordered sum $\sum_{n\in\mathbb{N}}\langle h, e_n\rangle e_n$ converges to v we have

$$\lim_{N\to\infty}\sum_{n=1}^{N}\langle h,e_n\rangle e_n = v.$$

If $m \in \mathbb{N}$ then it follows that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\langle h,e_n\rangle\langle e_n,e_m\rangle = \langle v,e_m\rangle,$$

which is

$$\langle h,e_m\rangle = \langle v,e_m\rangle.$$

Let Q be the orthogonal projection onto $\bigvee_{n\in\mathbb{N}}\{e_n\}$. On the one hand, because $\langle h, e\rangle = 0$ for $e \notin \{e_n : n \in \mathbb{N}\}$, we check that $Ph = Qh$. On the other hand, we check that $Qh = v$. Therefore, $v = Ph$, i.e.

$$\sum_{e\in\mathscr{E}} e\otimes e(h) = \sum_{e\in\mathscr{E}}\langle h,e\rangle e = \sum_{n\in\mathbb{N}}\langle h,e_n\rangle e_n = Ph,$$

showing that the unordered sum $\sum_{e\in\mathscr{E}} e \otimes e$ converges strongly to P. □

In particular, if $\mathscr{E}$ is an orthonormal basis for H, then $\sum_{e\in\mathscr{E}} e \otimes e$ converges strongly to id_H.