# The uniform metric on product spaces

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## 1 Metric topology

If $(X,d)$ is a metric space, $a \in X$, and $r > 0$, then the *open ball with center $a$ and radius $r$* is

$$B_r^d(a) = \{x \in X : d(x,a) < r\}.$$

The set of all open balls is a basis for the *metric topology induced by $d$*.

If $(X,d)$ is a metric space, define

$$\overline{d}(a,b) = d(a,b) \wedge 1, \qquad a,b \in X,$$

where $x \wedge y = \min\{x,y\}$. It is straightforward to check that $\overline{d}$ is a metric on $X$, and one proves that $d$ and $\overline{d}$ induce the same metric topologies.[1] The *diameter* of a subset $S$ of a metric space $(X,d)$ is

$$\mathrm{diam}(S,d) = \sup_{a,b \in S} d(a,b).$$

The subset $S$ is said to be *bounded* if its diameter is finite. The metric space $(X,d)$ might be unbounded, but the diameter of the metric space $(X,\overline{d})$ is

$$\mathrm{diam}(X,\overline{d}) = \sup_{a,b \in X} \overline{d}(a,b) = \mathrm{diam}(X,d) \wedge 1,$$

and thus the metric space $(X,\overline{d})$ is bounded.

## 2 Product topology

If $J$ is a set and $X_j$ are topological spaces for each $j \in J$, let $X = \prod_{j \in J} X_j$ and let $\pi_j : X \to X_j$ be the projection maps. A basis for the *product topology* on $X$ are those sets of the form $\bigcap_{j \in J_0} \pi_j^{-1}(U_j)$, where $J_0$ is a finite subset of $J$ and $U_j$ is an open subset of $X_j$, $j \in J_0$. Equivalently, the product topology is the initial topology for the projection maps $\pi_j : X \to X_j$, $j \in J$, i.e. the coarsest topology on $X$ such that each projection map is continuous. Each of the projection maps is open.[2] The following theorem characterizes convergent nets in the product topology.[3]

[1] James Munkres, *Topology*, second ed., p. 121, Theorem 20.1.
[2] John L. Kelley, *General Topology*, p. 90, Theorem 2.
[3] John L. Kelley, *General Topology*, p. 91, Theorem 4.

**Theorem 1.** *Let $J$ be a set and for each $j \in J$ let $X_j$ be a topological space. If $X = \prod_{j \in J} X_j$ has the product topology and $(x_\alpha)_{\alpha \in I}$ is a net in $X$, then $x_\alpha \to x$ if and only if $\pi_j(x_\alpha) \to \pi_j(x)$ for each $j \in J$.*

*Proof.* Let $(x_\alpha)_{\alpha \in I}$ be a net that converges to $x \in X$. Because each projection map is continuous, if $j \in J$ then $\pi_j(x_\alpha) \to \pi_j(x)$. On the other hand, suppose that $(x_\alpha)_{\alpha \in I}$ is a net, that $x \in X$, and that $\pi_j(x_\alpha) \to \pi_j(x)$ for each $j \in J$. Let $\mathscr{O}_j$ be the set of open neighborhoods of $\pi_j(x) \in X_j$. For $j \in J$ and $U \in \mathscr{O}_j$, because $\pi_j(x_\alpha) \to \pi_j(x)$ we have that $\pi_j(x_\alpha)$ is eventually in $U$. It follows that if $j \in J$ and $U \in \mathscr{O}_j$ then $x_\alpha$ is eventually in $\pi_j^{-1}(U)$. Therefore, if $J_0$ is a finite subset of $J$ and $U_j \in \mathscr{O}_j$ for each $j \in J_0$, then $x_\alpha$ is eventually in $\bigcap_{j \in J_0} \pi_j^{-1}(U_j)$. This means that the net $(x_\alpha)_{\alpha \in I}$ is eventually in every basic open neighborhood of $x$, which implies that $x_\alpha \to x$. □

The following theorem states that if $J$ is a countable set and $(X, d)$ is a metric space, then the product topology on $X^J$ is metrizable.[4]

**Theorem 2.** *If $J$ is a countable set and $(X, d)$ is a metric space, then*

$$\rho(x, y) = \sup_{j \in J} \frac{\overline{d}(x_j, y_j)}{j} = \sup_{j \in J} \frac{d(x_j, y_j) \wedge 1}{j}$$

*is a metric on $X^J$ that induces the product topology.*

A topological space is *first-countable* if every point has a countable local basis; a *local basis* at a point $p$ is a set $\mathscr{B}$ of open sets each of which contains $p$ such that each open set containing $p$ contains an element of $\mathscr{B}$. It is a fact that a metrizable topological space is first-countable. In the following theorem we prove that the product topology on an uncountable product of Hausdorff spaces each of which has at least two points is not first-countable.[5] From this it follows that if $(X, d)$ is a metric space with at least two points and $J$ is an uncountable set, then the product topology on $X^J$ is not metrizable.

**Theorem 3.** *If $J$ is an uncountable set and for each $j \in J$ we have that $X_j$ is a Hausdorff space with at least two points, then the product topology on $\prod_{j \in J} X_j$ is not first-countable.*

*Proof.* Write $X = \prod_{j \in J} X_j$, and suppose that $x \in X$ and that $U_n, n \in \mathbb{N}$, are open subsets of $X$ containing $x$. Since $U_n$ is an open subset of $X$ containing $x$, there is a basic open set $B_n$ satisfying $x \in B_n \subseteq U_n$: by saying that $B_n$ is a basic open set we mean that there is a finite subset $F_n$ of $J$ and open subsets $U_{n,j}$ of $X_j$, $j \in F_n$, such that

$$B_n = \bigcap_{j \in F_n} \pi_j^{-1}(U_{n,j}).$$

[4] James Munkres, *Topology*, second ed., p. 125, Theorem 20.5.
[5] cf. John L. Kelley, *General Topology*, p. 92, Theorem 6.

Let $F = \bigcup_{n\in\mathbb{N}} F_n$, and because $J$ is uncountable there is some $k \in J \setminus F$; this is the only place in the proof at which we use that $J$ is uncountable. As $X_k$ has at least two points and $x(k) \in X_k$, there is some $a \in X_k$ with $x(k) \neq a$. Since $X_k$ is a Hausdorff space, there are disjoint open subsets $N_1, N_2$ of $X_k$ with $x(k) \in N_1$ and $a \in N_2$. Define

$$V_j = \begin{cases} N_1 & j = k \\ X_j & j \neq k \end{cases}$$

and let $V = \prod_{j\in J} V_j$. We have $x \in V$. But for each $n \in \mathbb{N}$, there is some $y_n \in B_n$ with $y_n(k) = a \in N_2$, hence $y_n(k) \notin N_1$ and so $y_n \notin V$. Thus none of the sets $B_n$ is contained in $V$, and hence none of the sets $U_n$ is contained in $V$. Therefore $\{U_n : n \in \mathbb{N}\}$ is not a local basis at $x$, and as this was an arbitrary countable set of open sets containing $x$, there is no countable local basis at $x$, showing that $X$ is not first-countable. (In fact, we have proved there is no countable local basis at any point in $X$; not to be first-countable merely requires that there be at least one point at which there is no countable local basis.) $\square$

# 3 Uniform metric

If $J$ is a set and $(X, d)$ is a metric space, we define the *uniform metric* on $X^J$ by

$$d_J(x, y) = \sup_{j\in J} \overline{d}(x_j, y_j) = \sup_{j\in J} d(x_j, y_j) \wedge 1.$$

It is apparent that $d_J(x, y) = 0$ if and only if $x = y$ and that $d_J(x, y) = d_J(y, x)$. If $x, y, z \in X$ then,

$$\begin{aligned} d_J(x, z) &= \sup_{j\in J} \overline{d}(x_j, z_j) \\ &\leq \sup_{j\in J} \overline{d}(x_j, y_j) + \overline{d}(y_j, z_j) \\ &\leq \sup_{j\in J} \overline{d}(x_j, y_j) + \sup_{j\in J} \overline{d}(y_j, z_j) \\ &= d_J(x, y) + d_J(y, z), \end{aligned}$$

showing that $d_J$ satisfies the triangle inequality and thus that it is indeed a metric on $X^J$. The *uniform topology* on $X^J$ is the metric topology induced by the uniform metric.

If $(X, d)$ is a metric space, then $X$ is a topological space with the metric topology, and thus $X^J = \prod_{j\in J} X$ is a topological space with the product topology. The following theorem shows that the uniform topology on $X^J$ is finer than the product topology on $X^J$.[6]

[6] James Munkres, *Topology*, second ed., p. 124, Theorem 20.4.

**Theorem 4.** *If $J$ is a set and $(X, d)$ is a metric space, then the uniform topology on $X^J$ is finer than the product topology on $X^J$.*

*Proof.* If $x \in X^J$, let $U = \prod_{j \in J} U_j$ be a basic open set in the product topology with $x \in U$. Thus, there is a finite subset $J_0$ of $J$ such that if $j \in J \setminus J_0$ then $U_j = X$. If $j \in J_0$, then because $U_j$ is an open subset of $(X, d)$ with the metric topology and $x_j \in U_j$, there is some $0 < \epsilon_j < 1$ such that $B^d_{\epsilon_j}(x_j) \subseteq U_j$. Let $\epsilon = \min_{j \in J_0} \epsilon_j$. If $d_J(x, y) < \epsilon$ then $d(x_j, y_j) < \epsilon$ for all $j \in J$ and hence $d(x_j, y_j) < \epsilon_j$ for all $j \in J_0$, which implies that $y_j \in B^d_{\epsilon_j}(x_j) \subseteq U_j$ for all $j \in J_0$. If $j \in J \setminus J_0$ then $U_j = X$ and of course $y_j \in U_j$. Therefore, if $y \in B^{d_J}_{\epsilon}(x)$ then $y \in U$, i.e. $B^{d_J}_{\epsilon}(x) \subseteq U$. It follows that the uniform topology on $X^J$ is finer than the product topology on $X^J$. $\square$

The following theorem shows that if we take the product of a complete metric space with itself, then the uniform metric on this product space is complete.[7]

**Theorem 5.** *If $J$ is a set and $(X, d)$ is a complete metric space, then $X^J$ with the uniform metric is a complete metric space.*

*Proof.* It is straightforward to check that $(X, d)$ being a complete metric space implies that $(X, \overline{d})$ is a complete metric space. Let $f_n$ be a Cauchy sequence in $(X^J, d_J)$: if $\epsilon > 0$ then there is some $N$ such that $n, m \geq N$ implies that

$$d_J(f_n, f_m) < \epsilon.$$

Thus, if $\epsilon > 0$, then there is some $N$ such that $n, m \geq N$ and $j \in J$ implies that $\overline{d}(f_n(j), f_m(j)) \leq d_J(f_n, f_m) < \epsilon$. Thus, if $j \in J$ then $f_n(j)$ is a Cauchy sequence in $(X, \overline{d})$, which therefore converges to some $f(j) \in X$, and thus $f \in X^J$. If $n, m \geq N$ and $j \in J$, then

$$\begin{aligned} \overline{d}(f_n(j), f(j)) &\leq \overline{d}(f_n(j), f_m(j)) + \overline{d}(f_m(j), f(j)) \\ &\leq d_J(f_n, f_m) + \overline{d}(f_m(j), f(j)) \\ &< \epsilon + \overline{d}(f_m(j), f(j)). \end{aligned}$$

As the left-hand side does not depend on $m$ and $\overline{d}(f_m(j), f(j)) \to 0$, we get that if $n \geq N$ and $j \in J$ then

$$\overline{d}(f_n(j), f(j)) \leq \epsilon.$$

Therefore, if $n \geq N$ then

$$d_J(f_n, f) \leq \epsilon.$$

This means that $f_n$ converges to $f$ in the uniform metric, showing that $(X^J, d_J)$ is a complete metric space. $\square$

[7] James Munkres, *Topology*, second ed., p. 267, Theorem 43.5.

# 4 Bounded functions and continuous functions

If $J$ is a set and $(X, d)$ is a metric space, a function $f : J \to X$ is said to be *bounded* if its image is a bounded subset of $X$, i.e. $f(J)$ has a finite diameter. Let $B(J, X)$ be the set of bounded functions $J \to (X, d)$; $B(J, X)$ is a subset of $X^J$. Since the diameter of $(X, \overline{d})$ is $\leq 1$, any function $J \to (X, \overline{d})$ is bounded, but there might be unbounded functions $J \to (X, d)$. We prove in the following theorem that $B(J, X)$ is a closed subset of $X^J$ with the uniform topology.[8]

**Theorem 6.** *If $J$ is a set and $(X, d)$ is a metric space, then $B(J, X)$ is a closed subset of $X^J$ with the uniform topology.*

*Proof.* If $f_n \in B(J, Y)$ and $f_n$ converges to $f \in X^J$ in the uniform topology, then there is some $N$ such that $d_J(f_N, f) < \frac{1}{2}$. Thus, for all $j \in J$ we have $\overline{d}(f_N(j), f(j)) < \frac{1}{2}$, which implies that

$$d(f_N(j), f(j)) = \overline{d}(f_N(j), f(j)) < \frac{1}{2}.$$

If $i, j \in J$, then

$$\begin{aligned} d(f(i), f(j)) &\leq d(f(i), f_N(i)) + d(f_N(i), f_N(j)) + d(f_N(j), f(j)) \\ &\leq \frac{1}{2} + \operatorname{diam}(f_N(J), d) + \frac{1}{2}. \end{aligned}$$

$f_N \in B(J, X)$ means that $\operatorname{diam}(f_N(J), d) < \infty$, and it follows that $\operatorname{diam}(f(J), d) \leq \operatorname{diam}(f_N(J), d) + 1 < \infty$, showing that $f \in B(J, X)$. Therefore if a sequence of elements in $B(J, X)$ converges to an element of $X^J$, that limit is contained in $B(J, X)$. This implies that $B(J, X)$ is a closed subset of $X^J$ in the uniform topology, as in a metrizable space the closure of a set is the set of limits of sequences of points in the set. $\square$

If $J$ is a set and $Y$ is a complete metric space, we have shown in Theorem 5 that $Y^J$ is a complete metric space with the uniform metric. If $X$ and $Y$ are topological spaces, we denote by $C(X, Y)$ the set of continuous functions $X \to Y$. $C(X, Y)$ is a subset of $Y^X$, and we show in the following theorem that if $Y$ is a metric space then $C(X, Y)$ is a closed subset of $Y^X$ in the uniform topology.[9] Thus, if $Y$ is a complete metric space then $C(X, Y)$ is a closed subset of the complete metric space $Y^X$, and is therefore itself a complete metric space with the uniform metric.

**Theorem 7.** *If $X$ is a topological space and let $(Y, d)$ is a metric space, then $C(X, Y)$ is a closed subset of $Y^X$ with the uniform topology.*

*Proof.* Suppose that $f_n \in C(X, Y)$ and $f_n \to f \in Y^X$ in the uniform topology. Thus, if $\epsilon > 0$ then there is some $N$ such that $n \geq N$ implies that $d_J(f_n, f) < \epsilon$, and so if $n \geq N$ and $x \in X$ then

$$\overline{d}(f_n(x), f(x)) \leq d_J(f_n, f) < \epsilon.$$

[8] James Munkres, *Topology*, second ed., p. 267, Theorem 43.6.
[9] James Munkres, *Topology*, second ed., p. 267, Theorem 43.6.

This means that the sequence $f_n$ converges uniformly in $X$ to $f$ in the uniform metric, and as each $f_n$ is continuous this implies that $f$ is continuous.[10] We have shown that if $f_n \in C(X,Y)$ and $f_n \to f \in Y^X$ in the uniform topology then $f \in C(X,Y)$, and therefore $C(X,Y)$ is a closed subset of $Y^X$ in the uniform topology. $\square$

# 5 Topology of compact convergence

Let $X$ be a topological space and $(Y,d)$ be a metric space. If $f \in Y^X$, $C$ is a compact subset of $X$, and $\epsilon > 0$, we denote by $B_C(f,\epsilon)$ the set of those $g \in Y^X$ such that

$$\sup\{d(f(x),g(x)) : x \in C\} < \epsilon.$$

A basis for the *topology of compact convergence* on $Y^X$ are those sets of the form $B_C(f,\epsilon)$, $f \in Y^X$, $C$ a compact subset of $X$, and $\epsilon > 0$. It can be proved that the uniform topology on $Y^X$ is finer than the topology of compact convergence on $Y^X$, and that the topology of compact convergence on $Y^X$ is finer than the product topology on $Y^X$.[11] Indeed, we have already shown in Theorem 4 that the uniform topology on $Y^X$ is finer than the product topology on $Y^X$. The significance of the topology of compact convergence on $Y^X$ is that a sequence of functions $f_n : X \to Y$ converges in the topology of compact convergence to a function $f : X \to Y$ if and only if for each compact subset $C$ of $X$ the sequence of functions $f_n|C : C \to Y$ converges uniformly in $C$ to the function $f|C : C \to Y$.

[10] See James Munkres, *Topology*, second ed., p. 132, Theorem 21.6.
[11] James Munkres, *Topology*, second ed., p. 285, Theorem 46.7.