Spectral theory, Volterra integral operators and the Sturm-Liouville theorem

Jordan Bell

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1 Banach algebras

Let A be a complex Banach algebra with unit element e. Let $G(A)$ be the set of invertible elements of A. For $x \in A$, the **resolvent set of** x is

$$\rho(x) = \{\lambda \in \mathbb{C} : \lambda e - x \in G(A)\}.$$

The **spectrum of** x is

$$\sigma(x) = \mathbb{C} \setminus \rho(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin G(A)\}.$$

The **spectral radius of** x is

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

One proves that $\sigma(x) \subset \mathbb{C}$ is compact and nonempty and

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n},$$

the **spectral radius formula**.[1] If $r(x) = 0$ we say that x is **quasinilpotent**.[2] $x \in A$ is quasinilpotent if and only if $\sigma(x) = \{0\}$.

Lemma 1. *If $x \in A$ is quasinilpotent and $|\lambda| > 0$, then $S_n = \sum_{j=0}^n \lambda^j x^j \in A$ is a Cauchy sequence, and*

$$(e - \lambda x) \sum_{n=0}^{\infty} \lambda^n x^n = e.$$

Proof. Let $0 < \epsilon < |\lambda|^{-1}$. There is some n_ϵ such that $\|x^n\|^{1/n} \le \epsilon$ for $n \ge n_\epsilon$. For $n > m \ge n_\epsilon$,

$$\|S_n - S_m\| \le \sum_{j=m+1}^{n} |\lambda|^j \left\|x^j\right\| \le \sum_{j=m+1} |\lambda|^j \epsilon^j,$$

[1] Walter Rudin, *Functional Analysis*, second ed., p. 253, Theorem 10.13.

[2] We say that $x \in A$ is **nilpotent** if there is some $n \ge 1$ such that $x^n = 0$, and if x is nilpotent then by the spectral radius formula, x is quasinilpotent.

and because $|\lambda|\epsilon < 1$, it follows that $S_n \in A$ is a Cauchy sequence and so converges to some $S \in A$, $S = \sum_{n=0}^{\infty} \lambda^k x^k$. Now,

$$\begin{aligned}(e-\lambda x)S &= (e-\lambda x)S_n + (e-\lambda x)(S-S_n)\\ &= S_n - \lambda x S_n + (e-\lambda x)(S-S_n)\\ &= S_n - \sum_{j=1}^{n+1} \lambda^j x^j + (e-\lambda x)(S-S_n)\\ &= e - \lambda^{n+1}x^{n+1} + (e-\lambda x)(S-S_n).\end{aligned}$$

Because x is quasinilpotent it follows that $\|(e-\lambda x)S - e\| \to 0$. □

For $x \in A$ and $\lambda \in \rho(x)$, let

$$R_x(\lambda) = (x-\lambda e)^{-1}.$$

Lemma 2. *If $x \in A$ is quasinilpotent and $\lambda \in \mathbb{C}$ then*

$$(e-\lambda x)^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n$$

and if $|\lambda| > 0$ then

$$R_x(\lambda) = -\lambda^{-1}(e-\lambda^{-1}x)^{-1} = -\lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-n} x^n.$$

2 Volterra integral operators

Let $I = [0,1]$ and let μ be Lebesgue measure on I. $C(I)$ is a Banach space with the norm

$$\|f\|_\infty = \sup_{x\in I} |f(x)|, \qquad f \in C(I).$$

$L^1(I)$ is a Banach space with the norm

$$\|f\|_{L^1} = \int_I |f(x)|dx, \qquad f \in L^1(I).$$

For $f : I \to \mathbb{C}$, let

$$|f|_{\mathrm{Lip}} = \sup_{x,y\in I, x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

Let $\mathrm{Lip}(I)$ be the set of those $f : I \to \mathbb{C}$ with $|f|_{\mathrm{Lip}} < \infty$. It is a fact that $\mathrm{Lip}(I)$ is a Banach space with the norm $\|f\|_{\mathrm{Lip}} = \|f\|_\infty + |f|_{\mathrm{Lip}}$.[3]

$$\mathrm{Lip}(I) \subset C(I) \subset L^1(I).$$

[3] Walter Rudin, *Real and Complex Analysis*, third ed., p. 113, Exercise 11.

$A = \mathscr{L}(C(I))$ is a Banach algebra with unit element $e(f) = f$ and with the operator norm:

$$\|T\| = \sup_{f \in C(I), \|f\|_\infty \leq 1} \|Tf\|_\infty, \qquad T \in A.$$

For $K : I \times I \to \mathbb{C}$ and for $x, y \in I$ define

$$K_x(y) = K(x, y), \qquad K^y(x) = K(x, y).$$

Let $K \in C(I \times I)$. For $f \in L^1(I)$ define $V_K f : I \to \mathbb{C}$ by

$$V_K f(x) = \int_0^x K(x, y) f(y) dy, \qquad x \in I.$$

Lemma 3. *If $K \in C(I \times I)$ and $f \in C(I)$ then $V_K f \in C(I)$.*

Proof. For $x_1, x_2 \in I$, $x_1 > x_2$,

$$\begin{aligned} V_K f(x_1) - V_K f(x_2) &= \int_0^{x_1} K(x_1, y) f(y) dy - \int_0^{x_1} K(x_2, y) f(y) dy \\ &+ \int_0^{x_1} K(x_2, y) f(y) dy - \int_0^{x_2} K(x_2, y) f(y) dy \\ &= \int_0^{x_1} \Big[K(x_1, y) - K(x_2, y) \Big] f(y) dy + \int_{x_2}^{x_1} K(x_2, y) f(y) dy. \end{aligned}$$

Let $\epsilon > 0$. Because $K : I \times I \to \mathbb{C}$ is uniformly continuous, there is some $\delta_1 > 0$ such that $|(x_1, y_1) - (x_2, y_2)| \leq \delta_1$ implies $|K(x_1, y_1) - K(x_2, y_2)| \leq \epsilon$. By the absolute continuity of the Lebesgue integral, there is some $\delta_2 > 0$ such that $\mu(E) \leq \delta_2$ implies $\int_E |f| d\mu \leq \epsilon$.[4] Therefore if $|x_1 - x_2| < \delta = \min(\delta_1, \delta_2)$ then

$$\begin{aligned} |V_K f(x_1) - V_K f(x_2)| &\leq \int_0^{x_1} \epsilon |f(y)| dy + \|K\|_\infty \int_{x_2}^{x_1} |f(y)| dy \\ &\leq \epsilon \|f\|_{L^1} + \|K\|_\infty \epsilon. \end{aligned}$$

It follows that $V_K f : I \to \mathbb{C}$ is uniformly continuous, so $V_K f \in C(I)$. □

$\|V_K f\|_\infty \leq \|K\|_\infty \|f\|_\infty$ so $\|V_K\| \leq \|K\|_\infty$, hence $V_K : C(I) \to C(I)$ is a bounded linear operator, namely $V_K \in A$. We call V_K a **Volterra integral operator**.

For $x \in I$,

$$V_K^2 f(x) = \int_0^x K(x, y_1) V_K f(y_1) dy_1 = \int_0^x K(x, y_1) \left(\int_0^{y_1} K(y_1, y_2) f(y_2) dy_2 \right) dy_1.$$

[4] `http://individual.utoronto.ca/jordanbell/notes/L0.pdf`, p. 8, Theorem 8.

$$
\begin{aligned}
V_K^3 f(x) &= V_K^2 V_K f(x) \\
&= \int_0^x K(x,y_1) \int_0^{y_1} K(y_1,y_2) V_K f(y_2) dy_2 dy_1 \\
&= \int_0^x K(x,y_1) \int_0^{y_1} K(y_1,y_2) \int_0^{y_2} K(y_2,y_3) f(y_3) dy_3 dy_2 dy_1.
\end{aligned}
$$

For $n \geq 2$,

$$
V_K^n f(x) = \int_{y_1=0}^x \int_{y_2=0}^{y_1} \cdots \int_{y_n=0}^{y_{n-1}} K(x,y_1)K(y_1,y_2)\cdots K(y_{n-1},y_n) f(y_n) dy_n \cdots dy_1.
$$

We prove that V_K is quasinilpotent.[5]

Theorem 4. *If $K \in C(I \times I)$ then*

$$
\|V_K^n\| \leq \frac{\|K\|_\infty^n}{n!},
$$

and thus $V_K \in A = \mathscr{L}(C(I))$ is quasinilpotent.

Proof. Let

$$
\begin{aligned}
\Phi_n(x) &= \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} dy_n \cdots dy_1 \\
&= \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-2}} y_{n-1} dy_{n-1} \cdots dy_1 \\
&= \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-3}} \frac{y_{n-2}^2}{2} dy_{n-2} \cdots dy_1 \\
&= \int_0^x \frac{y_1^{n-1}}{(n-1)!} dy_1 \\
&= \frac{x^n}{n!}.
\end{aligned}
$$

For $x \in I$,

$$
\begin{aligned}
|V_K^n f(x)| &\leq \|K\|_\infty^n \|f\|_\infty \int_0^x \int_0^{y_1} \cdots \int_0^{y_{n-1}} dy_n \cdots dy_1 \\
&= \|K\|_\infty^n \|f\|_\infty \Phi_n(x) \\
&= \|K\|_\infty^n \|f\|_\infty \frac{x^n}{n!}.
\end{aligned}
$$

Hence

$$
\|V_K^n\| \leq \frac{\|K\|_\infty^n}{n!}.
$$

[5] Barry Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4*, p. 53, Example 2.2.13.

Then

$$\|V_K^n\|^{1/n} \le \frac{\|K\|_\infty}{(n!)^{1/n}}.$$

Using $(n!)^{1/n} \to \infty$ we get $\|V_K^n\|^{1/n} \to 0$. Thus $V_K \in A$ is quasinilpotent. □

Theorem 4 tells us that V_K is quasinilpotent and then Lemma 2 then tells us that for $\lambda \in \mathbb{C}$,

$$(e - \lambda V_K)^{-1} = \sum_{n=0}^{\infty} \lambda^n V_K^n \in A. \tag{1}$$

3 Sturm-Liouville theory

Let $Q \in C(I)$ and for $u \in C^2(I)$ define

$$L_Q u = -u'' + Qu.$$

Lemma 5. *If $u \in C^2(I)$ and*

$$L_Q u = 0, \qquad u(0) = 0, \quad u'(0) = 1,$$

then

$$u(x) = x + \int_0^x (x - y)Q(y)u(y)dy, \qquad x \in I.$$

Proof. For $y \in I$, by the fundamental theorem of calculus[6] and using $u'(0) = 1$,

$$\int_0^y u''(t)dt = u'(y) - u'(0) = u'(y) - 1.$$

Using $L_Q u = 0$,

$$u'(y) = 1 + \int_0^y u''(t)dt = 1 + \int_0^y Q(t)u(t)dt.$$

For $x \in I$, by the fundamental theorem of calculus and using $u(0) = 0$,

$$\int_0^x u'(y)dy = u(x) - u(0) = u(x).$$

Thus

$$\begin{aligned} u(x) &= \int_0^x u'(y)dy \\ &= \int_0^x \left(1 + \int_0^y Q(t)u(t)dt\right) dy \\ &= x + \int_0^x \left(\int_0^y Q(t)u(t)dt\right) dy. \end{aligned}$$

[6] Walter Rudin, *Real and Complex Analysis*, third ed., p. 149, Theorem 7.21.

Applying Fubini's theorem,

$$\begin{aligned} u(x) &= x + \int_0^x Q(t)u(t)\left(\int_t^x dy\right)dt \\ &= x + \int_0^x Q(t)u(t)(x-t)dt. \end{aligned}$$

□

Lemma 6. *If $u \in C(I)$ and*

$$u(x) = x + \int_0^x (x-y)Q(y)u(y)dy, \qquad x \in I,$$

then $u \in C^2(I)$ and

$$L_Q u = 0, \qquad u(0) = 0, \quad u'(0) = 1.$$

Proof.

$$u(x) = x + \int_0^x (x-y)Q(y)u(y)dy, \qquad x \in I,$$

then

$$u(x) = x + x\int_0^x Q(y)u(y)dy - \int_0^x yQ(y)u(y)dy,$$

and using the fundamental theorem of calculus,

$$u'(x) = 1 + \int_0^x Q(y)u(y)dy + xQ(x)u(x) - xQ(x)u(x) = 1 + \int_0^x Q(y)u(y)dy$$

hence

$$u''(x) = Q(x)u(x),$$

and so

$$L_Q u = -u'' + Qu = -Qu + Qu = 0.$$

$u(0) = 0$ and $u'(0) = 1$, so

$$L_Q u = 0, \qquad u(0) = 0, \quad u'(0) = 1.$$

□

Lemma 7. *Let $Q \in C(I)$ and let $K(x,y) = (x-y)Q(y)$, $K \in C(I \times I)$. Let $u_0(x) = x$, $u_0 \in C(I)$. Then $\sum_{j=0}^n V_K^j$ is a Cauchy sequence in $A = \mathscr{L}(C(I))$, and $u = \sum_{n=0}^\infty V_K^n u_0 \in C(I)$ satisfies $u = (e - V_K)^{-1}u_0$.*

Proof. $V_K \in C(I)$ is quasinilpotent so applying (1) with $\lambda = 1$,

$$(e - V_K)^{-1} = \lim_{n\to\infty} \sum_{j=0}^{n} V_K^j \in A.$$

Then

$$(e - V_K)^{-1} u_0 = \left(\lim_{n\to\infty} \sum_{j=0}^{n} V_K^j \right) u_0 = \lim_{n\to\infty} (V_K^j u_0) = \sum_{n=0}^{\infty} V_K^n u_0.$$

Hence $u = (1 - V_K)^{-1} u_0$, and so $(1 - V_K)u = u_0$, i.e. $u = u_0 + V_K u$, i.e. for $x \in I$,

$$u(x) = u_0(x) + \int_0^x K(x,y)u(y)dy.$$

□

Theorem 8. *Let $Q \in C(I)$ and let $K(x,y) = (x-y)Q(y)$, $K \in C(I \times I)$. Let $u_0(x) = x$, $u_0 \in C(I)$. Then $\sum_{j=0}^{n} V_K^j$ is a Cauchy sequence in $A = \mathscr{L}(C(I))$, and $u = \sum_{n=0}^{\infty} V_K^n u_0 \in C(I)$ satisfies $u \in C^2(I)$,*

$$L_Q u = 0, \qquad u(0) = 0, \quad u'(0) = 1.$$

Proof. By Lemma 7, $u = (e - V_K)^{-1} u_0$, i.e. $(e - V_K)u = u_0$, i.e. $u - V_K u = u_0$, i.e. for $x \in I$,

$$u(x) = x + V_K u(x) = x + \int_0^x K(x,y)u(y)dy = x + \int_0^x (x-y)Q(y)u(y)dy.$$

Lemma 6 then tells us that $u \in C^2(I)$ and

$$L_Q u = 0, \qquad u(0) = 0, \quad u'(0) = 1.$$

□

4 Gronwall's inequality

Let $f \in L^1(I)$. We say that $x \in I$ is a **Lebesgue point of** f if

$$\frac{1}{r} \int_x^{x+r} |f(y) - f(x)| dy \to 0, \qquad r \to 0,$$

which implies

$$\frac{1}{r} \int_x^{x+r} f(y) dy \to f(x), \qquad r \to 0.$$

The **Lebesgue differentiation theorem**[7] states that for almost all $x \in I$, x is a Lebesgue point of f. Let

$$F(x) = \int_0^x f(y)dy, \qquad x \in I,$$

so

$$F(x+r) - F(x) = \int_x^{x+r} f(y)dy.$$

If x is a Lebesgue point of f then

$$\frac{F(x+r) - F(x)}{r} = \frac{1}{r} \int_x^{x+r} f(y)dy \to f(x),$$

which means that if x is a Lebesgue point of f then

$$F'(x) = f(x).$$

We now prove **Gronwall's inequality**.[8]

Theorem 9 (Gronwall's inequality)**.** *Let $g \in L^1(I)$, $g \geq 0$ almost everywhere and let $f : I \to \mathbb{R}$ be continuous. If $y : I \to \mathbb{R}$ is continuous and*

$$y(t) \leq f(t) + \int_0^t g(s)y(s)ds, \qquad t \in I,$$

then

$$y(t) \leq f(t) + \int_0^t f(s)g(s) \exp\left(\int_s^t g(u)du\right) ds, \qquad t \in I.$$

If f is increasing then

$$y(t) \leq f(t) \exp\left(\int_0^t g(s)ds\right), \qquad t \in I.$$

Proof. Let $z(t) = g(t)y(t)$ and

$$Z(t) = \int_0^t z(s)ds, \qquad t \in I.$$

By hypothesis, $g \geq 0$ almost everywhere, and by the Lebesgue differentiation theorem, $Z'(t) = z(t)$ for almost all $t \in I$. Therefore for almost all $t \in I$,

$$Z'(t) = z(t) = g(t)y(t) \leq g(t)\left(f(t) + \int_0^t g(s)y(s)ds\right) = g(t)f(t) + g(t)Z(t).$$

That is, there is a Borel set $E \subset I$, $\mu(E) = 1$, such that for $t \in I$, Z is differentiable at t and

$$Z'(t) - g(t)Z(t) \leq g(t)f(t).$$

[7] Walter Rudin, *Real and Complex Analysis*, third ed., p. 138, Theorem 7.7

[8] Anton Zettl, *Sturm-Liouville Theory*, p. 8, Theorem 1.4.1.

For $s \in E$, using the product rule,

$$\left[\exp\left(-\int_0^s g(u)du\right) Z(s)\right]' = \exp\left(-\int_0^s g(u)du\right) \left[Z'(s) - g(t)Z(s)\right].$$

For $t \in I$, as $\mu(E) = 1$,

$$\begin{aligned}
&\int_0^t \left[\exp\left(-\int_0^s g(u)du\right) Z(s)\right]' ds \\
&= \int_{[0,t]\cap E} \left[\exp\left(-\int_0^s g(u)du\right) Z(s)\right]' ds \\
&= \int_{[0,t]\cap E} \exp\left(-\int_0^s g(u)du\right) \left[Z'(s) - g(s)Z(s)\right] ds \\
&\leq \int_{[0,t]\cap E} \exp\left(-\int_0^s g(u)du\right) g(s)f(s)ds \\
&= \int_0^t g(s)f(s)\exp\left(-\int_0^s g(u)du\right) ds.
\end{aligned}$$

But

$$\begin{aligned}
\int_0^t \left[\exp\left(-\int_0^s g(u)du\right) Z(s)\right]' ds &= \left[\exp\left(-\int_0^s g(u)du\right) Z(s)\right]\Bigg|_0^t \\
&= \exp\left(-\int_0^t g(u)du\right) Z(t).
\end{aligned}$$

So

$$\exp\left(-\int_0^t g(u)du\right) Z(t) \leq \int_0^t g(s)f(s)\exp\left(-\int_0^s g(u)du\right) ds.$$

Therefore,

$$\begin{aligned}
y(t) &\leq f(t) + \int_0^t g(s)y(s)ds \\
&= f(t) + Z(t) \\
&\leq f(t) + \exp\left(\int_0^t g(u)du\right) \int_0^t g(s)f(s)\exp\left(-\int_0^s g(u)du\right) ds \\
&= f(t) + \int_0^t g(s)f(s)\exp\left(\int_0^t g(u)du - \int_0^s g(u)du\right) ds \\
&= f(t) + \int_0^t g(s)f(s)\exp\left(\int_s^t g(u)du\right) ds.
\end{aligned}$$

Suppose that f is increasing. Let

$$G(s) = \int_0^s g(u)du, \qquad s \in I.$$

For $t \in I$,

$$\begin{aligned}
y(t) &\leq f(t) + \int_0^t g(s) f(s) \exp\left(\int_s^t g(u)du\right) ds \\
&\leq f(t) + \int_0^t g(s) f(t) \exp\left(\int_s^t g(u)du\right) ds \\
&= f(t)\left[1 + \int_0^t g(s) \exp\left(\int_s^t g(u)du\right) ds\right] \\
&= f(t)\left[1 + \int_0^t g(s) e^{G(t)-G(s)} ds\right] \\
&= f(t)\left[1 + e^{G(t)} \int_0^t g(s) e^{-G(s)} ds\right].
\end{aligned}$$

Let $H(s) = e^{-G(s)}$, with which

$$y(t) \leq f(t)\left[1 + \frac{1}{H(t)} \int_0^t g(s) H(s) ds\right].$$

If s is a Lebesgue point of g then

$$H'(s) = -G'(s) e^{-G(s)} = -g(s) H(s).$$

Hence

$$\begin{aligned}
y(t) &\leq f(t)\left[1 - \frac{1}{H(t)} \int_0^t H'(s) ds\right] \\
&= f(t)\left[1 - \frac{1}{H(t)} \left[H(t) - H(0)\right]\right] \\
&= f(t)\left[1 - 1 + \frac{H(0)}{H(t)}\right] \\
&= f(t) e^{G(t)} \\
&= f(t) \exp\left(\int_0^t g(u)du\right).
\end{aligned}$$

□

Let $K(x, y) = (x - y)Q(y)$. Let $u = \sum_{n=0}^{\infty} V_K^n u_0 \in C(I)$. Lemma 7 tells us that $u = (e - V_K)^{-1} u_0$, i.e. $(e - V_K)u = u_0$, i.e. $u = u_0 + V_K u$, i.e. for $x \in I$,

$$u(x) = x + \int_0^x (x - y) Q(y) u(y) dy.$$

Then

$$|u(x)| \leq x + \int_0^x |x - y||Q(y)||u(y)| dy \leq x + \int_0^x |Q(y)||u(y)| dy.$$

Applying Gronwall's inequality we get

$$|u(x)| \leq x \exp\left(\int_0^x |Q(y)| dy\right), \qquad x \in I. \tag{2}$$

5 The spectral theorem for positive compact operators

The following is the **spectral theorem for positive compact operators**.[9]

Theorem 10 (Spectral theorem for positive compact operators)**.** *Let H be a separable complex Hilbert space and let $T \in \mathscr{L}(H)$ be positive and compact. There are countable sets $\Phi, \Psi \subset H$ and $\lambda_\phi > 0$ for $\phi \in \Phi$ such that (i) $\Phi \cup \Psi$ is an orthonormal basis for H, (ii) $T\phi = \lambda_\phi \phi$ for $\phi \in \Phi$, (iii) $T\psi = 0$ for $\psi \in \Psi$, (iv) if Φ is infinite then 0 is a limit point of Λ and is the only limit point of Λ.*

Suppose that H is infinite dimensional and that T is a positive compact operator with $\ker(T) = 0$. The spectral theorem for positive compact operators then says that there is a a countable set $\Phi \subset H$ and $\lambda_\phi > 0$ for $\phi \in \Phi$ such that Φ is an orthonormal basis for H, $T\phi = \lambda_\phi \phi$ for $\phi \in \Phi$, and the unique limit point of $\{\lambda_\phi : \phi \in \Phi\}$ is 0. Let $\Phi = \{\phi_n : n \geq 1\}$, $\phi_n \neq \phi_m$ for $n \geq m$, such that $n \geq m$ implies $\lambda_{\phi_n} \leq \lambda_{\phi_m}$. Let $\lambda_n = \lambda_{\phi_n}$. Then $\lambda_n \downarrow 0$. Summarizing, there is an orthonormal basis $\{\phi_n : n \geq 1\}$ for H and $\lambda_n > 0$ such that $T\phi_n = \lambda_n \phi_n$ for $n \geq 1$ and $\lambda_n \downarrow 0$.

6 $Q > 0$, Green's function for L_Q

Suppose $Q \in C(I)$ with $Q(x) > 0$ for $0 < x < 1$. Let $K(x, y) = (x - y)Q(y)$, $K \in C(I \times I)$, and $u_0(x) = x$, $u_0 \in C(I)$. Let

$$u = \sum_{n=0}^{\infty} V_K^n u_0 \in C(I).$$

By Theorem 8, $u \in C^2(I)$ and

$$L_Q u = 0, \qquad u(0) = 0, \quad u'(0) = 1.$$

If $f \in C(I)$ and $f(x) > 0$ for $0 < x < 1$ then

$$V_K f(x) = \int_0^x (x - y)Q(y)f(y)dy > 0.$$

By induction, for $0 < x < 1$ and for $n \geq 1$ we have $V_K^n f(x) > 0$. Hence for $0 < x < 1$,

$$u(x) = \sum_{n=0}^{\infty} (V_K^n u_0)(x) > 0.$$

For $x \in I$,

$$u(x) = x + \int_0^x (x - y)Q(y)u(y)dy = x + x\int_0^x Q(y)u(y)dy - \int_0^x yQ(y)u(y)dy.$$

[9] Barry Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4*, p. 102, Theorem 3.2.1.

Using the fundamental theorem of calculus,

$$u'(x) = 1 + \int_0^x Q(y)u(y)dy.$$

Then because $Q(y) > 0$ for $0 < y < 1$ and $u(y) > 0$ for $0 < y < 1$,

$$u'(x) > 1, \qquad 0 < x < 1.$$

Using $u(x) = x + \int_0^x (x-y)Q(y)u(y)dy$ and $Q > 0$ we get

$$u(x) > x, \qquad 0 < x < 1.$$

Let $u_1(x) = u(x)$ and $u_2(x) = u(1-x)$. Then

$$L_Q u_1 = 0, \qquad u_1(0) = 0, \qquad u_1'(0) = 1$$

and

$$L_Q u_2 = 0, \qquad u_2(1) = 0, \qquad u_2'(1) = -1.$$

A fortiori,

$$u_1(x) > 0, \qquad u_1'(x) > 0, \qquad 0 < x < 1,$$

and as $u_2'(x) = -u'(1-x)$,

$$u_2(x) > 0, \qquad u_2'(x) < 0, \qquad 0 < x < 1.$$

For $0 < x < 1$ let

$$W(x) = u_1'(x)u_2(x) - u_1(x)u_2'(x).$$

$u_1' > 0, u_2 > 0$ so $u_1'u_2 > 0$. $u_1 > 0, u_2' < 0$ so $-u_1u_2' > 0$, hence $W > 0$.

$$\begin{aligned} W' &= (u_1'u_2 - u_1u_2')' \\ &= u_1''u_2 + u_1'u_2' - u_1'u_2' - u_1u_2'' \\ &= u_1''u_2 - u_1u_2'' \\ &= (Qu_1)u_2 - u_1(Qu_2) \\ &= 0. \end{aligned}$$

Therefore there is some $W_0 > 0$ such that $W(x) = W_0$ for all $0 < x < 1$.

Define

$$G(x,y) = \frac{u_1(x \wedge y)u_2(x \vee y)}{W_0}, \qquad (x,y) \in I \times I.$$

$x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$. Because $(x,y) \mapsto x \wedge y$ and $(x,y) \mapsto x \vee y$ are each continuous $I \times I \to I$, it follows that $G \in C(I \times I)$. $G(x,y) = G(y,x)$.

G is the **Green's function for** L_Q. Let $(x,y) \in I \times I$. If $x > y$ then

$$G^y(x) = \frac{u_1(y)u_2(x)}{W_0}$$

and so

$$L_Q G^y(x) = \frac{u_1(y)}{W_0} L_Q u_2(x) = 0.$$

If $x < y$ then

$$G^y(x) = \frac{u_1(x)u_2(y)}{W_0}$$

and so

$$L_Q G^y(x) = \frac{u_2(y)}{W_0} L_Q u_1(x) = 0.$$

7 $Q > 0$, $L^2(I)$

$L^2(I)$ is a separable complex Hilbert space with the inner product

$$\langle f, g \rangle = \int_I f\overline{g} d\mu, \qquad f, g \in L^2(I).$$

Define $T_Q : L^2(I) \to L^2(I)$ by

$$(T_Q g)(x) = \int_I G(x,y) g(y) dy.$$

$T_Q : L^2(I) \to L^2(I)$ is a Hilbert-Schmidt operator.[10]

It is immediate that $G(y,x) = G(x,y)$ and $\overline{G} = G$. Then by Fubini's theorem, for $f, g \in L^2(I)$,

$$\begin{aligned}
\langle T_Q g, f \rangle &= \int_I (T_Q g)(x) \overline{f(x)} dx \\
&= \int_I \left(\int_I G(x,y) g(y) dy \right) \overline{f(x)} dx \\
&= \int_I g(y) \overline{\left(\int_I G(y,x) f(x) dx \right)} dy \\
&= \int_I g(y) \overline{(T_Q f)(y)} dy \\
&= \langle g, T_Q f \rangle .
\end{aligned}$$

Therefore $T_Q : L^2(I) \to L^2(I)$ is self-adjoint.

We now establish properties of T_Q.[11] Let

$$N^k(I) = \{ f \in C^k(I) : f(0) = 0, f(1) = 0 \}.$$

[10]Barry Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4*, p. 96, Theorem 3.1.16.

[11]Barry Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4*, p. 106, Proposition 3.2.8.

Lemma 11. *Let $Q \in C(I)$, $Q(x) > 0$ for $0 < x < 1$. Let $g \in L^2(I)$ and let $f = T_Q g$,*

$$f(x) = (T_Q g)(x) = \int_I G(x,y)g(y)dy = \int_I G_x g d\mu.$$

Then $f \in N^0(I)$.

If $g \in C(I)$ then $f \in C^2(I)$ and

$$L_Q f = g.$$

Proof. For $x \in I$,

$$\begin{aligned} f(x) &= \int_0^x \frac{u_1(x \wedge y)u_2(x \vee y)}{W_0} g(y)dy + \int_x^1 \frac{u_1(x \wedge y)u_2(x \vee y)}{W_0} g(y)dy \\ &= \int_0^x \frac{u_1(y)u_2(x)}{W_0} g(y)dy + \int_x^1 \frac{u_1(x)u_2(y)}{W_0} g(y)dy \\ &= u_2(x)\int_0^x \frac{u_1(y)g(y)}{W_0}dy + u_1(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy. \end{aligned}$$

It follows that $f \in C(I)$.

Suppose $g \in C(I)$. Then by the fundamental theorem of calculus,

$$\begin{aligned} f'(x) = u_2'(x)&\int_0^x \frac{u_1(y)g(y)}{W_0}dy + u_2(x)\frac{u_1(x)g(x)}{W_0} \\ &+ u_1'(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy - u_1(x)\frac{u_2(x)g(x)}{W_0} \\ &= u_2'(x)\int_0^x \frac{u_1(y)g(y)}{W_0}dy + u_1'(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy. \end{aligned}$$

Because $u_1', u_2' \in C(I)$ it follows that $f' \in C(I)$, i.e. $f \in C^1(I)$. Then

$$\begin{aligned} f''(x) = u_2''(x)&\int_0^x \frac{u_1(y)g(y)}{W_0}dy + u_2'(x)\frac{u_1(x)g(x)}{W_0} \\ &+ u_1''(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy - u_1'(x)\frac{u_2(x)g(x)}{W_0} \\ &= u_2''(x)\int_0^x \frac{u_1(y)g(y)}{W_0}dy + u_1''(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy - \frac{W(x)g(x)}{W_0} \\ &= u_2''(x)\int_0^x \frac{u_1(y)g(y)}{W_0}dy + u_1''(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy - g(x). \end{aligned}$$

Because $g \in C(I)$ it follows that $f'' \in C(I)$, i.e. $f \in C^2(I)$. Furthermore, because $u_1'' = Qu_1$ and $u_2'' = Qu_2$,

$$\begin{aligned} f''(x) &= Q(x)u_2(x)\int_0^x \frac{u_1(y)g(y)}{W_0}dy + Q(x)u_1(x)\int_x^1 \frac{u_2(y)g(y)}{W_0}dy - g(x) \\ &= Q(x)f(x) - g(x). \end{aligned}$$

□

We now establish more facts about T_Q.[12]

Lemma 12. *Let $Q \in C(I)$, $Q(x) > 0$ for $0 < x < 1$.*

1. *If $f_1, f_2 \in N^2(I)$ then*
$$\int_I f_1 L_Q f_2 dx = \int_I (f_1' f_2' + Q f_1 f_2) dx.$$
2. *If $f \in N^2(I)$ and $L_Q f = 0$, then $f = 0$.*
3. *If $f \in N^2(I)$ then $f = T_Q L_Q f$.*
4. *$T_Q \geq 0$.*
5. *$\ker T_Q = 0$.*

Proof. First, doing integration by parts,

$$\begin{aligned}\int_I f_1(-f_2'' + Q f_2) dx &= -\int_{\partial I} f_1 f_2' + \int_I f_1' f_2' dx + \int_I Q f_1 f_2 dx \\ &= \int_I f_1' f_2' dx + \int_I Q f_1 f_2 dx \\ &= \int_I (f_1' f_2' + Q f_1 f_2) dx.\end{aligned}$$

Second, using the above with $f_1 = f$ and $f_2 = f$, with $f \in C^2(I)$ real-valued,

$$\int_I f(-f'' + Qf) dx = \int_I (|f'|^2 + Q|f|^2) dx.$$

Using $-f'' + Qf = 0$,

$$\int_I (|f'|^2 + Q|f|^2) dx = 0.$$

Because $Q(x) > 0$ for $0 < x < 1$, it follows that $|f| = 0$ almost everywhere. But f is continuous so $f = 0$. For $f = f_1 + i f_2$, if $-f'' + Qf = 0$ and $f(0) = 0, f(1) = 0$ then as Q is real-valued, we get $f_1 = 0$ and $f_2 = 0$ hence $f = 0$.

Third, say $f \in C^2(I)$ is real-valued, $f(0) = 0$, $f(1) = 0$, and $g = L_Q f = -f'' + Qf \in C(I)$. Let $h = T_Q g$. By Lemma 11, $h \in C^2(I)$ and

$$-h'' + Qh = g, \qquad h(0) = 0, \qquad h(1) = 0.$$

Let $F = f - h$. Then using $-f'' + Qf = g$ we get

$$F'' = f'' - h'' = (Qf - g) - (Qh - g) = Q(f - h) = QF.$$

[12] Barry Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4*, p. 107, Proposition 3.2.9.

Furthermore,

$$F(0) = f(0) - h(0) = 0 - 0 = 0, \qquad F(1) = f(1) - h(1) = 0 - 0 = 0.$$

Because f is real-valued so is g, and because g is real-valued it follows that $h = T_Q g$ is real-valued. Thus F is real-valued and so by the above, $F = 0$. That is, $f = h$, i.e. $f = T_Q g$. For $f = f_1 + if_2$, if $f(0) = 0$, $f(1) = 0$ and $g = -f'' + Qf$, let $g = g_1 + ig_2$. As Q is real-valued we get $g_1 = -f_1'' + Qf_1$ and $g_2 = -f_2'' + Qf_2$. Then $f_1 = T_Q g_1$ and $f_2 = T_Q g_2$. Thus

$$f = f_1 + if_2 = T_Q g_1 + iT_Q g_2 = T_Q(g_1 + ig_2) = T_Q g.$$

Fourth, let $g \in C(I)$ and let $f = T_Q g$. By Lemma 11, $f \in C^2(I)$ and

$$-f'' + Qf = g, \qquad f(0) = 0, \qquad f(1) = 0.$$

Then using the above,

$$\begin{aligned}
\langle g, T_Q g \rangle &= \langle -f'' + Qf, f \rangle \\
&= \int_I (-f'' + Qf)\overline{f} dx \\
&= \int_I (\overline{f}' f' + Q\overline{f} f) dx \\
&= \int_I (|f'|^2 + Q|f|^2) dx.
\end{aligned}$$

Because $Q \geq 0$ we have $\langle g, T_Q g \rangle \geq 0$. For $g \in L^2(I)$ let $g_n \in C(I)$ with $\|g_n - g\|_{L^2} \to 0$. Then $\langle g_n, T_Q g_n \rangle \to \langle g, T_Q g \rangle$ as $n \to \infty$, and because $\langle g_n, T_Q g_n \rangle \geq 0$ it follows that $\langle g, T_Q g \rangle \geq 0$. Therefore $T_Q \geq 0$, namely T_Q is a positive operator.

Let $f \in N^2$ and let $g = -f'' + Qf$. Then $f = T_Q g$. This means that $N^2 \subset \mathrm{Ran}(T_Q)$. One checks that N^2 is dense in $L^2(I)$, so $\mathrm{Ran}(T_Q)$ is dense in $L^2(I)$. If $f \in \ker(T_Q)$ and $g \in L^2(I)$ then $\langle f, T_Q^* g \rangle = \langle T_Q f, g \rangle = 0$. Hence $\ker(T_Q) \perp \mathrm{Ran}(T_Q^*)$. But T_Q is self-adjoint which implies that $\ker(T_Q) \perp \mathrm{Ran}(T_Q)$. Because $\mathrm{Ran}(T_Q)$ is dense in $L^2(I)$ it follows that $\ker(T_Q) = 0$. □

We now prove the **Sturm-Liouville theorem.**[13]

Theorem 13 (Sturm-Liouville theorem)**.** *Let $Q \in C(I)$, $Q(x) > 0$ for $0 < x < 1$. There is an orthonormal basis $\{u_n : n \geq 1\} \subset N^2(I)$ for $L^2(I)$ and $\lambda_n > 0$, $\lambda_m < \lambda_n$ for $m < n$ and $\lambda_n \to \infty$, such that*

$$L_Q u_n = \lambda_n u_n, \qquad n \geq 1.$$

[13] Barry Simon, *Operator Theory. A Comprehensive Course in Analysis, Part 4*, p. 105, Theorem 3.2.7, p. 110, Exercise 7.

Proof. We have established that T_Q is a positive compact operator with $\ker T_Q = 0$. The spectral theorem for positive compact operators then tells us that there is an orthonormal basis $\{\phi_n : n \geq 1\}$ for $L^2(I)$ and $\gamma_n > 0$ such that $T_Q\phi_n = \gamma_n\phi_n$ for $n \geq 1$ and $\gamma_n \downarrow 0$. By Lemma 11, $T_Q\phi_n \in N^0(I)$. Let

$$u_n = \frac{1}{\gamma_n} T_Q \phi_n \in N^0(I).$$

Because $T_Q\phi_n = \gamma_n\phi_n$ we have $u_n = \phi_n$ in $L^2(I)$ and so

$$u_n = \frac{1}{\gamma_n} T_Q u_n.$$

Let $v_n = T_Q u_n$. Because $u_n \in C(I)$, Lemma 11 tells us that $v_n \in N^2(I)$ and $L_Q v_n = u_n$. But $u_n = \frac{1}{\gamma_n} v_n$ so $u_n \in N^2(I)$ and

$$L_Q u_n = \frac{1}{\gamma_n} L_Q v_n = \frac{1}{\gamma_n} u_n.$$

Let $\lambda_n = \frac{1}{\gamma_n}$. Then $\lambda_n > 0$, $\lambda_m \leq \lambda_n$ for $m \leq n$, $\lambda_n \to \infty$, and

$$L_Q u_n = \lambda_n u_n, \qquad n \geq 1.$$

To prove the claim it remains to show that the sequence λ_n is strictly increasing.

Let $\lambda > 0$ and suppose that $f, g \in N^2(I)$ satisfy

$$L_Q f = \lambda f, \qquad L_Q g = \lambda g.$$

Let $W(x) = f(x)g'(x) - g(x)f'(x)$, the Wronskian of f and g. Either $W(x) = 0$ for all $x \in I$ or $W(x) \neq 0$ for all $x \in I$. Using $f(0) = 0$ and $g(0) = 0$ we get $W(0) = 0$. Therefore $W(x) = 0$ for all $x \in I$ and $W = 0$ implies that f, g are linearly dependent.

Suppose by contradiction that $\lambda_n = \lambda_m$ for some $n \neq m$. Applying the above with $\lambda = \lambda_n = \lambda_m$, $f = u_n, g = u_m$ we get that u_n, u_m are linearly dependent, contradicting that $\{u_n : n \geq 1\}$ is an orthonormal set. Therefore $m \neq n$ implies that $\lambda_m \neq \lambda_n$. □

8 Other results in Sturm-Liouville theory

[14]

[14] B. M. Levitan and I. S. Sargsjan, *Spectral Theory: Selfadjoint Ordinary Differential Operators*, p. 11.