The Stone-Čech compactification of Tychonoff spaces

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June 27, 2014

1 Completely regular spaces and Tychonoff spaces

A topological space X is said to be **completely regular** if whenever F is a nonempty closed set and $x \in X \setminus F$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$. A completely regular space need not be Hausdorff. For example, if X is any set with more than one point, then the trivial topology, in which the only closed sets are $\emptyset$ and X, is vacuously completely regular, but not Hausdorff. A topological space is said to be a **Tychonoff space** if it is completely regular and Hausdorff.

Lemma 1. *A topological space X is completely regular if and only if for any nonempty closed set F, any $x \in X \setminus F$, and any distinct $a, b \in \mathbb{R}$, there is a continuous function $f : X \to \mathbb{R}$ such $f(x) = a$ and $f(F) = \{b\}$.*

Theorem 2. *If X is a Hausdorff space and $A \subset X$, then A with the subspace topology is a Hausdorff space. If $\{X_i : i \in I\}$ is a family of Hausdorff spaces, then $\prod_{i \in I} X_i$ is Hausdorff.*

Proof. Suppose that a, b are distinct points in A. Because X is Hausdorff, there are disjoint open sets U, V in X with $a \in U, b \in V$. Then $U \cap A, V \cap A$ are disjoint open sets in A with the subspace topology and $a \in U \cap A, b \in V \cap A$, showing that A is Hausdorff.

Suppose that x, y are distinct elements of $\prod_{i \in I} X_i$. x and y being distinct means there is some $i \in I$ such that $x(i) \neq y(i)$. Then $x(i), y(i)$ are distinct points in X_i, which is Hausdorff, so there are disjoint open sets U_i, V_i in X_i with $x(i) \in U_i, y(i) \in V_i$. Let $U = \pi_i^{-1}(U_i), V = \pi_i^{-1}(V_i)$, where π_i is the projection map from the product to X_i. U and V are disjoint, and $x \in U, y \in V$, showing that $\prod_{i \in I} X_i$ is Hausdorff. $\square$

We prove that subspaces and products of completely regular spaces are completely regular.[1]

[1] Stephen Willard, *General Topology*, p. 95, Theorem 14.10.

Theorem 3. *If X is Hausdorff and $A \subset X$, then A with the subspace topology is completely regular. If $\{X_i : i \in I\}$ is a family of completely regular spaces, then $\prod_{i\in I} X_i$ is completely regular.*

Proof. Suppose that F is closed in A with the subspace topology and $x \in A \setminus F$. There is a closed set G in X with $F = G \cap A$. Then $x \notin G$, so there is a continuous function $f : X \to [0, 1]$ satisfying $f(x) = 0$ and $f(F) = \{1\}$. The restriction of f to A with the subspace topology is continuous, showing that A is completely regular.

Suppose that F is a closed subset of $X = \prod_{i\in I} X_i$ and that $x \in X \setminus F$. A base for the product topology consists of intersections of finitely many sets of the form $\pi_i^{-1}(U_i)$ where $i \in I$ and U_i is an open subset of X_i, and because $X \setminus F$ is an open neighborhood of x, there is a finite subset J of I and open sets U_j in X_j for $j \in J$ such that

$$x \in \bigcap_{j\in J} \pi_j^{-1}(U_j) \subset X \setminus F.$$

For each $j \in J$, $X_j \setminus U_j$ is closed in X_j and $x(j) \in U_j$, and because X_j is completely regular there is a continuous function $f_j : X_j \to [0, 1]$ such that $f_j(x(j)) = 0$ and $f_j(X_j \setminus U_j) = \{1\}$. Define $g : X \to [0, 1]$ by

$$g(y) = \max_{j\in J}(f_j \circ \pi_j)(y), \qquad y \in X.$$

In general, suppose that Y is a topological space and denote by $C(Y)$ the set of continuous functions $Y \to \mathbb{R}$. It is a fact that $C(Y)$ is a **lattice** with the partial order $F \leq G$ when $F(y) \leq G(y)$ for all $y \in Y$. Hence, the maximum of finitely many continuous functions is also a continuous functions, hence $g : X \to [0, 1]$ is continuous. Because $(f_j \circ \pi_j)(x) = 0$ for each $j \in J$, $g(x) = 0$. On the other hand, $F \subset X \setminus \bigcap_{j\in J} \pi_j^{-1}(U_j)$, so if $y \in F$ then there is some $j \in J$ such that $\pi_j(y) \in X_j \setminus U_j$ and then $(f_j \circ \pi_j)(y) = 1$. Hence, for any $y \in F$ we have $g(y) = 1$. Thus we have proved that $g : X \to [0, 1]$ is a continuous function such that $g(x) = 0$ and $g(F) = \{1\}$, which shows that X is completely regular. $\square$

Therefore, subspaces and products of Tychonoff spaces are Tychonoff.

If X is a normal topological space, it is immediate from **Urysohn's lemma** that X is completely regular. A metrizable space is normal and Hausdorff, so a metrizable space is thus a Tychonoff space. Let X be a locally compact Hausdorff space. Either X or the one-point compactification of X is a compact Hausdorff space Y of which X is a subspace. Y being a compact Hausdorff space implies that it is normal and hence completely regular. But X is a subspace of Y and being completely regular is a hereditary property, so X is completely regular, and therefore Tychonoff. Thus, we have proved that a locally compact Hausdorff space is Tychonoff.

2 Initial topologies

Suppose that X is a set, X_i, $i \in I$, are topological spaces, and $f_i : X \to X_i$ are functions. The **initial topology on X induced by** $\{f_i : i \in I\}$ is the coarsest topology on X such that each f_i is continuous. A subbase for the initial topology is the collection of those sets of the form $f_i^{-1}(U_i)$, $i \in I$ and U_i open in X_i.

If $f_i : X \to X_i$, $i \in I$, are functions, the **evaluation map** is the function $e : X \to \prod_{i \in I} X_i$ defined by

$$(\pi_i \circ e)(x) = f_i(x), \qquad i \in I.$$

We say that a collection $\{f_i : i \in I\}$ of functions on X **separates points** if $x \neq y$ implies that there is some $i \in I$ such that $f_i(x) \neq f_i(y)$. We remind ourselves that if X and Y are topological spaces and $\phi : X \to Y$ is a function, ϕ is called an **embedding** when $\phi : X \to \phi(X)$ is a homeomorphism, where $\phi(X)$ has the subspace topology inherited from Y. The following theorem gives conditions on when X can be embedded into the product of the codomains of the f_i.[2]

Theorem 4. *Let X be a topological space, let X_i, $i \in I$, be topological spaces, and let $f_i : X \to X_i$ be functions. The evaluation map $e : X \to \prod_{i \in I} X_i$ is an embedding if and only if both (i) X has the initial topology induced by the family $\{f_i : i \in I\}$ and (ii) the family $\{f_i : i \in I\}$ separates points in X.*

Proof. Write $P = \prod_{i \in I} X_i$ and let $p_i : e(X) \to X_i$ be the restriction of $\pi_i : X \to X_i$ to $e(X)$. A subbase for $e(X)$ with the subspace topology inherited from P consists of those sets of the form $\pi_i^{-1}(U_i) \cap e(X)$, $i \in I$ and U_i open in X_i. But $\pi_i^{-1}(U_i) \cap e(X) = p_i^{-1}(U_i)$, and the collection of sets of this form is a subbase for $e(X)$ with the initial topology induced by the family $\{p_i : i \in I\}$, so these topologies are equal.

Assume that $e : X \to e(X)$ is a homeomorphism. Because e is a homeomorphism and $f_i = \pi_i \circ e = p_i \circ e$, $e(X)$ having the initial topology induced by $\{p_i : i \in I\}$ implies that X has the initial topology induced by $\{f_i : i \in I\}$. If x, y are distinct elements of X then there is some $i \in I$ such that $p_i(e(x)) \neq p_i(e(y))$, i.e. $f_i(x) \neq f_i(y)$, showing that $\{f_i : i \in I\}$ separates points in X.

Assume that X has the initial topology induced by $\{f_i : i \in I\}$ and that the family $\{f_i : i \in I\}$ separates points in X. We shall prove that $e : X \to e(X)$ is a homeomorphism, for which it suffices to prove that $e : X \to P$ is one-to-one and continuous and that $e : X \to e(X)$ is open. If $x, y \in X$ are distinct then because the f_i separate points, there is some $i \in I$ such that $f_i(x) \neq f_i(y)$, and so $e(x) \neq e(y)$, showing that e is one-to-one.

For each $i \in I$, f_i is continuous and $f_i = \pi_i \circ e$. The fact that this is true for all $i \in I$ implies that $e : X \to P$ is continuous. (Because the product topology is the initial topology induced by the family of projection maps, a map to a product is continuous if and only if its composition with each projection map is continuous.)

[2] Stephen Willard, *General Topology*, p. 56, Theorem 8.12.

A subbase for the topology of X consists of those sets of the form $V = f_i^{-1}(U_i)$, $i \in I$ and U_i open in X_i. As $f_i = p_i \circ e$ we can write this as

$$V = (p_i \circ e)^{-1}(U_i) = e^{-1}(p_i^{-1}(U_i)),$$

which implies that $e(V) = p_i^{-1}(U_i)$, which is open in $e(X)$ and thus shows that $e : X \to e(X)$ is open. $\square$

We say that a collection $\{f_i : i \in I\}$ of functions on a topological space X **separates points from closed sets** if whenever F is a closed subset of X and $x \in X \setminus F$, there is some $i \in I$ such that $f_i(x) \notin \overline{f_i(F)}$, where $\overline{f_i(F)}$ is the closure of $f_i(F)$ in the codomain of f.

Theorem 5. *Assume that X is a topological space and that $f_i : X \to X_i$, $i \in I$, are continuous functions. This family separates points from closed sets if and only if the collection of sets of the form $f_i^{-1}(U_i)$, $i \in I$ and U_i open in X_i, is a base for the topology of X.*

Proof. Assume that the family $\{f_i : i \in I\}$ separates points from closed sets in X. Say $x \in X$ and that U is an open neighborhood of x. Then $F = X \setminus U$ is closed so there is some $i \in I$ such that $f_i(x) \notin \overline{f_i(F)}$. Thus $U_i = X_i \setminus \overline{f_i(F)}$ is open in X_i, hence $f_i^{-1}(U_i)$ is open in X. On the one hand, $f(x_i) \in U_i$ yields $x_i \in f_i^{-1}(U_i)$. On the other hand, if $y \in f_i^{-1}(U_i)$ then $f_i(y) \in U_i$, which tells us that $y \notin F$ and so $y \in U$, giving $f_i^{-1}(U_i) \subset U$. This shows us that the collection of sets of the form $f_i^{-1}(U_i)$, $i \in I$ and U_i open in X_i, is a base for the topology of X.

Assume that the collection of sets of the form $f_i^{-1}(U_i)$, $i \in I$ and U_i open in X_i, is a base for the topology of X, and suppose that F is a closed subset of X and that $x \in X \setminus F$. Because $X \setminus F$ is an open neighborhood of x, there is some $i \in I$ and open U_i in X_i such that $x \in f_i^{-1}(U_i) \subset X \setminus F$, so $f_i(x) \in U_i$. Suppose by contradiction that there is some $y \in F$ such that $f_i(y) \in U_i$. This gives $y \in f_i^{-1}(U_i) \subset X \setminus F$, which contradicts $y \in F$. Therefore $U_i \cap f_i(F) = \emptyset$, and hence $X_i \setminus U_i$ is a closed set that contains $f_i(F)$, which tells us that $\overline{f_i(F)} \subset X_i \setminus U_i$, i.e. $\overline{f_i(F)} \cap U_i = \emptyset$. But $f_i(x) \in U_i$, so we have proved that $\{f_i : i \in I\}$ separates points from closed sets. $\square$

A T_1 **space** is a topological space in which all singletons are closed.

Theorem 6. *If X is a T_1 space, X_i, $i \in I$, are topological spaces, $f_i : X \to X_i$ are continuous functions, and $\{f_i : i \in I\}$ separates points from closed sets in X, then the evaluation map $e : X \to \prod_{i \in I} X_i$ is an embedding.*

Proof. By Theorem 5, there is a base for the topology of X consisting of sets of the form $f_i^{-1}(U_i)$, $i \in I$ and U_i open in X_i. Since this collection of sets is a base it is a fortiori a subbase, and the topology generated by this subbase is the initial topology for the family of functions $\{f_i : i \in I\}$. Because X is T_1, singletons are closed and therefore the fact that $\{f_i : i \in I\}$ separates points and closed sets implies that it separates points in X. Therefore we can apply Theorem 4, which tells us that the evaluation map is an embedding. $\square$

3 Bounded continuous functions

For any set X, we denote by $\ell^\infty(X)$ the set of all bounded functions $X \to \mathbb{R}$, and we take as known that $\ell^\infty(X)$ is a Banach space with the **supremum norm**

$$\|f\|_\infty = \sup_{x \in X} |f(x)|, \qquad f \in \ell^\infty(X).$$

If X is a topological space, we denote by $C_b(X)$ the set of bounded continuous functions $X \to \mathbb{R}$. $C_b(X) \subset \ell^\infty(X)$, and it is apparent that $C_b(X)$ is a linear subspace of $\ell^\infty(X)$. One proves that $C_b(X)$ is closed in $\ell^\infty(X)$ (i.e., that if a sequence of bounded continuous functions converges to some bounded function, then this function is continuous), and hence with the supremum norm, $C_b(X)$ is a Banach space.

The following result shows that the Banach space $C_b(X)$ of bounded continuous functions $X \to \mathbb{R}$ is a useful collection of functions to talk about.[3]

Theorem 7. *Let X be a topological space. X is completely regular if and only if X has the initial topology induced by $C_b(X)$.*

Proof. Assume that X is completely regular. If F is a closed subset of X and $x \in X \setminus F$, then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$. Then $f \in C_b(X)$, and $f(x) = 0 \notin \{1\} = \overline{f(F)}$. This shows that $C_b(X)$ separates points from closed sets in X. Applying Theorem 5, we get that X has the initial topology induced by $C_b(X)$. (This would follow if the collection that Theorem 5 tells us is a base were merely a subbase.)

Assume that X has the initial topology induced by $C_b(X)$. Suppose that F is a closed subset of X and that $x \in U = X \setminus F$. A subbase for the initial topology induced by $C_b(X)$ consists of those sets of the form $f^{-1}(V)$ for $f \in C_b(X)$ and V an **open ray** in $\mathbb{R}$ (because the open rays are a subbase for the topology of $\mathbb{R}$), so because U is an open neighborhood of x, there is a finite subset J of $C_b(X)$ and open rays V_f in $\mathbb{R}$ for each $f \in J$ such that

$$x \in \bigcap_{f \in J} f^{-1}(V_f) \subset U.$$

If some V_j is of the form $(-\infty, a_f)$, then with $g = -f$ we have $f^{-1}(-\infty, a_f) = g^{-1}(-a_f, \infty)$. We therefore suppose that in fact $V_f = (a_f, \infty)$ for each $f \in J$. For each $f \in J$, define $g_f : X \to \mathbb{R}$ by

$$g_f(x) = \sup\{f(x) - a_f, 0\},$$

which is continuous and ≥ 0, and satisfies $f^{-1}(a_f, \infty) = g_f^{-1}(0, \infty)$, so that

$$x \in \bigcap_{f \in J} g_f^{-1}(0, \infty) \subset U.$$

[3] Stephen Willard, *General Topology*, p. 96, Theorem 14.12.

Define $g = \prod_{f \in J} g_f$, which is continuous because each factor is continuous. This function satisfies $g(x) = \prod_{f \in J} g_f(x) > 0$ because this is a product of finitely many factors each of which are > 0. If $y \in g^{-1}(0, \infty)$ then $y \in \bigcap_{f \in J} g_f^{-1}(0, \infty) \subset U$, so $g^{-1}(0, \infty) \subset U$. But g is nonnegative, so this tells us that $g(X \setminus U) = \{0\}$, i.e. $g(F) = \{0\}$. By Lemma 1 this suffices to show that X is completely regular. $\square$

A **cube** is a topological space that is homeomorphic to a product of compact intervals in $\mathbb{R}$. Any product is homeomorphic to the same product without singleton factors, (e.g. $\mathbb{R} \times \mathbb{R} \times \{3\}$ is homeomorphic to $\mathbb{R} \times \mathbb{R}$) and a product of nonsingleton compact intervals with index set I is homeomorphic to $[0, 1]^I$. We remind ourselves that to say that a topological space is homeomorphic to a subspace of a cube is equivalent to saying that the space can be embedded into the cube.

Theorem 8. *A topological space X is a Tychonoff space if and only if it is homeomorphic to a subspace of a cube.*

Proof. Suppose that I is a set and that X is homeomorphic to a subspace Y of $[0, 1]^I$. $[0, 1]$ is Tychonoff so the product $[0, 1]^I$ is Tychonoff, and hence the subspace Y is Tychonoff, thus X is Tychonoff.

Suppose that X is Tychonoff. By Theorem 7, X has the initial topology induced by $C_b(X)$. For each $f \in C_b(X)$, let $I_f = [-\|f\|_\infty, \|f\|_\infty]$, which is a compact interval in $\mathbb{R}$, and $f : X \to I_f$ is continuous. Because X is Tychonoff, it is T_1 and the functions $f : X \to I_f$, $f \in C_b(X)$, separate points and closed sets, we can now apply Theorem 6, which tells us that the evaluation map $e : X \to \prod_{f \in C_b(X)} I_f$ is an embedding. $\square$

4 Compactifications

In §1 we talked about the one-point compactification of a locally compact Hausdorff space. A **compactification** of a topological space X is a pair (K, h) where (i) K is a compact Hausdorff space, (ii) $h : X \to K$ is an embedding, and (iii) $h(X)$ is a dense subset of K. For example, if X is a compact Hausdorff space then (X, id_X) is a compactification of X, and if X is a locally compact Hausdorff space, then the one-point compactification $X^* = X \cup \{\infty\}$, where ∞ is some symbol that does not belong to X, together with the inclusion map $X \to X^*$ is a compactification.

Suppose that X is a topological space and that (K, h) is a compactification of X. Because K is a compact Hausdorff space it is normal, and then Urysohn's lemma tells us that K is completely regular. But K is Hausdorff, so in fact K is Tychonoff. A subspace of a Tychonoff space is Tychonoff, so $h(X)$ with the subspace topology is Tychonoff. But X and $h(X)$ are homeomorphic, so X is Tychonoff. Thus, if a topological space has a compactification then it is Tychonoff.

In Theorem 8 we proved that any Tychonoff space can be embedded into a cube. Here review our proof of this result. Let X be a Tychonoff space, and for each $f \in C_b(X)$ let $I_f = [-\|f\|_\infty, \|f\|_\infty]$, so that $f : X \to I_f$ is continuous, and the family of these functions separates points in X. The evaluation map for this family is $e : X \to \prod_{f \in C_b(X)} I_f$ defined by $(\pi_f \circ e)(x) = f(x)$ for $f \in C_b(X)$, and Theorem 6 tells us that $e : X \to \prod_{f \in C_b(X)} I_f$ is an embedding. Because each interval I_f is a compact Hausdorff space (we remark that if $f = 0$ then $I_f = \{0\}$, which is indeed compact), the product $\prod_{f \in C_b(X)} I_f$ is a compact Hausdorff space, and hence any closed subset of it is compact. We define βX to be the closure of $e(X)$ in $\prod_{f \in C_b(X)} I_f$, and the **Stone-Čech compactification of** X is the pair $(\beta X, e)$, and what we have said shows that indeed this is a compactification of X.

The Stone-Čech compactification of a Tychonoff space is useful beyond displaying that every Tychonoff space has a compactification. We prove in the following that any continuous function from a Tychonoff space to a compact Hausdorff space factors through its Stone-Čech compactification.[4]

Theorem 9. *If X is a Tychonoff space, K is a compact Hausdorff space, and $\phi : X \to K$ is continuous, then there is a unique continuous function $\Phi : \beta X \to K$ such that $\phi = \Phi \circ e$.*

Proof. K is Tychonoff because a compact Hausdorff space is Tychonoff, so the evaluation map $e_K : K \to \prod_{g \in C_b(K)} I_g$ is an embedding. Write $F = \prod_{f \in C_b(X)} I_f$, $G = \prod_{g \in C_b(K)} I_g$, and let $p_f : F \to I_f$, $q_g : G \to I_g$ be the projection maps.

We define $H : F \to G$ for $t \in F$ by $(q_g \circ H)(t) = t(g \circ \phi) = p_{g \circ \phi}(t)$. For each $g \in G$, the map $q_g \circ H : F \to I_{g \circ \phi}$ is continuous, so H is continuous.

For $x \in X$, we have

$$\begin{aligned}(q_g \circ H \circ e)(x) &= (q_g \circ H)(e(x)) \\ &= p_{g \circ \phi}(e(x)) \\ &= (p_{g \circ \phi} \circ e)(x) \\ &= (g \circ \phi)(x) \\ &= g(\phi(x)) \\ &= (q_g \circ e_K)(\phi(x)) \\ &= (q_g \circ e_K \circ \phi)(x),\end{aligned}$$

so

$$H \circ e = e_K \circ \phi. \tag{1}$$

On the one hand, because K is compact and e_K is continuous, $e_K(K)$ is compact and hence is a closed subset of G (G is Hausdorff so a compact subset is closed). From (1) we know $H(e(X)) \subset e_K(K)$, and thus

$$\overline{H(e(X))} \subset \overline{e_K(K)} = e_K(K).$$

[4] Stephen Willard, *General Topology*, p. 137, Theorem 19.5.

On the other hand, because βX is compact and H is continuous, $H(\beta X)$ is compact and hence is a closed subset of G. As $e(X)$ is dense in βX and H is continuous, $H(e(X))$ is dense in $H(\beta X)$, and thus

$$\overline{H(e(X))} = \overline{H(\beta X)} = H(\beta X).$$

Therefore we have

$$H(\beta X) \subset e_K(K).$$

Let h be the restriction of H to βX, and define $\Phi : \beta X \to K$ by $\Phi = e_K^{-1} \circ h$, which makes sense because $e_K : K \to e_K(K)$ is a homeomorphism and h takes values in $e_K(K)$. Φ is continuous, and for $x \in X$ we have, using (1),

$$(\Phi \circ e)(x) = (e_K^{-1} \circ h \circ e)(x) = (e_K^{-1} \circ H \circ e)(x) = \phi(x),$$

showing that $\Phi \circ e = \phi$.

If $\Psi : \beta X \to K$ is a continuous function satisfying $f = \Psi \circ e$, let $y \in e(X)$. There is some $x \in X$ such that $y = e(x)$, and $f(x) = (\Psi \circ e)(x) = \Psi(y)$, $f(x) = (\Phi \circ e)(x) = \Phi(y)$, showing that for all $y \in e(X)$, $\Psi(y) = \Phi(y)$. Since Ψ and Φ are continuous and are equal on $e(X)$, which is a dense subset of βX, we get $\Psi = \Phi$, which completes the proof. $\square$

If X is a Tychonoff space with Stone-Čech compactification $(\beta X, e)$, then because βX is a compact space, $C(\beta X)$ with the supremum norm is a Banach space. We show in the following that the extension in Theorem 9 produces an isometric isomorphism $C_b(X) \to C(\beta X)$.

Theorem 10. *If X is a Tychonoff space with Stone-Čech compactification $(\beta X, e)$, then there is an isomorphism of Banach spaces $C_b(X) \to C(\beta X)$.*

Proof. Let $f, g \in C_b(X)$, let α be a scalar, and let $K = [-|\alpha| \|f\| - \|g\|, |\alpha| \|f\| + \|g\|]$, which is a compact set. Define $\phi = \alpha f + g$, and then Theorem 9 tells us that there is a unique continuous function $F : \beta X \to K$ such that $f = F \circ e$, a unique continuous function $G : \beta X \to K$ such that $g = G \circ e$, and a unique continuous function $\Phi : \beta X \to K$ such that $\phi = \Phi \circ e$. For $y \in e(X)$ and $x \in X$ such that $y = e(x)$,

$$\Phi(y) = \phi(x) = \alpha f(x) + g(x) = \alpha F(y) + G(y).$$

Since Φ and $\alpha F + G$ are continuous functions $\beta X \to K$ that are equal on the dense set $e(X)$, we get $\Phi = \alpha F + G$. Therefore, the map that sends $f \in C_b(X)$ to the unique $F \in C(\beta X)$ such that $f = F \circ e$ is linear.

Let $f \in C_b(X)$ and let F be the unique element of $C(\beta X)$ such that $f = F \circ e$. For any $x \in X$, $|f(x)| = |(F \circ e)(x)|$, so

$$\|f\|_\infty = \sup_{x \in X} |f(x)| = \sup_{x \in X} |(F \circ e)(x)| = \sup_{y \in e(X)} |F(y)|.$$

Because F is continuous and $e(X)$ is dense in βX,

$$\sup_{y \in e(X)} |F(y)| = \sup_{y \in \beta X} |F(y)| = \|F\|_\infty,$$

so $\|f\|_\infty = \|F\|_\infty$, showing that $f \mapsto F$ is an isometry.

For $\Phi \in C(\beta X)$, define $\phi = \Phi \circ e$. Φ is bounded so ϕ is also, and ϕ is a composition of continuous functions, hence $\phi \in C_b(X)$. Thus $\phi \mapsto \Phi$ is onto, completing the proof. $\square$

5 Spaces of continuous functions

If X is a topological space, we denote by $C(X)$ the set of continuous functions $X \to \mathbb{R}$. For K a compact set in X (in particular a singleton) and $f \in C(X)$, define $p_K(f) = \sup_{x \in K} |f(x)|$. The collection of p_K for all compact subsets of K of X is a **separating family of seminorms**, because if f is nonzero there is some $x \in X$ for which $f(x) \neq 0$ and then $p_{\{x\}}(f) > 0$. Hence $C(X)$ with the topology induced by this family of seminorms is a locally convex space. (If X is **σ-compact** then the seminorm topology is induced by countably many of the seminorms, and then $C(X)$ is metrizable.) However, since we usually are not given that X is compact (in which case $C(X)$ is normable with p_X) and since it is often more convenient to work with normed spaces than with locally convex spaces, we shall talk about subsets of $C(X)$.

For X a topological space, we say that a function $f : X \to \mathbb{R}$ **vanishes at infinity** if for each $\epsilon > 0$ there is a compact set K such that $|f(x)| < \epsilon$ whenever $x \in X \setminus K$, and we denote by $C_0(X)$ the set of all continuous functions $X \to \mathbb{R}$ that vanish at infinity.

The following theorem shows first that $C_0(X)$ is contained in $C_b(X)$, second that $C_0(X)$ is a linear space, and third that it is a closed subset of $C_b(X)$. With the supremum norm $C_b(X)$ is a Banach space, so this shows that $C_0(X)$ is a Banach subspace. We work through the proof in detail because it is often proved with unnecessary assumptions on the topological space X.

Theorem 11. *Suppose that X is a topological space. Then $C_0(X)$ is a closed linear subspace of $C_b(X)$.*

Proof. If $f \in C_0(X)$, then there is a compact set K such that $x \in X \setminus K$ implies that $|f(x)| < 1$. On the other hand, because f is continuous, $f(K)$ is a compact subset of the scalar field and hence is bounded, i.e., there is some $M \geq 0$ such that $x \in K$ implies that $|f(x)| \leq M$. Therefore f is bounded, showing that $C_0(X) \subset C_b(X)$.

Let $f, g \in C_0(X)$ and let $\epsilon > 0$. There is a compact set K_1 such that $x \in X \setminus K_1$ implies that $|f(x)| < \frac{\epsilon}{2}$ and a compact set K_2 such that $x \in X \setminus K_2$ implies that $|g(x)| < \frac{\epsilon}{2}$. Let $K = K_1 \cup K_2$, which is a union of two compact sets hence is itself compact. If $x \in X \setminus K$, then $x \in X \setminus K_1$ implying $|f(x)| < \frac{\epsilon}{2}$ and $x \in X \setminus K_2$ implying $|g(x)| < \frac{\epsilon}{2}$, hence $|f(x) + g(x)| \leq |f(x)| + |g(x)| < \epsilon$. This shows that $f + g \in C_0(X)$.

If $f \in C_0(X)$ and α is a nonzero scalar, let $\epsilon > 0$. There is a compact set K such that $x \in X \setminus K$ implies that $|f(x)| < \frac{\epsilon}{|\alpha|}$, and hence $|(\alpha f)(x)| = |\alpha||f(x)| < \epsilon$, showing that $\alpha f \in C_0(X)$. Therefore $C_0(X)$ is a linear subspace of $C_b(X)$.

Suppose that f_n is a sequence of elements of $C_0(X)$ that converges to some $f \in C_b(X)$. For $\epsilon > 0$, there is some n_ϵ such that $n \geq n_\epsilon$ implies that $\|f_n - f\|_\infty < \frac{\epsilon}{2}$, that is,

$$\sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

For each n, let K_n be a compact set in X such that $x \in X \setminus K_n$ implies that $|f_n(x)| < \frac{\epsilon}{2}$; there are such K_n because $f_n \in C_0(X)$. If $x \in X \setminus K_{n_\epsilon}$, then

$$|f(x)| \leq |f_{n_\epsilon}(x) - f(x)| + |f_{n_\epsilon}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

showing that $f \in C_0(X)$. □

If X is a topological space and $f : X \to \mathbb{R}$ is a function, the **support of** f is the set

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}.$$

If $\operatorname{supp} f$ is compact we say that f has **compact support**, and we denote by $C_c(X)$ the set of all continuous functions $X \to \mathbb{R}$ with compact support.

Suppose that X is a topological space and let $f \in C_c(X)$. For any $\epsilon > 0$, if $x \in X \setminus \operatorname{supp} f$ then $|f(x)| = 0 < \epsilon$, showing that $f \in C_0(X)$. Therefore

$$C_c(X) \subset C_0(X),$$

and this makes no assumptions about the topology of X.

We can prove that if X is a locally compact Hausdorff space then $C_c(X)$ is dense in $C_0(X)$.[5]

Theorem 12. *If X is a locally compact Hausdorff space, then $C_c(X)$ is a dense subset of $C_0(X)$.*

Proof. Let $f \in C_0(X)$, and for each $n \in \mathbb{N}$ define

$$C_n = \left\{ x \in X : |f(x)| \geq \frac{1}{n} \right\}.$$

For $n \in \mathbb{N}$, because $f \in C_0(X)$ there is a compact set K_n such that $x \in X \setminus K_n$ implies that $|f(x)| < \frac{1}{n}$, and hence $C_n \subset K_n$. Because $x \mapsto |f_n(x)|$ is continuous, C_n is a closed set in X, and it follows that C_n, being contained in the compact set K_n, is compact. (This does not use that X is Hausdorff.)

Let $n \in \mathbb{N}$. Because X is a locally compact Hausdorff space and C_n is compact, Urysohn's lemma[6] tells us that there is a compact set D_n containing C_n and a continuous function $g_n : X \to [0, 1]$ such that $g_n(C_n) = \{1\}$ and $g_n(X \setminus D_n) \subset \{0\}$. That is, $g_n \in C_c(X)$, $0 \leq g_n \leq 1$, and $g_n(C_n) = \{1\}$. Define

[5] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, p. 132, Proposition 4.35.

[6] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, p. 131, Lemma 4.32.

$f_n = g_n f \in C_c(X)$. (A product of continuous functions is continuous, and because f is bounded and g_n has compact support, $g_n f$ has compact support.) For $x \in C_n$, $f_n(x) - f(x) = (g_n(x)-1)f(x) = 0$, and for $x \in X \setminus C_n$, $|f_n(x) - f(x)| = |g_n(x)-1||f(x)| \leq 1 \cdot \frac{1}{n}$. Therefore

$$\|f_n - f\|_\infty \leq \frac{1}{n},$$

and hence f_n is a sequence in $C_c(X)$ that converges to f, showing that $C_c(X)$ is dense in $C_0(X)$. □

If X is a Hausdorff space, then we prove that $C_c(X)$ is a linear subspace of $C_0(X)$. When X is a locally compact Hausdorff space then combined with the above this shows that $C_c(X)$ is a dense linear subspace of $C_0(X)$.

Lemma 13. *Suppose that X is a Hausdorff space. Then $C_c(X)$ is a linear subspace of $C_0(X)$.*

Proof. If $f, g \in C_c(X)$ and α is a scalar, let $K = \operatorname{supp} f \cup \operatorname{supp} g$, which is a union of two compact sets hence compact. If $x \in X \setminus K$, then $f(x) = 0$ because $x \notin \operatorname{supp} f$ and $g(x) = 0$ because $x \notin \operatorname{supp} g$, so $(\alpha f + g)(x) = 0$. Therefore $\{x \in X : (\alpha f + g)(x) \neq 0\} \subset K$ and hence $\operatorname{supp}(\alpha f + g) \subset \overline{K}$. But as X is Hausdorff, K being compact implies that K is closed in X, so we get $\operatorname{supp}(\alpha f + g) \subset K$. Because $\operatorname{supp}(\alpha f + g)$ is closed and is contained in the compact set K, it is itself compact, so $\alpha f + g \in C_c(X)$. □

Let X be a topological space, and for $x \in X$ define $\delta_x : C_b(X) \to \mathbb{R}$ by $\delta_x(f) = f(x)$. For each $x \in X$, δ_x is linear and $|\delta_x(f)| = |f(x)| \leq \|f\|_\infty$, so δ_x is continuous and hence belongs to the dual space $C_b(X)^*$. Moreover, the constant function $f(x) = 1$ shows that $\|\delta_x\| = 1$. We define $\Delta : X \to C_b(X)^*$ by $\Delta(x) = \delta_x$. Suppose that x_i is a net in X that converges to some $x \in X$. Then for every $f \in C_b(X)$ we have $f(x_i) \to f(x)$, and this means that δ_{x_i} weak-* converges to δ_x in $C_b(X)^*$. This shows that with $C_b(X)^*$ assigned the weak-* topology, $\Delta : X \to C_b(X)^*$ is continuous. We now characterize when Δ is an embedding.[7]

Theorem 14. *Suppose that X is a topological space and assign $C_b(X)^*$ the weak-* topology. Then the map $\Delta : X \to \Delta(X)$ is a homeomorphism if and only if X is Tychonoff, where $\Delta(X)$ has the subspace topology inherited from $C_b(X)^*$.*

Proof. Suppose that X is Tychonoff. If $x, y \in X$ are distinct, then there is some $f \in C_b(X)$ such that $f(x) = 0$ and $f(y) = 1$, and then $\delta_x(f) = 0 \neq 1 = \delta_y(f)$, so $\Delta(x) \neq \Delta(y)$, showing that Δ is one-to-one. To show that $\Delta : X \to \Delta(X)$ is a homeomorphism, it suffices to prove that Δ is an open map, so let U be an open subset of X. For $x_0 \in U$, because $X \setminus U$ is closed there is some $f \in C_b(X)$ such that $f(x_0) = 0$ and $f(X \setminus U) = \{1\}$. Let

$$V_1 = \{\mu \in C_b(X)^* : \mu(f) < 1\}.$$

[7] John B. Conway, *A Course in Functional Analysis*, second ed., p. 137, Proposition 6.1.

This is an open subset of $C_b(X)^*$ as it is the inverse image of $(-\infty, 1)$ under the map $\mu \mapsto \mu(f)$, which is continuous $C_b(X)^* \to \mathbb{R}$ by definition of the weak-* topology. Then

$$V = V_1 \cap \Delta(X) = \{\delta_x : f(x) < 1\}$$

is an open subset of the subspace $\Delta(X)$, and we have both $\delta_{x_0} \in V$ and $V \subset \Delta(U)$. This shows that for any element δ_{x_0} of $\Delta(U)$, there is some open set V in the subspace $\Delta(X)$ such that $\delta_{x_0} \in V \subset \Delta(U)$, which tells us that $\Delta(U)$ is an open set in the subspace $\Delta(U)$, showing that Δ is an open map and therefore a homeomorphism.

Suppose that $\Delta : X \to \Delta(X)$ is a homeomorphism. By the Banach-Alaoglu theorem we know that the closed unit ball B_1 in $C_b(X)^*$ is compact. (We remind ourselves that we have assigned $C_b(X)^*$ the weak-* topology.) That is, with the subspace topology inherited from $C_b(X)^*$, B_1 is a compact space. It is Hausdorff because $C_b(X)^*$ is Hausdorff, and a compact Hausdorff space is Tychonoff. But $\Delta(X)$ is contained in the surface of B_1, in particular $\Delta(X)$ is contained in B_1 and hence is itself Tychonoff with the subspace topology inherited from B_1, which is equal to the subspace topology inherited from $C_b(X)^*$. Since $\Delta : X \to \Delta(X)$ is a homeomorphism, we get that X is a Tychonoff space, completing the proof. $\square$

The following result shows when the Banach space $C_b(X)$ is separable.[8]

Theorem 15. *Suppose that X is a Tychonoff space. Then the Banach space $C_b(X)$ is separable if and only if X is compact and metrizable.*

Proof. Assume that X is compact and metrizable, with a compatible metric d. For each $n \in \mathbb{N}$ there are open balls $U_{n,1}, \ldots, U_{n,N_n}$ of radius $\frac{1}{n}$ that cover X. As X is metrizable it is normal, so there is a **partition of unity subordinate to the cover** $\{U_{n,k} : 1 \leq k \leq N_n\}$.[9] That is, there are continuous functions $f_{n,1}, \ldots, f_{n,N_n} : X \to [0,1]$ such that $\sum_{k=1}^{N_n} f_{n,k} = 1$ and such that $x \in X \setminus U_{n,k}$ implies that $f_{n,k}(x) = 0$. Then $\{f_{n,k} : n \in \mathbb{N}, 1 \leq k \leq N_n\}$ is countable, so its span D over $\mathbb{Q}$ is also countable. We shall prove that D is dense in $C(X) = C_b(X)$, which will show that $C_b(X)$ is separable.

Let $f \in C(X)$ and let $\epsilon > 0$. Because (X, d) is a compact metric space, f is uniformly continuous, so there is some $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let $n \in \mathbb{N}$ be $> \frac{2}{\delta}$, and for each $1 \leq k \leq N_n$ let $x_k \in U_{n,k}$. For each k there is some $\alpha_k \in \mathbb{Q}$ such that $|\alpha_k - f(x_k)| < \frac{\epsilon}{2}$, and we define

$$g = \sum_{k=1}^{N_n} \alpha_k f_{n,k} \in D.$$

Because $\sum_{k=1}^{N_n} f_{n,k} = 1$ we have $f = \sum_{k=1}^{N_n} f f_{n,k}$. Let $x \in X$, and then

$$|f(x) - g(x)| = \left| \sum_{k=1}^{N_n} (f(x) - \alpha_k) f_{n,k}(x) \right| \leq \sum_{k=1}^{N_n} |f(x) - \alpha_k| f_{n,k}(x).$$

[8] John B. Conway, *A Course in Functional Analysis*, second ed., p. 140, Theorem 6.6.
[9] John B. Conway, *A Course in Functional Analysis*, second ed., p. 139, Theorem 6.5.

For each $1 \leq k \leq N_n$, either $x \in U_{n,k}$ or $x \notin U_{n,k}$. In the first case, since x and x_k are then in the same open ball of radius $\frac{1}{n}$, $d(x, x_k) < \frac{2}{n} < \delta$, so

$$|f(x) - \alpha_k| \leq |f(x) - f(x_k)| + |f(x_k) - \alpha_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

In the second case, $f_{n,k}(x) = 0$. Therefore,

$$\sum_{k=1}^{N_n} |f(x) - \alpha_k| f_{n,k}(x) \leq \sum_{k=1}^{N_n} \epsilon f_{n,k}(x) = \epsilon,$$

showing that $|f(x) - g(x)| \leq \epsilon$. This shows that D is dense in $C(X)$, and therefore that $C_b(X) = C(X)$ is separable.

Suppose that $C_b(X)$ is separable. Because X is Tychonoff, by Theorem 10 there is an isometric isomorphism between the Banach spaces $C_b(X)$ and $C(\beta X)$, where $(\beta X, e)$ is the Stone-Čech compactification of X. Hence $C(\beta X)$ is separable. But it is a fact that a compact Hausdorff space Y is metrizable if and only if the Banach space $C(Y)$ is separable.[10] (This is proved using the Stone-Weierstrass theorem.) As βX is a compact Hausdorff space and $C(\beta X)$ is separable, we thus get that βX is metrizable.

It is a fact that if Y is a Banach space and B_1 is the closed unit ball in the dual space Y^*, then B_1 with the subspace topology inherited from Y^* with the weak-* topology is metrizable if and only if Y is separable.[11] Thus, the closed unit ball B_1 in $C_b(X)^*$ is metrizable. Theorem 14 tells us there is an embedding $\Delta : X \to B_1$, and B_1 being metrizable implies that $\Delta(X)$ is metrizable. As $\Delta : X \to \Delta(X)$ is a homeomorphism, we get that X is metrizable.

Because βX is compact and metrizable, to prove that X is compact and metrizable it suffices to prove that $\beta X \setminus e(X) = \emptyset$, so we suppose by contradiction that there is some $\tau \in \beta X \setminus e(X)$. $e(X)$ is dense in βX, so there is a sequence $x_n \in X$, for which we take $x_n \neq x_m$ when $n \neq m$, such that $e(x_n) \to \tau$. If x_n had a subsequence $x_{a(n)}$ that converged to some $y \in X$, then $e(x_{a(n)}) \to e(y)$ and hence $e(y) = \tau$, a contradiction. Therefore the sequence x_n has no limit points, so the sets $A = \{x_n : n \text{ odd}\}$ and $B = \{x_n : n \text{ even}\}$ are closed and disjoint. Because X is metrizable it is normal, hence by Urysohn's lemma there is a continuous function $\phi : X \to [0, 1]$ such that $\phi(a) = 0$ for all $a \in A$ and $\phi(b) = 1$ for all $b \in B$. Then, by Theorem 9 there is a unique continuous $\Phi : X \to [0, 1]$ such that $\phi = \Phi \circ e$. Then we have, because a subsequence of a

[10] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 353, Theorem 9.14.

[11] John B. Conway, *A Course in Functional Analysis*, second ed., p. 134, Theorem 5.1.

convergent sequence has the same limit,

$$\begin{aligned}
\Phi(\tau) &= \Phi\left(\lim_{n\to\infty} e(x_n)\right) \\
&= \Phi\left(\lim_{n\to\infty} e(x_{2n+1})\right) \\
&= \lim_{n\to\infty} (\Phi \circ e)(x_{2n+1}) \\
&= \lim_{n\to\infty} \phi(x_{2n+1}) \\
&= 0,
\end{aligned}$$

and likewise

$$\Phi(\tau) = \lim_{n\to\infty} \phi(x_{2n}) = 1,$$

a contradiction. This shows that $\beta X \setminus e(X) = \emptyset$, which completes the proof. □

6 C^*-algebras and the Gelfand transform

A C^***-algebra** is a complex Banach algebra A with a map $^* : A \to A$ such that

1. $a^{**} = a$ for all $a \in A$ (namely, * is an **involution**),
2. $(a+b)^* = a^* + b^*$ and $(ab)^* = b^* a^*$ for all $a \in A$,
3. $(\lambda a)^* = \overline{\lambda} a^*$ for all $a \in A$ and $\lambda \in \mathbb{C}$,
4. $\|a^* a\| = \|a\|^2$ for all $a \in A$.

We do not require that a C^*-algebra be unital. If $a = 0$ then $\|a^*\| = \|0\| = \|a\|$. Otherwise,

$$\|a\|^2 = \|a^* a\| \leq \|a^*\| \, \|a\|$$

gives $\|a\| \leq \|a^*\|$ and

$$\|a^*\|^2 = \|a^{**} a^*\| = \|a a^*\| \leq \|a\| \, \|a^*\|$$

gives $\|a^*\| \leq \|a\|$, showing that * is an isometry.

We now take $C_b(X)$ to denote $C_b(X, \mathbb{C})$ rather than $C_b(X, \mathbb{R})$, and likewise for $C(X), C_0(X)$, and $C_c(X)$. It is routine to verify that everything we have asserted about these spaces when the codomain is $\mathbb{R}$ is true when the codomain is $\mathbb{C}$, but this is not obvious. In particular, $C_b(X)$ is a Banach space with the supremum norm and $C_0(X)$ is a closed linear subspace, whatever the topological space X. It is then straightforward to check that with the involution $f^* = \overline{f}$ they are commutative C^*-algebras.

A **homomorphism of** C^***-algebras** is an algebra homomorphism $f : A \to B$, where A and B are C^*-algebras, such that $f(a^*) = f(a)^*$ for all $a \in A$. It can be proved that $\|f\| \leq 1$.[12] We define an **isomorphism of** C^***-algebras** to

[12] José M. Gracia-Bondía, Joseph C. Várilly and Héctor Figueroa, *Elements of Noncommutative Geometry*, p. 29, Lemma 1.16.

be an algebra isomorphism $f : A \to B$ such that $f(a^*) = f(a)^*$ for all $a \in A$. It follows that $\|f\| \leq 1$ and because f is bijective, the inverse f^{-1} is a C^*-algebra homomorphism, giving $\|f^{-1}\| \leq 1$ and therefore $\|f\| = 1$. Thus, an isomorphism of C^*-algebras is an isometric isomorphism.

Suppose that A is a commutative C^*-algebra, which we do not assume to be unital. A **character of** A is a nonzero algebra homomorphism $A \to \mathbb{C}$. We denote the set of characters of A by $\sigma(A)$, which we call the **Gelfand spectrum of** A. We make some assertions in the following text that are proved in Folland.[13] It is a fact that for every $h \in \sigma(A)$, $\|h\| \leq 1$, so $\sigma(A)$ is contained in the closed unit ball of A^*, where A^* denotes the dual of the Banach space A. Furthermore, one can prove that $\sigma(A) \cup \{0\}$ is a weak-* closed set in A^*, and hence is weak-* compact because it is contained in the closed unit ball which we know to be weak-* compact by the Banach-Alaoglu theorem. We assign $\sigma(A)$ the subspace topology inherited from A^* with the weak-* topology. Depending on whether 0 is or is not an isolated point in $\sigma(A) \cup \{0\}$, $\sigma(A)$ is a compact or a locally compact Hausdorff space; in any case $\sigma(A)$ is a locally compact Hausdorff space.

The **Gelfand transform** is the map $\Gamma : A \to C_0(\sigma(A))$ defined by $\Gamma(a)(h) = h(a)$; that $\Gamma(a)$ is continuous follows from $\sigma(A)$ having the weak-* topology, and one proves that in fact $\Gamma(a) \in C_0(\sigma(A))$.[14] The **Gelfand-Naimark theorem**[15] states that $\Gamma : A \to C_0(\sigma(A))$ is an isomorphism of C^*-algebras.

It can be proved that two commutative C^*-algebras are isomorphic as C^*-algebras if and only if their Gelfand spectra are homeomorphic.[16]

7 Multiplier algebras

An **ideal of a C^*-algebra** A is a closed linear subspace I of A such that $IA \subset I$ and $AI \subset I$. An ideal I is said to be **essential** if $I \cap J \neq \{0\}$ for every nonzero ideal J of A. In particular, A is itself an essential ideal.

Suppose that A is a C^*-algebra. The **multiplier algebra of** A, denoted $M(A)$, is a C^*-algebra containing A as an essential ideal such that if B is a C^*-algebra containing A as an essential ideal then there is a unique homomorphism of C^*-algebras $\pi : B \to M(A)$ whose restriction to A is the identity. We have not shown that there is a multiplier algebra of A, but we shall now prove that this definition is a **universal property**: that any C^*-algebra satisfying the definition is isomorphic as a C^*-algebra to $M(A)$, which allows us to talk about "the" multiplier algebra rather than "a" multiplier algebra.

Suppose that C is a C^*-algebra containing A as an essential ideal such that if B is a C^*-algebra containing A as an essential ideal then there is a unique C^*-algebra homomorphism $\pi : B \to C$ whose restriction to A is the identity.

[13] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 12, §1.3.
[14] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 15.
[15] Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, p. 16, Theorem 1.31.
[16] José M. Gracia-Bondía, Joseph C. Várilly and Héctor Figueroa, *Elements of Noncommutative Geometry*, p. 11, Proposition 1.5.

Hence there is a unique homomorphism of C^*-algebras $\pi_1 : C \to M(A)$ whose restriction to A is the identity, and there is a unique homomorphism of C^*-algebras $\pi_2 : M(A) \to C$ whose restriction to A is the identity. Then $\pi_2 \circ \pi_1 : C \to C$ and $\pi_1 \circ \pi_2 : M(A) \to M(A)$ are homomorphisms of C^*-algebras whose restrictions to A are the identity. But the identity maps $\mathrm{id}_C : C \to C$ and $\mathrm{id}_{M(A)} : M(A) \to M(A)$ are also homomorphisms of C^*-algebras whose restrictions to A are the identity. Therefore, by uniqueness we get that $\pi_2 \circ \pi_1 = \mathrm{id}_C$ and $\pi_1 \circ \pi_2 = \mathrm{id}_{M(A)}$. Therefore $\pi_1 : C \to M(A)$ is an isomorphism of C^*-algebras.

One can prove that if A is unital then $M(A) = A$.[17] It can be proved that for any C^*-algebra A, the multiplier algebra $M(A)$ is unital.[18] For a locally compact Hausdorff space X, it can be proved that $M(C_0(X)) = C_b(X)$.[19] This last assertion is the reason for my interest in multiplier algebras. We have seen that if X is a locally compact Hausdorff space then $C_c(X)$ is a dense linear subspace of $C_0(X)$, and for any topological space $C_0(X)$ is a closed linear subspace of $C_b(X)$, but before talking about multiplier algebras we did not have a tight fit between the C^*-algebras $C_0(X)$ and $C_b(X)$.

8 Riesz representation theorem for compact Hausdorff spaces

There is a proof due to D. J. H. Garling of the Riesz representation theorem for compact Hausdorff spaces that uses the Stone-Čech compactification of discrete topological spaces. This proof is presented in Carothers' book.[20]

[17] Paul Skoufranis, *An Introduction to Multiplier Algebras*, `http://www.math.ucla.edu/~pskoufra/OANotes-MultiplierAlgebras.pdf`, p. 4, Lemma 1.9.

[18] Paul Skoufranis, *An Introduction to Multiplier Algebras*, `http://www.math.ucla.edu/~pskoufra/OANotes-MultiplierAlgebras.pdf`, p. 9, Corollary 2.8.

[19] Eberhard Kaniuth, *A Course in Commutative Banach Algebras*, p. 29, Example 1.4.13; José M. Gracia-Bondía, Joseph C. Várilly and Héctor Figueroa, *Elements of Noncommutative Geometry*, p. 14, Proposition 1.10.

[20] N. L. Carothers, *A Short Course on Banach Space Theory*, Chapter 16, pp. 156–165.