Germs of smooth functions

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1 Sheafs

Let $M = \mathbb{R}^m$. For an open set U in M, write $\mathcal{F}(U) = C^\infty(U)$, which is a commutative ring with unity $1_M(x) = 1$. For open sets $V \subset U$ in M, define $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ by $r_{U,V} f = f|_V$, which is a homomorphism of rings. $\mathcal{F}$ is a **presheaf**, a contravariant functor from the category of open sets in M to the category of commutative unital rings. For $\mathcal{F}$ to be a **sheaf** means the following:

1. If U_i, $i \in I$, is an open cover of an open set U and if $f, g \in \mathcal{F}(U)$ satisfy $r_{U,U_i} f = r_{U,U_i} g$ for all $i \in I$, then $f = g$.

2. If U_i, $i \in I$, is an open cover of an open set U and for each $i \in I$ there is some $f_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$, $r_{U_i,U_i \cap U_j} f_i = r_{U_j,U_i \cap U_j} f_j$, then there is some $f \in \mathcal{F}(U)$ such that $r_{U,U_i} f = f_i$ for each $i \in I$.

For the first condition, let $p \in U$. As U_i is an open cover of U, there is some i for which $p \in U_i$. As $f|_{U_i} = g|_{U_i}$, $f(p) = g(p)$. Therefore $f = g$. For the second condition, let $p \in U$. If $p \in U_i$ and $p \in U_j$, then $f_i(p) = f_j(p)$. This shows that it makes sense to define $f : U \to \mathbb{R}$ by $f(p) = f_i(p)$, for any i such that $p \in U_i$. Then $f|_{U_i} = f_i$, which implies that $f \in \mathcal{F}(U)$: for each $p \in U$, there is some open neighborhood U_i of p on which f is smooth. Therefore $\mathcal{F}$ is a sheaf.

2 Stalks and germs

For $p \in M$, let $\mathcal{U}_p$ be the set of open neighborhoods of p. For $U, V \in \mathcal{U}_p$, say $U \leq V$ when $V \subset U$. For $U \leq V \leq W$ and $f \in \mathcal{F}(U)$,

$$(r_{V,W} \circ r_{U,V})(f) = r_{V,W} f|_V = f_W = r_{U,W} f.$$

For $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, say $f \sim_p g$ if there is some $W \in \mathcal{U}_p$, $W \geq U$, $W \geq V$, such that $r_{U,W} f = r_{V,W} g$. Let

$$\mathcal{R}_p = \bigsqcup_{U \in \mathcal{U}_p} \mathcal{F}(U),$$

and let $\mathcal{F}_p$ be the direct limit of the direct system $\mathcal{F}(U)$, $r_{U,V}$ of commutative unital rings:

$$\mathcal{F}_p = \mathcal{R}_p / \sim_p .$$

We call $\mathcal{F}_p$ the **stalk of $\mathcal{F}$ at** p. An element of $\mathcal{F}_p$ is called a **germ of $\mathcal{F}$ at** p. In other words, for $f \in \mathcal{R}_p$, let $[f]_p$ be the set of those $g \in \mathcal{R}_p$ such that $f \sim_p g$, equivalently, $f|_{U_f \cap U_g} = g|_{U_f \cap U_g}$. A germ of $\mathcal{F}$ at p is such an equivalence class $[f]_p$, and

$$\mathcal{F}_p = \{[f]_p : f \in \mathcal{R}_p\} .$$

3 Maximal ideals

For $p \in M$, and $f, g \in \mathcal{R}_p$ with $f \sim_p g$, $f(p) = g(p)$. Thus it makes sense to define $\mathrm{ev}_p : \mathcal{F}_p \to \mathbb{R}$ by $\mathrm{ev}_p[f]_p = f(p)$. Now, for $[f]_p, [g]_p \in \mathcal{F}_p$,

$$\mathrm{ev}_p([f]_p + [g]_p) = \mathrm{ev}_p([f + g]_p) = (f + g)(p) = f(p) + g(p) = \mathrm{ev}_p[f]_p + \mathrm{ev}_p[g]_p,$$

$$\mathrm{ev}_p([f]_p[g]_p) = \mathrm{ev}_p([fg]_p) = (fg)(p) = f(p)g(p) = \mathrm{ev}_p[f]_p \cdot \mathrm{ev}_p[g]_p,$$

$\mathrm{ev}_p[1_M]_p = 1$. This means that $\mathrm{ev}_p : \mathcal{F}_p \to \mathbb{R}$ is a homomorphism of unital rings. It is straightforward that ev_p is surjective. Write $\mathfrak{m}_p = \ker \mathrm{ev}_p$. By the first isomorphism theorem, there is an isomorphism of unital rings $\mathcal{F}_p/\mathfrak{m}_p \to \mathbb{R}$. Therefore $\mathfrak{m}_p$ is a maximal ideal in $\mathcal{F}_p$. Now, if $[f]_p \in \mathcal{F}_p \setminus \mathfrak{m}_p$ then $\mathrm{ev}_p[f]_p \neq 0$, hence $f(p) \neq 0$. Then there is some $U \in \mathcal{U}_p$ such that $f(x) \neq 0$ for $x \in U$, and $(1/f)(x) = \frac{1}{f(x)}$ belongs to $\mathcal{F}(U)$. Then $[1/f]_p \in \mathcal{F}_p$ and $[f]_p \cdot [1/f]_p = [f \cdot 1/f]_p = [1_M]_p$, which shows that if $[f]_p \in \mathcal{F}_p \setminus \mathfrak{m}_p$ then $[f]_p$ has an inverse $[1/f]_p$ in $\mathcal{F}_p$. This means $\mathfrak{m}_p$ is the set of noninvertible elements of $\mathcal{F}_p$, which means that $\mathcal{F}_p$ is a **local ring**.

For $1 \leq i \leq m$ define the coordinate function $x^i : M \to \mathbb{R}$ by $x^i(p) = p_i$, which belongs to $\mathcal{F}(M)$. Because $\mathrm{ev}_0 x^i = 0$, $[x^i]_0 \in \mathfrak{m}_0$. We prove **Hadamard's lemma**, that the ring $\mathfrak{m}_0$ is generated by the germs of the coordinate functions at 0.[1]

Lemma 1 (Hadamard's lemma)**.** *The ideal $\mathfrak{m}_0$ is generated by the set $\{[x^i]_0 : 1 \leq i \leq m\}$.*

Proof. Let $[f]_0 \in \mathfrak{m}_0$ with $f \in \mathcal{F}(B_r)$ for some $r > 0$. For $y \in B_r$, using the fundamental theorem of calculus and using the chain rule,

$$f(y) = f(y) - f(0) = \int_0^1 \frac{d}{ds} f(sy) ds = \int_0^1 \sum_{i=1}^m x^i(y)(\partial_i f)(sy) ds = \sum_{i=1}^m x^i(y) u_i(y),$$

and $u_i \in \mathcal{F}(B_r)$. This means that $[f]_0 = \sum_{i=1}^m [x^i]_0 [u_i]_0$, which shows that $[f]_0$ belongs to the ideal generated by the set $\{[x^i]_0 : 1 \leq i \leq m\}$. $\square$

[1] Liviu Nicolaescu, *An Invitation to Morse Theory*, second ed., p. 14, Lemma 1.13.

For a multi-index $\alpha \in \mathbb{Z}^m_{\geq 0}$, write

$$|\alpha| = \sum_{i=1}^m \alpha_i, \qquad \alpha! = \alpha_1! \cdots \alpha_m!$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}, \qquad x^\alpha = (x^1)^{\alpha_1} \cdots (x^m)^{\alpha_m},$$

and say $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each i. We shall use the fact that

$$\partial^\alpha x^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha} & \alpha \leq \beta \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. *For $f \in \mathcal{R}_0$, if $(\partial^\alpha f)(0) = 0$ for all $|\alpha| < k$, then $[f]_0 \in \mathfrak{m}_0^k$.*

Proof. For $k = 1$, if $(\partial^\alpha f)(0) = 0$ for $\alpha = (0, \dots, 0)$ then $\mathrm{ev}_0 f = f(0) = 0$, hence $[f]_0 \in \mathfrak{m}_0$. Suppose the claim is true for some $k \geq 1$, and suppose that $f \in \mathcal{R}_0$ and that $(\partial^\alpha f)(0) = 0$ for all $|\alpha| < k+1$. A fortiori, $(\partial^\alpha f)(0) = 0$ for all $|\alpha| < k$ and then by the induction hypothesis we get $[f]_0 \in \mathfrak{m}_0^k$. Now, Lemma 1 tells us that the ideal $\mathfrak{m}_0$ is generated by the set $\{[x^i]_0 : 1 \leq i \leq m\}$, and then the product ideal $\mathfrak{m}_0^k$ is generated by the set

$$\begin{aligned} \{[x^{i_1}]_0 \cdots [x^{i_k}]_0 : 1 \leq i_1, \dots, i_k \leq m\} &= \{[x^{i_1} \cdots x^{i_k}]_0 : 1 \leq i_1, \dots, i_k \leq m\} \\ &= \{[x^\alpha]_0 : |\alpha| = k\}, \end{aligned}$$

for $x^\alpha = (x^1)^{\alpha_1} \cdots (x^m)^{\alpha_m}$. As $[f]_0 \in \mathfrak{m}^k$, there are $[u_\alpha]_0 \in \mathcal{F}_0$, $|\alpha| = k$, such that

$$[f]_0 = \sum_{|\alpha|=k} [u_\alpha]_0 [x^\alpha]_0.$$

For $|\alpha| = k$, on some set in $\mathcal{U}_0$, using the Leibniz rule,

$$\partial^\alpha f = \sum_{|\beta|=k} \partial^\alpha (u_\beta x^\beta) = \sum_{|\beta|=k} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (\partial^{\alpha-\gamma} u_\beta)(\partial^\gamma x^\beta).$$

And for $\gamma \neq \beta$, $(\partial^\gamma x^\beta)(0) = 0$, so

$$\partial^\alpha f \in u_\alpha \partial^\alpha x^\alpha + h, \qquad [h]_0 \in \mathfrak{m}_0.$$

But $(\partial^\alpha f)(0) = 0$, so $u_\alpha(0) = 0$, which means that $u_\alpha \in \mathfrak{m}_0$. And

$$[x^\alpha]_0 = [x^1]_0^{\alpha_1} \cdots [x^m]_0^{\alpha_m} \in \mathfrak{m}_0^{|\alpha|} = \mathfrak{m}_0^k,$$

so $[u_\alpha]_0 [x^\alpha]_0 \in \mathfrak{m}_0^{k+1}$, showing that $[f]_0 \in \mathfrak{m}_0^{k+1}$. This completes the proof by induction. $\square$

4 Hessians

For an open set U in $\mathbb{R}^m$ and $\phi \in \mathcal{F}(U)$, $\phi' : U \to \mathscr{L}(\mathbb{R}^m, \mathbb{R})$, and $\nabla\phi : U \to \mathbb{R}^m$ satisfies

$$\langle \nabla\phi(x), v\rangle = \phi'(x)(v), \qquad x \in U, \quad v \in \mathbb{R}^m.$$

$x \in U$ is a **critical point of** ϕ if $\phi'(x) = 0$, equivalently $\nabla\phi(x) = 0$. Define $\mathrm{Hess}\,\phi : U \to \mathscr{L}(\mathbb{R}^m, \mathbb{R}^m)$ by

$$\mathrm{Hess}\,\phi = (\nabla\phi)'.$$

This satisfies[2]

$$\phi''(x)(u)(v) = \langle v, \mathrm{Hess}\,\phi(x)(u)\rangle, \qquad x \in U, \qquad u, v, \in \mathbb{R}^m.$$

A critical point x of ϕ is called **nondegenerate** if $\mathrm{Hess}\,\phi(x)$ is invertible in $\mathscr{L}(\mathbb{R}^m, \mathbb{R}^m)$.

For $\phi \in \mathcal{R}_p$, let J_ϕ be the ideal in the ring $\mathcal{F}_p$ generated by the set

$$\{[\partial_i\phi]_p : 1 \le i \le m\}.$$

We call J_ϕ the **Jacobian ideal of** ϕ **at** p. If p is a critical point of ϕ, then $(\partial_i\phi)(p) = 0$ for each i, hence $[\partial_i\phi]_p \in \mathfrak{m}_p$ for each i.

If 0 is a nondegenerate critical point of ϕ, we prove that $\mathfrak{m}_0 \subset J_\phi$.[3]

Theorem 3. *Let U be an open set in $\mathbb{R}^m$ containing 0 and let $\phi \in \mathcal{F}(U)$. If 0 is a nondegenerate critical point of ϕ, then $J_\phi = \mathfrak{m}_0$.*

Proof. Let $f = \nabla\phi$, which is a smooth function $U \to \mathbb{R}^m$. Because 0 is a nondegenerate critical point of ϕ, $f'(0)$ is invertible in $\mathscr{L}(\mathbb{R}^m, \mathbb{R}^m)$ and hence by the **inverse function theorem**,[4] f is a local C^∞ isomorphism at x: there is some open set V, $x \in V$ and $V \subset U$, such that $W = f(V)$ is open in $\mathbb{R}^m$, and there is a smooth function $g : W \to V$ such that $g \circ f = \mathrm{id}_V$ and $f \circ g = \mathrm{id}_W$. $\square$

[2] `http://individual.utoronto.ca/jordanbell/notes/gradienthilbert.pdf`
[3] Liviu Nicolaescu, *An Invitation to Morse Theory*, second ed., p. 15, Lemma 1.15.
[4] Serge Lang, *Real and Functional Analysis*, third ed., p. 361, chapter XIV, Theorem 1.2.