L^p norms of a sine sum

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April 3, 2014

George Pólya and Gábor Szegő, *Problems and theorems in analysis, Volume II*, volume 216 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, 1976, translated from the German by C. E. Billigheimer.

The following is an estimate of the L^p norms of a sum with nonnegative terms (p. 77, no. 38).

Let $\Gamma_n(t) = \sum_{k=1}^n \frac{|\sin kt|}{k}$. As $|\sin kt| \leq 1$, we have $\Gamma_n(t) \leq \sum_{k=1}^n \frac{1}{k}$, but we can give a sharper upper bound for $\Gamma_n(t)$ using the following two results. First, if $B_n(t) = \sum_{k=1}^n \frac{\cos kt}{k}$, then $B_n(t) \geq -1$ for all t (p. 75, no. 28). Second , for all t (p. 76, no. 34),

$$|\sin t| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{\cos 2jt}{4j^2-1}.$$

Therefore

$$\begin{aligned}
\Gamma_n(t) &= \sum_{k=1}^n \frac{1}{k} \left(\frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{\cos 2jkt}{4j^2-1} \right) \\
&= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{1}{4j^2-1} \sum_{k=1}^n \frac{\cos 2jkt}{k} \\
&= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} - \frac{4}{\pi} \sum_{j=1}^\infty \frac{B_n(2jt)}{4j^2-1} \\
&\leq \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} + \frac{4}{\pi} \sum_{j=1}^\infty \frac{1}{4j^2-1} \\
&= \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} + \frac{2}{\pi},
\end{aligned}$$

using

$$\sum_{j=1}^\infty \frac{1}{4j^2-1} = \sum_{j=1}^\infty \frac{1}{2} \cdot \frac{1}{2j-1} - \frac{1}{2} \cdot \frac{1}{2j+1} = \frac{1}{2}.$$

Thus $\|\Gamma_n\|_p \leq \|\Gamma_n\|_\infty \leq \frac{2}{\pi}\sum_{k=1}^n \frac{1}{k} + \frac{2}{\pi}$. On the other hand,

$$\begin{aligned}
\|\Gamma_n\|_1 &= \frac{1}{2\pi}\int_0^{2\pi} \Gamma_n(t)dt \\
&= \frac{1}{2\pi}\sum_{k=1}^n \frac{1}{k}\int_0^{2\pi} |\sin kt|dt \\
&= \frac{1}{2\pi}\sum_{k=1}^n \frac{1}{k^2}\int_0^{2\pi k} |\sin t|dt \\
&= \frac{1}{2\pi}\sum_{k=1}^n \frac{1}{k}\int_0^{2\pi} |\sin t|dt \\
&= \frac{1}{2\pi}\sum_{k=1}^n \frac{1}{k}\cdot 4 \\
&= \frac{2}{\pi}\sum_{k=1}^n \frac{1}{k}.
\end{aligned}$$

Therefore $\|\Gamma_n\|_p \geq \|\Gamma_n\|_1 = \frac{2}{\pi}\sum_{k=1}^n \frac{1}{k}$.

Hence for $1 \leq p \leq \infty$,

$$\frac{2}{\pi}\sum_{k=1}^n \frac{1}{k} \leq \|\Gamma_n\|_p \leq \frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^n \frac{1}{k}.$$

So,

$$\|\Gamma_n\|_p = \frac{2}{\pi}\log n + O(1).$$