The spectrum of a self-adjoint operator is a compact subset of $\mathbb{R}$

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Abstract

In these notes I prove that the spectrum of a bounded linear operator from a Hilbert space to itself is a nonempty compact subset of $\mathbb{C}$, and that if the operator is self-adjoint then the spectrum is contained in $\mathbb{R}$. To show that the spectrum is nonempty I prove various facts about resolvents.

1 Adjoints

1.1 Operator norm

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$, and define $I : H \to H$ by $Ix = x$, $x \in H$.. For $v \in H$, let $\|v\| = \sqrt{\langle v, v \rangle}$, and if $T : H \to H$ is a bounded linear map, let

$$\|T\| = \sup_{\|v\| \leq 1} \|Tv\| .$$

namely, the *operator norm* of T.

1.2 Definition of adjoint

The *Riesz representation theorem* states that if $\phi : H \to \mathbb{C}$ is a bounded linear map then there is a unique $v_\phi \in H$ such that

$$\phi(x) = \langle x, v_\phi \rangle$$

for all $x \in H$. Let $T : H \to H$ be a bounded linear map, and for $y \in H$, define $\phi_y : H \to \mathbb{C}$ by

$$\phi_y(x) = \langle Tx, y \rangle .$$

$\phi_y : H \to \mathbb{C}$ is a bounded linear map, so by the Riesz representation theorem there is a unique v_y such that

$$\phi_y(x) = \langle x, v_y \rangle$$

for all $x \in H$. Define $T^* : H \to H$ by

$$T^* y = v_y.$$

$T^* y$ is well-defined because of the uniqueness in the Riesz representation theorem. For all $x, y \in H$,

$$\langle x, T^* y \rangle = \langle x, v_y \rangle = \phi_y(x) = \langle Tx, y \rangle .$$

We call $T^* : H \to H$ the *adjoint* of $T : H \to H$.

1.3 Adjoint is linear

For $y_1, y_2 \in H$, we have for all $x \in H$ that

$$\begin{aligned} \langle x, T^*(y_1 + y_2) \rangle &= \langle Tx, y_1 + y_2 \rangle \\ &= \langle Tx, y_1 \rangle + \langle Tx, y_2 \rangle \\ &= \langle x, T^* y_1 + T^* y_2 \rangle . \end{aligned}$$

Hence for all $x \in H$,

$$\langle x, T^*(y_1 + y_2) - T^* y_1 - T^* y_2 \rangle = 0.$$

In particular this is true for $x = T^*(y_1 + y_2) - T^* y_1 - T^* y_2$, so by the nondegeneracy of $\langle \cdot, \cdot \rangle$ we get

$$T^*(y_1 + y_2) - T^* y_1 - T^* y_2 = 0.$$

We similarly obtain for all $\lambda \in \mathbb{C}$ and all $y \in H$ that

$$T^*(\lambda y) - \lambda T^* y = 0.$$

Hence $T^* : H \to H$ is a linear map.

1.4 Adjoint is bounded

For $x, y \in H$, by the Cauchy-Schwarz inequality we have

$$|\phi_y(x)| = |\langle x, v_y \rangle| \leq \|x\| \, \|v_y\| ,$$

so $\|\phi_y\| \leq \|v_y\|$, i.e. the operator norm of ϕ_y is less than or equal to the norm of v_y. If $v_y \neq 0$, then $\left\| \frac{v_y}{\|v_y\|} \right\| = 1$ and

$$\left| \phi_y \left(\frac{v_y}{\|v_y\|} \right) \right| = \left\langle \frac{v_y}{\|v_y\|}, v_y \right\rangle = \|v_y\| .$$

It follows that

$$\|\phi_y\| = \|v_y\| .$$

Then for $y \in H$, by the Cauchy-Schwarz inequality and because T is bounded we have

$$\begin{aligned}
\|T^* y\| &= \|v_y\| \\
&= \|\phi_y\| \\
&= \sup_{\|x\| \leq 1} \|\phi_y(x)\| \\
&= \sup_{\|x\| \leq 1} |\langle Tx, y \rangle| \\
&\leq \sup_{\|x\| \leq 1} \|T\| \, \|x\| \, \|y\| \\
&\leq \|T\| \, \|y\| .
\end{aligned}$$

Therefore T^* is bounded. Thus if $T : H \to H$ is a bounded linear map then its adjoint $T^* : H \to H$ is a bounded linear map.

1.5 Adjoint is involution

Because $T^* : H \to H$ is a bounded linear map, it has an adjoint $T^{**} : H \to H$, and T^{**} is itself a bounded linear map. For all $x, y \in H$,

$$\begin{aligned}
\langle Tx, y \rangle &= \langle x, T^* y \rangle \\
&= \overline{\langle T^* y, x \rangle} \\
&= \overline{\langle y, T^{**} x \rangle} \\
&= \langle T^{**} x, y \rangle .
\end{aligned}$$

Hence for all $x, y \in H$,

$$\langle Tx - T^{**}x, y \rangle = 0.$$

This is true in particular for $y = Tx - T^{**}x$, so by the nondegeneracy of $\langle \cdot, \cdot \rangle$ we obtain

$$Tx - T^{**}x = 0, \qquad x \in H.$$

Thus for any bounded linear map $T : H \to H$, $T^{**} = T$. In words, if T is a bounded linear map from a Hilbert space to itself, then the adjoint of its adjoint is itself. We have shown already that $\|T^*\| \leq \|T\|$. Hence also $\|T\| = \|T^{**}\| \leq \|T^*\|$, so

$$\|T\| = \|T^*\| .$$

If $T^* = T$, we say that T is *self-adjoint*.

2 Bounded linear operators

Let $\mathscr{B}(H)$ be the set of bounded linear maps $H \to H$. With the operator norm, one checks that $\mathscr{B}(H)$ is a Banach space. We define a product on $\mathscr{B}(H)$ by $T_1 T_2 = T_1 \circ T_2$, and thus $\mathscr{B}(H)$ is an algebra. We have

$$\|T_1 T_2\| = \sup_{\|x\| \leq 1} \|T_1(T_2 x)\| \leq \sup_{\|x\| \leq 1} \|T_1\| \, \|T_2 x\| = \|T_1\| \sup_{\|x\| \leq 1} \|T_2 x\| \leq \|T_1\| \, \|T_2\| ,$$

and thus $\mathscr{B}(H)$ is a *Banach algebra.*[1] Let $\mathscr{B}_{\mathrm{sa}}(H)$ be the set of all $T \in \mathscr{B}(H)$ that are self-adjoint.

Theorem 1. *If $T \in \mathscr{B}(H)$, then T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.*

Proof. If $T \in \mathscr{B}_{\mathrm{sa}}(H)$, then for all $x \in H$,

$$\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle},$$

so $\langle Tx, x\rangle \in \mathbb{R}$.

If $T \in \mathscr{B}(H)$ and $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$, then

$$\langle Tx, x\rangle = \langle x, T^*x\rangle = \overline{\langle T^*x, x\rangle} = \langle T^*x, x\rangle\,,$$

so, putting $A = T - T^*$, for all $x \in H$ we have

$$\langle Ax, x\rangle = 0.$$

Thus, for all $x, y \in H$ we have

$$\langle Ax, x\rangle = 0, \qquad \langle Ay, y\rangle = 0, \qquad \langle A(x+y), x+y\rangle = 0,$$

and combining these three equations,

$$0 = \langle Ax, x\rangle + \langle Ax, y\rangle + \langle Ay, x\rangle + \langle Ay, y\rangle = 0 + \langle Ax, y\rangle + \langle Ay, x\rangle + 0.$$

But $A^* = -A$, so we get

$$\langle Ax, y\rangle + \langle y, -Ax\rangle = 0,$$

hence

$$\langle Ax, y\rangle - \overline{\langle Ax, y\rangle} = 0. \tag{1}$$

As well, for all $x, y \in H$ we have

$$\langle Ax, -iy\rangle - \overline{\langle Ax, -iy\rangle} = 0,$$

so

$$\langle Ax, y\rangle + \overline{\langle Ax, y\rangle} = 0. \tag{2}$$

By (1) and (2), for all $x, y \in H$ we have

$$\langle Ax, y\rangle = 0,$$

and thus $A = 0$, i.e. $T = T^*$. $\square$

[1] The adjoint map $* : \mathscr{B}(H) \to \mathscr{B}(H)$ satisfies, for $\lambda \in \mathbb{C}$ and $T_1, T_2 \in \mathscr{B}(H)$,

$$T^{**} = T, \quad (T_1 + T_2)^* = T_1^* + T_2^*, \quad (\lambda T)^* = \overline{\lambda} T^*, \quad \|T^*T\| = \|T\|^2\,.$$

Thus $\mathscr{B}(H)$ is a C^**-algebra.* $I \in \mathscr{B}(H)$, so we say that $\mathscr{B}(H)$ is *unital.*

Using the above characterization of bounded self-adjoint operators, we can prove that a limit of bounded self-adjoint operators is itself a bounded self-adjoint operator.

Theorem 2. *$\mathscr{B}_{\mathrm{sa}}(H)$ is a closed subset of $\mathscr{B}(H)$.*

Proof. If $T_n \in \mathscr{B}_{\mathrm{sa}}(H)$ and $T_n \to T \in \mathscr{B}(H)$, then for $x \in H$ we have

$$\langle Tx, x \rangle = \lim_{n\to\infty} \langle T_n x, x \rangle \in \mathbb{R},$$

hence $T \in \mathscr{B}_{\mathrm{sa}}(H)$. □

If $T \in \mathscr{B}_{\mathrm{sa}}(H)$ and $\langle Tx, x \rangle \geq 0$ for all $x \in H$, we say that T is *positive*. Let $\mathscr{B}_+(H)$ be the set of all positive $T \in \mathscr{B}_{\mathrm{sa}}(H)$. For $S, T \in \mathscr{B}_{\mathrm{sa}}(H)$, if

$$T - S \in \mathscr{B}_+(H)$$

we write $S \leq T$. Thus, we can talk about one self-adjoint operator being greater than or equal to another self-adjoint operator. $S \leq T$ is equivalent to

$$\langle Sx, x \rangle \leq \langle Tx, x \rangle$$

for all $x \in H$.

3 A condition for invertibility

Theorem 3. *If $T \in \mathscr{B}(H)$ and there is some $\alpha > 0$ such that $\alpha I \leq TT^*$ and $\alpha I \leq T^*T$, then $T^{-1} \in \mathscr{B}(H)$.*

Proof. By $\alpha I \leq T^*T$, we have for all $x \in H$,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \geq \langle \alpha x, x \rangle = \alpha \|x\|^2,$$

so $\|Tx\| \geq \sqrt{\alpha}\, \|x\|$. This implies that T is injective. By $\alpha I \leq TT^*$, we have for all $x \in H$,

$$\|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle TT^*x, x \rangle \geq \langle \alpha x, x \rangle = \alpha \|x\|^2,$$

so $\|T^*x\| \geq \sqrt{\alpha}\, \|x\|$, and hence T^* is injective. Let $Tx_n \to y \in H$. Then,

$$\|Tx_n - Tx_m\|^2 = \|T(x_n - x_m)\|^2 \geq \alpha \|x_n - x_m\|^2.$$

Since Tx_n converges it is a Cauchy sequence, and from the above inequality it follows that x_n is a Cauchy sequence, hence there is some $x \in H$ with $x_n \to x$. As T is continuous, $y = Tx \in T(H)$, showing that $T(H)$ is a closed subset of H. But it is a fact that if $T \in \mathscr{B}(H)$ then the closure of $T(H)$ is equal to $(\ker T^*)^\perp$.[2] Thus, as we have shown that T^* is injective,

$$T(H) = (\ker T^*)^\perp = \{0\}^\perp = H,$$

[2] It is straightforward to show that if v is in the closure of $T(H)$ and $w \in \ker T^*$ then $\langle v, w \rangle = 0$. It is less straightforward to show the opposite inclusion.

i.e. T is surjective. Hence $T : H \to H$ is bijective. It is a fact that if $T \in \mathscr{B}(H)$ is bijective then $T^{-1} \in \mathscr{B}(H)$, completing the proof.[3] $\square$

4 Spectrum

For $T \in \mathscr{B}(H)$, we define the *spectrum* $\sigma(T)$ of T to be the set of all $\lambda \in \mathbb{C}$ such $T-\lambda I$ is not bijective, and we define the *resolvent set* of T to be $\rho(T) = \mathbb{C} \setminus \sigma(T)$. To say that $\lambda \in \rho(T)$ is to say that $T - \lambda I$ is a bijection, and if $T - \lambda I$ is a bijection it follows from the open mapping theorem that its inverse function is an element of $\mathscr{B}(H)$: the inverse of a linear bijection is itself linear, but the inverse of a continuous bijection need not itself be continuous, which is where we use the open mapping theorem.

We prove that the spectrum of a bounded self-adjoint operator is real.

Theorem 4. *If $T \in \mathscr{B}_{\mathrm{sa}}(H)$, then $\sigma(T) \subseteq \mathbb{R}$.*

Proof. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lambda = a + ib$, $b \neq 0$, and $X = T - \lambda I$, then

$$\begin{aligned}
XX^* &= (T - \lambda I)(T - \lambda I)^* \\
&= (T - (a + ib)I)(T - (a - ib)I) \\
&= T^2 - (a - ib)T - (a + ib)T + (a^2 + b^2)I \\
&= (a^2 + b^2)I - 2aT + T^2 \\
&= b^2 I + (aI - T)^2 \\
&= b^2 I + (aI - T)(aI - T)^* \\
&\geq b^2 I.
\end{aligned}$$

$X^*X = XX^* \geq b^2 I$ and $b > 0$, so by Theorem 3, $X = T - \lambda I$ has an inverse $(T - \lambda I)^{-1} \in \mathscr{B}(H)$, showing $\lambda \notin \sigma(T)$. $\square$

5 The spectrum of a bounded linear map is bounded

If $\lambda \in \rho(T)$ then we define $R_\lambda = (T - \lambda I)^{-1} \in \mathscr{B}(H)$, called the *resolvent* of T.

Theorem 5. *If $T \in \mathscr{B}(H)$ and $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$.*

Proof. Define $R_{\lambda,N} \in \mathscr{B}(H)$ by

$$R_{\lambda,N} = -\frac{1}{\lambda} \sum_{n=0}^{N} \frac{T^n}{\lambda^n}.$$

[3] $T^{-1} : H \to H$ is linear. The *open mapping theorem* states that if X and Y are Banach spaces and $S : X \to Y$ is a bounded linear map that is surjective, then S is an open map, i.e., if U is an open subset of X then $S(U)$ is an open subset of Y. Here, $T \in \mathscr{B}(H)$ and T is bijective, and so by the open mapping theorem T is open, from which it follows that $T^{-1} : H \to H$ is continuous, and so bounded (a linear map between normed vector spaces is continuous if and only if it is bounded).

As $\frac{\|T\|}{|\lambda|} < 1$, the geometric series $\sum_{n=0}^{\infty} \frac{\|T\|^n}{|\lambda|^n}$ converges, from which it follows that $R_{\lambda,N}$ is a Cauchy sequence in $\mathscr{B}(H)$ and so converges to some $S_\lambda \in \mathscr{B}(H)$. We have

$$\begin{aligned}
\|S_\lambda(T-\lambda I) - I\| &\leq \|S_\lambda(T-\lambda I) - R_{\lambda,N}(T-\lambda I)\| \\
&\quad + \|R_{\lambda,N}(T-\lambda I) - I\| \\
&\leq \|S_\lambda - R_{\lambda,N}\| \, \|T-\lambda I\| + \left\| -\frac{T}{\lambda}\sum_{n=0}^{N}\frac{T^n}{\lambda^n} + \sum_{n=0}^{N}\frac{T^n}{\lambda^n} - I \right\| \\
&= \|S_\lambda - R_{\lambda,N}\| \, \|T-\lambda I\| + \left\| -\frac{T^{N+1}}{\lambda^{N+1}} \right\| \\
&\leq \|S_\lambda - R_{\lambda,N}\| \, \|T-\lambda I\| + \left(\frac{\|T\|}{|\lambda|}\right)^{N+1},
\end{aligned}$$

which tends to 0 as $N \to \infty$. Therefore $S_\lambda(T-\lambda I) = I$. And,

$$\begin{aligned}
\|(T-\lambda I)S_\lambda - I\| &\leq \|(T-\lambda I)S_\lambda - (T-\lambda I)R_{\lambda,N}\| \\
&\quad + \|(T-\lambda I)R_{\lambda,N} - I\| \\
&\leq \|T-\lambda I\| \, \|S_\lambda - R_{\lambda,N}\| + \left(\frac{\|T\|}{|\lambda|}\right)^{N+1},
\end{aligned}$$

whence $(T-\lambda I)S_\lambda = I$, showing that

$$S_\lambda = (T-\lambda I)^{-1}.$$

Thus, if $|\lambda| > \|T\|$ then $\lambda \in \rho(T)$. □

The above theorem shows that $\sigma(T)$ is a bounded set: it is contained in the closed disc $|\lambda| \leq \|T\|$. Moreover, if $|\lambda| > \|T\|$ then we have an explicit expression for the resolvent R_λ:

$$R_\lambda = -\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{T^n}{\lambda^n}.$$

6 The spectrum of a bounded linear map is closed

Theorem 6. *If $T \in \mathscr{B}(H)$, then $\rho(T)$ is an open subset of $\mathbb{C}$.*

Proof. If $\lambda \in \rho(T)$, let $|\mu - \lambda| < \|R_\lambda\|^{-1}$, and define $R_{\mu,N} \in \mathscr{B}(H)$ by

$$R_{\mu,N} = R_\lambda \sum_{n=0}^{N} (\mu-\lambda)^n R_\lambda^n.$$

Because $|\mu - \lambda| < \|R_\lambda\|^{-1}$, $R_{\mu,N}$ is a Cauchy sequence in $\mathscr{B}(H)$ and converges to some $S_\mu \in \mathscr{B}(H)$. We have, as $R_\lambda = (T - \lambda I)^{-1}$,

$$\begin{aligned}
\|S_\mu(T-\mu I) - I\| &\leq \|S_\mu(T-\mu I) - R_{\mu,N}(T-\mu I)\| \\
&\quad + \|R_{\mu,N}(T-\mu I + \lambda I - \lambda I) - I\| \\
&\leq \|S_\mu - R_{\mu,N}\| \, \|T-\mu I\| \\
&\quad + \|R_{\mu,N}(T-\lambda I) - R_{\mu,N}(\mu-\lambda) - I\| \\
&= \|S_\mu - R_{\mu,N}\| \, \|T-\mu I\| \\
&\quad + \left\| \sum_{n=0}^{N} (\mu-\lambda)^n R_\lambda^n - (\mu-\lambda) R_\lambda \sum_{n=0}^{N} (\mu-\lambda)^n R_\lambda^n - I \right\| \\
&= \|S_\mu - R_{\mu,N}\| \, \|T-\mu I\| + \left\| -(\mu-\lambda)^{N+1} R_\lambda^{N+1} \right\| \\
&= \|S_\mu - R_{\mu,N}\| \, \|T-\mu I\| + |\mu-\lambda|^{N+1} \|R_\lambda\|^{N+1},
\end{aligned}$$

which tends to 0 as $N \to \infty$. Therefore $S_\mu(T - \mu I) = I$. One checks likewise that $(T - \mu I)S_\mu = I$, and hence that

$$(T - \mu I)^{-1} = S_\mu,$$

showing that $\mu \in \rho(T)$. $\square$

As $\sigma(T)$ is bounded and closed, it is a compact set in $\mathbb{C}$. Moreover, if $\lambda \notin \sigma(T)$ and $|\mu - \lambda| < \|R_\lambda\|^{-1}$, then

$$R_\mu = R_\lambda \sum_{n=0}^{\infty} (\mu-\lambda)^n R_\lambda^n.$$

7 The spectrum of a bounded linear map is nonempty

Theorem 7. *If $T \in \mathscr{B}(H)$ is self-adjoint, then $\sigma(T) \neq \emptyset$.*

Proof. Suppose by contradiction that $\sigma(T) = \emptyset$.[4] If $\lambda, \mu \in \mathbb{C}$, then

$$\begin{aligned}
(T-\lambda I)(R_\lambda - R_\mu)(T-\mu I) &= (I - (T-\lambda I)R_\mu)(T-\mu I) \\
&= T - \mu I - (T - \lambda I) \\
&= (\lambda - \mu) I,
\end{aligned}$$

so

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu, \tag{3}$$

the *resolvent identity*. Thus

$$\|R_\lambda - R_\mu\| \leq |\lambda - \mu| \, \|R_\lambda\| \, \|R_\mu\|,$$

[4]For each $v, w \in H$ we are going to construct a bounded entire function $\mathbb{C} \to \mathbb{C}$ depending on v and w, which by Liouville's theorem must be constant, and it will turn out to be 0. This will lead to a contradiction.

and together with $\|R_\mu\| - \|R_\lambda\| \leq \|R_\mu - R_\lambda\|$ we get

$$\|R_\mu\| (1 - |\lambda - \mu| \|R_\lambda\|) \leq \|R_\lambda\| .$$

If $|\lambda - \mu| \leq \frac{1}{2} \cdot \|R_\lambda\|^{-1}$, then

$$\|R_\mu\| \leq 2 \|R_\lambda\| ,$$

whence, for $|\lambda - \mu| \leq \frac{1}{2} \cdot \|R_\lambda\|^{-1}$,

$$\|R_\lambda - R_\mu\| \leq 2|\lambda - \mu| \|R_\lambda\|^2 .$$

Therefore, $\lambda \mapsto R_\lambda$ is a continuous function $\mathbb{C} \to \mathscr{B}(H)$. From this and (3) it follows that for each $\lambda \in \mathbb{C}$,[5]

$$\lim_{\mu \to \lambda} \frac{R_\lambda - R_\mu}{\lambda - \mu} = R_\lambda^2.$$

Let $v, w \in H$ and define $f_{v,w} : \mathbb{C} \to \mathbb{C}$ by

$$f_{v,w}(\lambda) = \langle R_\lambda v, w \rangle , \qquad \lambda \in \mathbb{C}.$$

For $\lambda \in \mathbb{C}$,

$$\lim_{\mu \to \lambda} \frac{f_{v,w}(\lambda) - f_{v,w}(\mu)}{\lambda - \mu} = \lim_{\mu \to \lambda} \left\langle \frac{R_\lambda - R_\mu}{\lambda - \mu} v, w \right\rangle = \left\langle R_\lambda^2 v, w \right\rangle .$$

Thus $f_{v,w}$ is an entire function. For $|\lambda| > \|T\|$, $R_\lambda = -\frac{1}{\lambda} \sum_{n=0}^\infty \frac{T^n}{\lambda^n}$, so, for $r = \frac{\|T\|}{|\lambda|}$,

$$\begin{aligned}
\|R_\lambda\| &= \frac{1}{|\lambda|} \left\| \sum_{n=0}^\infty \frac{T^n}{\lambda} \right\| \\
&\leq \frac{1}{|\lambda|} \sum_{n=0}^\infty r^n \\
&= \frac{1}{|\lambda|} \frac{1}{1-r} \\
&= \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|T\|}{|\lambda|}} \\
&= \frac{1}{|\lambda| - \|T\|}.
\end{aligned}$$

Hence, for $|\lambda| > \|T\|$,

$$\begin{aligned}
|f_{v,w}(\lambda)| &= |\langle R_\lambda v, w \rangle | \\
&\leq \|R_\lambda\| \|v\| \|w\| \\
&\leq \frac{\|v\| \|w\|}{|\lambda| - \|T\|},
\end{aligned}$$

[5] There are no complications that appear if we do complex analysis on functions from $\mathbb{C}$ to a complex Banach algebra rather than on functions from $\mathbb{C}$ to $\mathbb{C}$. Thus this statement is that $\lambda \to R_\lambda$ is a holomorphic function $\mathbb{C} \to \mathscr{B}(H)$.

from which it follows that $f_{v,w}$ is bounded and that $\lim_{|\lambda|\to\infty} f_{v,w}(\lambda) = 0$. Therefore by Liouville's theorem, $f_{v,w}(\lambda) = 0$ for all λ. Let's recap: for all $v, w \in H$ and for all $\lambda \in \mathbb{C}$, $\langle R_\lambda v, w\rangle = 0$. Switching the order of the universal quantifiers, for all $\lambda \in \mathbb{C}$ and for all $v, w \in H$ we have $\langle R_\lambda v, w\rangle = 0$, which implies that for all $\lambda \in \mathbb{C}$ we have $R_\lambda = 0$. But by assumption R_λ is invertible, so this is a contradiction. Hence $\sigma(T)$ is nonempty. $\square$