Real reproducing kernel Hilbert spaces

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1 Reproducing kernels

We shall often speak about functions $F : X \times X \to \mathbb{R}$, where X is a nonempty set. For $x \in X$, we define $F_x : X \to \mathbb{R}$ by $F_x(y) = F(x, y)$ and for $y \in X$ we define $F^y : X \to \mathbb{R}$ by $F^y(x) = F(x, y)$. F is said to be **symmetric** if $F(x, y) = F(y, x)$ for all $x, y \in X$ and **positive-definite** if for any $x_1, \dots, x_n \in X$ and $c_1, \dots, c_n \in \mathbb{R}$ it holds that

$$\sum_{1 \le i,j \le n} c_i c_j F(x_i, x_j) \ge 0.$$

Lemma 1. *If $F : X \times X \to \mathbb{R}$ is symmetric and positive-definite then*

$$F(x, y)^2 \le F(x, x)F(y, y), \qquad x, y \in X.$$

Proof. For $\alpha, \beta \in \mathbb{R}$ define[1]

$$\begin{aligned} C(\alpha, \beta) &= \alpha^2 F(x, x) + \alpha\beta F(x, y) + \beta\alpha F(y, x) + \beta^2 F(y, y) \\ &= \alpha^2 F(x, x) + 2\alpha\beta F(x, y) + \beta^2 F(y, y), \end{aligned}$$

which is ≥ 0. Let

$$\begin{aligned} P(\alpha) &= C(\alpha, F(x, y)) \\ &= \alpha^2 F(x, x) + 2\alpha F(x, y)^2 + F(x, y)^2 F(y, y), \end{aligned}$$

which is ≥ 0. In the case $F(x, x) = 0$, the fact that $P \ge 0$ implies that $F(x, y) = 0$. In the case $F(x, y) \ne 0$, $P(\alpha)$ is a quadratic polynomial and because $P \ge 0$ it follows that the discriminant of P is ≤ 0:

$$4F(x, y)^4 - 4 \cdot F(x, x) \cdot F(x, y)^2 F(y, y) \le 0.$$

That is, $F(x, y)^4 \le F(x, y)^2 F(x, x)F(y, y)$, and this implies that $F(x, y)^2 \le F(x, x)F(y, y)$. ∎

[1] See Alain Berlinet and Christine Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, p. 13, Lemma 3.

A **real reproducing kernel Hilbert space** is a Hilbert space H contained in $\mathbb{R}^X$, where X is a nonempty set, such that for each $x \in X$ the map $\Lambda_x f = f(x)$ is continuous $H \to \mathbb{R}$. In this note we speak always about real Hilbert spaces.

Let $H \subset \mathbb{R}^X$ be a reproducing kernel Hilbert space. Because H is a Hilbert space, the Riesz representation theorem states that $\Phi : H \to H^*$ defined by

$$(\Phi g)(f) = \langle f, g \rangle_H, \qquad g, f \in H$$

is an isometric isomorphism. Because H is a reproducing kernel Hilbert space, $\Lambda_x \in H^*$ for each $x \in X$ and we define $T_x = \Phi^{-1}\Lambda_x \in H$, which satisfies

$$f(x) = \Lambda_x(f) = \langle f, T_x \rangle_H, \qquad f \in H.$$

In particular, because $T_x \in H$, for $y \in X$ it holds that

$$T_x(y) = \Lambda_y(T_x) = \langle T_x, T_y \rangle_H.$$

Define $K : X \times X \to \mathbb{R}$ by

$$K(x, y) = \langle T_x, T_y \rangle_H,$$

called **the reproducing kernel of** H. For $x, y \in X$,

$$T_x(y) = \langle T_x, T_y \rangle_H = K(x, y) = K_x(y),$$

which means that $T_x = K_x$.

A reproducing kernel is symmetric and positive-definite:

$$K(x, y) = \langle T_x, T_y \rangle_H = \langle T_y, T_x \rangle_H = K(y, x)$$

and

$$\begin{aligned}\sum_{1 \le i,j \le n} c_i c_j K(x_i, x_j) &= \sum_{1 \le i,j \le n} \left\langle c_i T_{x_i}, c_j T_{x_j} \right\rangle_H \\ &= \left\langle \sum_{1 \le i \le n} c_i T_{x_i}, \sum_{1 \le j \le n} c_j T_{x_j} \right\rangle_H \\ &\ge 0.\end{aligned}$$

Lemma 2. *If E is an orthonormal basis for a reproducing kernel Hilbert space $H \subset \mathbb{R}^X$ with reproducing kernel $K : X \times X \to \mathbb{R}$, then*

$$K(x, y) = \sum_{e \in E} e(x) e(y), \qquad x, y \in X.$$

Proof. Because E is an orthonormal basis for H, Parseval's identity tell us

$$\langle T_x, T_y \rangle_H = \sum_{e \in E} \langle T_x, e \rangle \langle T_y, e \rangle = \sum_{e \in E} \langle e, T_x \rangle \langle e, T_y \rangle = \sum_{e \in E} e(x) e(y).$$

□

If $H \subset \mathbb{R}^X$ is a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \to \mathbb{R}$ and V is a closed linear subspace of H, then V is itself a reproducing kernel Hilbert space, with some reproducing kernel $G : X \times X \to \mathbb{R}$. The following theorem expresses G in terms of K.[2]

Theorem 3. *Let $H \subset \mathbb{R}^X$ be a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \to \mathbb{R}$, let V be a closed linear subspace of H with reproducing kernel $G : X \times X \to \mathbb{R}$, and let $P_V : H \to V$ be the projection onto V. Then*

$$G_x = P_V K_x, \qquad x \in X.$$

Proof. $H = V \oplus V^\perp$, thus for $f \in H$ there are unique $g \in V, h \in V^\perp$ such that $f = g+h$, and $P_V f = g$.[3] Then $f - P_V f \in V^\perp$. Therefore for $y \in X$, as $G_y \in V$ it holds that

$$\langle f, G_y \rangle_H = \langle f - P_V f + P_V f, G_y \rangle_H = \langle P_V f, G_y \rangle_H = (P_V f)(y).$$

In particular, for $x, y \in X$ and $f = K_x$,

$$(P_V K_x)(y) = \langle K_x, G_y \rangle_H = \langle G_y, T_x \rangle_H = G_y(x) = G(y,x) = G(x,y) = G_x(y),$$

which means that $P_V K_x = G_x$, proving the claim. □

The **Moore-Aronszajn theorem** states that if X is a nonempty set and $K : X \times X \to \mathbb{R}$ is a symmetric and positive-definite function, then there is a unique reproducing kernel Hilbert space $H \subset \mathbb{R}^X$ for which K is the reproducing kernel.

We now prove that given a symmetric positive-definite kernel there is a unique reproducing Hilbert space for which it is the reproducing kernel.[4]

2 Sobolev spaces on $[0, T]$

Let $f \in \mathbb{R}^{[0,T]}$. The following are equivalent:[5]

1. f is absolutely continuous.
2. f is differentiable at almost all $t \in [0, T]$, $f' \in L^1$, and
$$f(t) = f(0) + \int_0^t f'(s) ds, \qquad t \in [0, T].$$
3. There is some $g \in L^1$ such that
$$f(t) = f(0) + \int_0^t g(s) ds, \qquad t \in [0, T].$$

[2] Ward Cheney and Will Light, *A Course in Approximation Theory*, p. 234, Chapter 31, Theorem 4.

[3] `http://individual.utoronto.ca/jordanbell/notes/pvm.pdf`

[4] Alain Berlinet and Christine Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, p. 19, Theorem 3.

[5] Elias M. Stein and Rami Shakarchi, *Real Analysis*, p. 130, Theorem 3.11.

In particular, if f is absolutely continuous and $f' = 0$ almost everywhere then $\int_0^t f'(s)ds = 0$ and so $f(t) = f(0)$ for all $t \in [0,T]$. That is, if f is absolutely continuous and $f' = 0$ almost everywhere then f is constant.

Let H be the set of those absolutely continuous functions $f \in \mathbb{R}^{[0,T]}$ such that $f(0) = 0$ and $f' \in L^2$. For $f, g \in H$ define

$$\langle f, g \rangle_H = \int_0^T f'(s)g'(s)ds.$$

If $\|f\|_H = 0$ then $\int_0^T f'(s)^2 ds = 0$, which implies that $f' = 0$ almost everywhere and hence that f is constant, and therefore $f = 0$. Thus $\langle \cdot, \cdot \rangle_H$ is indeed an inner product on H.

If f_n is a Cauchy sequence in H then f_n' is a Cauchy sequence in L^2 and hence converges to some $g \in L^2$. Then the function $f \in \mathbb{R}^{[0,T]}$ defined by

$$f(t) = \int_0^t g(s)ds, \qquad t \in [0,T],$$

is absolutely continuous, $f(0) = 0$, and satisfies $f' = g$ almost everywhere, which shows that $f \in H$. Then $f_n \to f$ in H, which proves that H is a Hilbert space. For $t \in [0,T]$, by the Cauchy-Schwarz inequality,

$$|f(t)|^2 = \left| \int_0^t f'(s)ds \right|^2 \leq \left| \int_0^T f'(s)ds \right|^2 \leq T \int_0^T f'(s)^2 ds = T \|f\|_H^2,$$

i.e. $|L_t f| \leq T^{1/2} \|f\|_H$, which shows that $L_t \in H^*$. Therefore H is a reproducing kernel Hilbert space.

For $a \in [0,T]$ define $h_a : [0,T] \to \mathbb{R}$ by $h_a(s) = 1_{[0,a]}(s)$, which belongs to L^2, and define $g_a : [0,T] \to \mathbb{R}$ by

$$g_a(t) = \int_0^t h_a(s)ds = \min(t,a),$$

which belongs to H. For $f \in H$,

$$\langle f, g_a \rangle_H = \int_0^T f'(s)g_a'(s)ds = \int_0^T f'(s)1_{[0,a]}(s)ds = \int_0^a f'(s)ds = f(a).$$

This means that $K_a = g_a$. For $a, b \in [0,T]$,

$$\langle K_a, K_b \rangle_H = \int_0^T g_a'(s)g_b'(s)ds = \int_0^T 1_{[0,a]}(s)1_{[0,b]}(s)ds = \int_0^T 1_{[0,\min(a,b)]}(s)ds.$$

That is, the reproducing kernel of H is $K : [0,T] \times [0,T] \to \mathbb{R}$,

$$K(a,b) = \langle K_a, K_b \rangle_H = \min(a,b).$$

3 Sobolev spaces on $\mathbb{R}$

Let λ be Lebesgue measure on $\mathbb{R}$. Let $\mathscr{L}^2(\lambda)$ be the collection of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f|^2$ is integrable, and let $L^2(\lambda)$ be the Hilbert space of equivalence classes of elements of $\mathscr{L}^2(\lambda)$ where $f \sim g$ when $f = g$ almost everywhere, with

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} fg d\lambda.$$

Let $H^1(\mathbb{R})$ be the set of locally absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f, f' \in L^2(\lambda)$. This is a Hilbert space with the inner product[6]

$$\langle f, g \rangle_{H^1} = \langle f, g \rangle_{L^2} + \langle f', g' \rangle_{L^2}.$$

Define $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$K(x, y) = \frac{1}{2} \exp(-|x - y|), \qquad x, y \in \mathbb{R}.$$

Let $x \in \mathbb{R}$. For $y < x$, $K_x'(y) = K_x(y)$ and for $y > x$, $K_x'(y) = -K_x(y)$, which shows that $K_x \in H^1(\mathbb{R})$. For $f \in H^1(\mathbb{R})$, doing integration by parts,

$$\begin{aligned}
\langle f, K_x \rangle_{H^1} &= \int_{-\infty}^{\infty} f K_x d\lambda + \int_{-\infty}^{x} f'(y) K_x(y) d\lambda(y) - \int_{x}^{\infty} f'(y) K_x(y) d\lambda(y) \\
&= \int_{-\infty}^{\infty} f K_x d\lambda + f(x) K_x(x) - \int_{-\infty}^{x} f(y) K_x'(y) d\lambda(y) \\
&+ f(x) K_x(x) + \int_{x}^{\infty} f(y) K_x'(y) d\lambda(y) \\
&= \int_{-\infty}^{\infty} f K_x d\lambda + \frac{1}{2} f(x) - \int_{-\infty}^{x} f(y) K_x(y) d\lambda(y) \\
&+ \frac{1}{2} f(x) - \int_{x}^{\infty} f(y) K_x(y) d\lambda(y) \\
&= f(x) \\
&= T_x f.
\end{aligned}$$

This shows that $H^1(\mathbb{R})$ is a reproducing kernel Hilbert space. We calculate, for $x < y$,

$$\begin{aligned}
\langle T_x, T_y \rangle_{H^1} &= \int_{-\infty}^{x} K_x K_y d\lambda + \int_{x}^{y} K_x K_y d\lambda + \int_{y}^{\infty} K_x K_y d\lambda \\
&+ \int_{-\infty}^{x} K_x K_y d\lambda - \int_{x}^{y} K_x K_y d\lambda + \int_{y}^{\infty} K_x K_y d\lambda \\
&= 4 \cdot \frac{1}{8} \exp(x - y) \\
&= K(x, y).
\end{aligned}$$

[6] http://individual.utoronto.ca/jordanbell/notes/sobolev1d.pdf

This shows that $K(x, y) = \frac{1}{2}\exp(-|x - y|)$ is the reproducing kernel of $H^1(\mathbb{R})$.[7]

[7] cf. Alain Berlinet and Christine Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, pp. 8–9, Example 5.