Polish spaces and Baire spaces

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1 Introduction

These notes consist of me working through those parts of the first chapter of Alexander S. Kechris, *Classical Descriptive Set Theory*, that I think are important in analysis. Denote by $\mathbb{N}$ the set of positive integers. I do not talk about universal spaces like the Cantor space $2^{\mathbb{N}}$, the Baire space $\mathbb{N}^{\mathbb{N}}$, and the Hilbert cube $[0,1]^{\mathbb{N}}$, or "localization", or about Polish groups.

If (X, τ) is a topological space, the **Borel σ-algebra of** X, denoted by $\mathscr{B}_X$, is the smallest σ-algebra of subsets of X that contains τ. $\mathscr{B}_X$ contains τ, and is closed under complements and countable unions, and rather than talking merely about **Borel sets** (elements of the Borel σ-algebra), we can be more specific by talking about open sets, closed sets, and sets that are obtained by taking countable unions and complements.

Definition 1. An F_σ **set** is a countable union of closed sets.

A G_δ **set** is a complement of an F_σ set. Equivalently, it is a countable intersection of open sets.

If (X, d) is a metric space, the **topology induced by the metric** d is the topology generated by the collection of open balls. If (X, τ) is a topological space, a metric d on the set X is said to be **compatible with** τ if τ is the topology induced by d. A **metrizable space** is a topological space whose topology is induced by some metric, and a **completely metrizable space** is a topological space whose topology is induced by some complete metric. One proves that being metrizable and being completely metrizable are topological properties, i.e., are preserved by homeomorphisms.

If X is a topological space, a **subspace of** X is a subset of X which is a topogical space with the subspace topology inherited from X. Because any topological space is a closed subset of itself, when we say that a **subspace is closed** we mean that it is a closed subset of its parent space, and similarly for open, F_σ, G_δ. A subspace of a compact Hausdorff space is compact if and only if it is closed; a subspace of a metrizable space is metrizable; and a subspace of a completely metrizable space is completely metrizable if and only if it is closed.

A topological space is said to be **separable** if it has a countable dense subset, and **second-countable** if it has a countable basis for its topology. It is straightforward to check that being second-countable implies being separable, but a separable topological space need not be second-countable. However, one checks that a separable metrizable space is second-countable. A subspace of a second-countable topological space is second-countable, and because a subspace of a metrizable space is metrizable, it follows that a subspace of a separable metrizable space is separable.

A **Polish space** is a separable completely metrizable space. My own interest in Polish spaces is because one can prove many things about Borel probability measures on a Polish space that one cannot prove for other types of topological spaces. Using the fact (the **Heine-Borel theorem**) that a compact metric space is complete and totally bounded, one proves that a compact metrizable space is Polish, but for many purposes we do not need a metrizable space to be compact, only Polish, and using compact spaces rather than Polish spaces excludes, for example, $\mathbb{R}$.

2 Separable Banach spaces

Let K denote either $\mathbb{R}$ or $\mathbb{C}$. If X and Y are Banach spaces over K, we denote by $\mathscr{B}(X,Y)$ the set of bounded linear operators $X \to Y$. With the operator norm, this is a Banach space. We shall be interested in the **strong operator topology**, which is the initial topology on $\mathscr{B}(X,Y)$ induced by the family $\{T \mapsto Tx : x \in X\}$. One proves that the strong operator topology on $\mathscr{B}(X,Y)$ is induced by the family of seminorms $\{T \mapsto \|Tx\| : x \in X\}$, and because this is a separating family of seminorms, $\mathscr{B}(X,Y)$ with the strong operator topology is a **locally convex space**. A basis of convex sets for the strong operator topology consists of those sets of the form

$$\{S \in \mathscr{B}(X,Y) : \|Sx_1 - T_1x_1\| < \epsilon, \dots, \|Sx_n - T_nx_n\| < \epsilon\},$$

for $x_1, \dots, x_n \in X$, $\epsilon > 0$, $T_1, \dots, T_n \in \mathscr{B}(X,Y)$.

We prove conditions under which the closed unit ball in $\mathscr{B}(X,Y)$ with the strong operator topology is Polish.[1]

Theorem 2. Suppose that X and Y are separable Banach spaces. Then the closed unit ball

$$B_1 = \{T \in \mathscr{B}(X,Y) : \|T\| \le 1\}$$

with the subspace topology inherited from $\mathscr{B}(X,Y)$ with the strong operator topology is Polish.

Proof. Let E be $\mathbb{Q}$ or $\{a + ib : a, b \in \mathbb{Q}\}$, depending on whether K is $\mathbb{R}$ or $\mathbb{C}$, let D_0 be a countable dense subset of X, and let D be the span of D_0 over K. D is countable and Y is Polish, so the product Y^D is Polish. Define $\Phi : B_1 \to Y^D$ by $\Phi(T) = T \circ \iota$, where $\iota : D \to X$ is the inclusion map. If $\Phi(S) = \Phi(T)$, then because D is dense in X and $S, T : X \to Y$ are continuous, $X = Y$,

[1] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 14.

showing that Φ is one-to-one. We check that $\Phi(B_1)$ consists of those $f \in Y^D$ such that both (i) if $x, y \in D$ and $a, b \in E$ then $f(ax + by) = af(x) + bf(y)$, and (ii) if $x \in D$ then $\|f(x)\| \leq \|x\|$. One proves that $\Phi(B_1)$ is a closed subset of Y^D, and because Y^D is Polish this implies that $\Phi(B_1)$ with the subspace topology inherited from Y^D is Polish. Then one proves that $\Phi : B_1 \to \Phi(B_1)$ is a homeomorphism, where B_1 has the subspace topology inherited from $\mathscr{B}(X, Y)$ with the strong operator topology, which tells us that B_1 is Polish. □

If X is a Banach space over K, where K is $\mathbb{R}$ or $\mathbb{C}$, we write $X^* = \mathscr{B}(X, K)$. The strong operator topology on $\mathscr{B}(X, K)$ is called the **weak-*** topology on X^*. **Keller's theorem**[2] states that if X is a separable infinite-dimensional Banach space, then the closed unit ball in X^* with the subspace topology inherited from X^* with the weak-* topology is homeomorphic to the Hilbert cube $[0, 1]^{\mathbb{N}}$.

3 G-delta sets

If (X, d) is a metric space and A is a subset of X, we define

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\},$$

with $\operatorname{diam}(\emptyset) = 0$, and if $x \in X$ we define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

with $d(x, \emptyset) = \infty$. We also define

$$B_d(A, \epsilon) = \{x \in X : d(x, A) < \epsilon\}.$$

If X and Y are topological spaces and $f : X \to Y$ is a function, the **set of continuity** of f is the set of all points in X at which f is continuous. To say that f is continuous is equivalent to saying that its set of continuity is X.

If X is a topological space, (Y, d) is a metric space, $A \subset X$, and $f : A \to Y$ is a function, for $x \in X$ we define the **oscillation of f at x** as

$$\operatorname{osc}_f(x) = \inf\{\operatorname{diam}(f(U \cap A)) : U \text{ is an open neighborhood of } x\}.$$

To say that $f : A \to Y$ is continuous at $x \in A$ means that for every $\epsilon > 0$ there is some open neighborhood U of x such that $y \in U \cap A$ implies that $d(f(y), f(x)) < \epsilon$, and this implies that $\operatorname{diam}(f(U \cap A)) \leq 2\epsilon$. Hence if f is continuous at x then $\operatorname{osc}_f(x) = 0$. On the other hand, suppose that $\operatorname{osc}_f(x) = 0$ and let $\epsilon > 0$. There is then some open neighborhood U of x such that $\operatorname{diam}(f(U \cap A)) < \epsilon$, and this implies that $d(f(y), f(x)) < \epsilon$ for every $y \in U \cap A$, showing that f is continuous at x. Therefore, the set of continuity of $f : A \to Y$ is

$$\{x \in A : \operatorname{osc}_f(x) = 0\}.$$

[2] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 64, Theorem 9.19.

As well, if $x \in X \setminus \overline{A} = \overline{A}^c$, then $\overline{A}^c$ is an open neighborhood of x and $f(\overline{A}^c \cap A) = f(\emptyset) = \emptyset$ and $\mathrm{diam}(\emptyset) = 0$, so in this case $\mathrm{osc}_f(x) = 0$.

The following theorem shows that the set of points where a function taking values in a metrizable space has zero oscillation is a G_δ set.[3]

Theorem 3. Suppose that X is a topological space, Y is a metrizable space, $A \subset X$, and $f : A \to Y$ is a function. Then $\{x \in X : \mathrm{osc}_f(x) = 0\}$ is a G_δ set.

Proof. Let d be a metric on Y that induces its topology and let $A_\epsilon = \{x \in X : \mathrm{osc}_f(x) < \epsilon\}$. For $x \in A_\epsilon$, there is an open neighborhood U of x such that $\mathrm{osc}_f(x) \leq \mathrm{diam}(f(U \cap A)) < \epsilon$. But if $y \in U$ then U is an open neighborhood of y and $\mathrm{diam}(f(U \cap A)) < \epsilon$, so $\mathrm{osc}_f(y) < \epsilon$ and hence $y \in A_\epsilon$, showing that A_ϵ is open. Finally,

$$\{x \in X : \mathrm{osc}_f(x) = 0\} = \bigcap_{n \in \mathbb{N}} A_{1/n},$$

which is a G_δ set, completing the proof. □

In a metrizable space, the only closed sets that are open are $\emptyset$ and the space itself, but we can show that any closed set is a countable intersection of open sets.[4]

Theorem 4. If X is a metrizable space, then any closed subset of X is a G_δ set.

Proof. Let d be a metric on X that induces its topology. Suppose that A is a nonempty subset of X and that $x, y \in X$. We have $d(x, A) \leq d(x, y) + d(y, A)$ and $d(y, A) \leq d(y, x) + d(x, A)$, so

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

It follows that $B_d(A, \epsilon)$ is open. But if F is a closed subset of X then check that

$$F = \bigcap_{n \in \mathbb{N}} B_d(F, 1/n),$$

which is an F_σ set, completing the proof. (If we did not know that F was closed then F would be contained in this intersection, but need not be equal to it.) □

Kechris attributes the following theorem[5] to Kuratowski. It and the following theorem are about extending continuous functions from a set to a G_δ set that contains it, and we will use the following theorem in the proof of Theorem 7.

[3] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 15, Proposition 3.6.
[4] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 15, Proposition 3.7.
[5] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 16, Theorem 3.8.

Theorem 5. Suppose that X is metrizable, Y is completely metrizable, A is a subspace of X, and $f : A \to Y$ is continuous. Then there is a G_δ set G in X such that $A \subset G \subset \overline{A}$ and a continuous function $g : G \to Y$ whose restriction to A is equal to f.

Proof. Let $G = \overline{A} \cap \{x \in X : \mathrm{osc}_f(x) = 0\}$. Theorem 4 tells us that the first set is G_δ and Theorem 3 tells us that the second set is G_δ, so G is G_δ. Because $f : A \to Y$ is continuous, $A \subset \{x \in X : \mathrm{osc}_f(x) = 0\}$, and hence $A \subset G$.

Let $x \in G \subset \overline{A}$, and let $x_n, t_n \in A$ with $x_n \to x$ and $t_n \to x$. Because $\mathrm{osc}_f(x) = 0$, for every $\epsilon > 0$ there is some open neighborhood U of x such that $\mathrm{diam}(f(U \cap A)) < \epsilon$. But then there is some n such that $k \geq n$ implies that $x_k, t_k \in U$, and thus $\mathrm{diam}(f(\{x_k, t_k : k \geq n\})) < \epsilon$. Hence $\mathrm{diam}(f(\{x_k, t_k : k \geq n\})) \to 0$ as $n \to \infty$, and this is equivalent to the sequence $f(x_1), f(t_1), f(x_2), f(t_2), \ldots$ being Cauchy. Because Y is completely metrizable this sequence converges to some $y \in Y$ and therefore the subsequence $f(x_n)$ and the subsequence $f(t_n)$ both converge to y. Thus it makes sense to define $g : G \to Y$ by

$$g(x) = \lim_{n \to \infty} f(x_n),$$

and the restriction of g to A is equal to f. It remains to prove that g is continuous.

If U is an open subset of X, then $g(U \cap G) \subset \overline{f(U \cap A)}$, hence

$$\mathrm{diam}(g(U \cap G)) \leq \mathrm{diam}(\overline{f(U \cap A)}) = \mathrm{diam}(f(U \cap A)).$$

For any $x \in G$ this and $\mathrm{osc}_f(x) = 0$ yield

$$\mathrm{osc}_g(x) \leq \mathrm{osc}_f(x) = 0,$$

showing that the set of continuity of g is G, i.e. that g is continuous. □

The following shows that a homeomorphism between subsets of metrizable spaces can be extended to a homeomorphism of G_δ sets.[6]

Theorem 6 (Lavrentiev's theorem)**.** Suppose that X and Y are completely metrizable spaces, that A is a subspace of X, and that B is a subspace of Y. If $f : A \to B$ is a homeomorphism, then there are G_δ sets $G \supset A$ and $H \supset B$ and a homeomorphism $G \to H$ whose restriction to A is equal to f.

Proof. Theorem 5 tells us that there is a G_δ set $G_1 \supset A$ and a continuous function $g_1 : G_1 \to Y$ whose restriction to A is equal to f, and there is a G_δ set $H_1 \supset B$ and a continuous function $h_1 : H_1 \to X$ whose restriction to B is equal to f^{-1}. Let

$$R = \{(x, y) \in G_1 \times Y : y = g_1(x)\}, \qquad S = \{(x, y) \in X \times H_1 : x = h_1(y)\}.$$

[6] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 16, Theorem 3.9.

Because $g_1 : G_1 \to Y$ is continuous, R is a closed subset of $X \times Y$, and because $h_1 : H_1 \to X$ is continuous, S is a closed subset of $X \times Y$. Let

$$G = \pi_X(R \cap S), \qquad H = \pi_Y(R \cap S),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projection maps. If $x \in A$ then $h_1(g_1(x)) = f^{-1}(f(x)) = x$, and hence $x \in G$, and if $y \in B$ then $g_1(h_1(y)) = f(f^{-1}(y)) = y$, and hence $y \in H$, so we have

$$A \subset G \subset G_1, \qquad B \subset H \subset H_1.$$

The map $E_1 : G_1 \to X \times Y$ defined by $E_1(x) = (x, g_1(x))$ is continuous because $g_1 : G_1 \to Y$ is continuous, and hence

$$E_1^{-1}(S) = \{x \in G_1 : x = h_1(g_1(x))\} = G$$

is a closed subset of G_1, and thus by Theorem 4 is a G_δ set in G_1. But G_1 is a G_δ subset of X, so G is a G_δ set in X also. Define $E_2 : H_1 \to X \times Y$ by $E_2(y) = (h_1(y), y)$, which is continuous because h_1 is continuous. Then

$$E_2^{-1}(R) = \{y \in H_1 : y = g_1(h_1(y))\} = H$$

is a closed subset of H_1, and hence is G_δ in H_1. But H_1 is a G_δ subset of Y, so H_1 is a G_δ set in Y also.

Check that the restriction of g_1 to G_1 is a homeomorphism $G_1 \to H_1$ whose restriction to A is equal to f, completing the proof. $\square$

If a topological space has some property and Y is a subset of X, one wants to know conditions under which Y with the subspace topology inherited from X has the same property. For example, a subspace of a compact Hausdorff space is compact if and only if it is closed, and a subspace of a completely metrizable space is completely metrizable if and only if it is closed. The following theorem shows in particular that a subspace of a Polish space is Polish if and only if it is G_δ.[7] (The statement of the theorem is about completely metrizable spaces and we obtain the conclusion about Polish spaces because any subspace of a separable metrizable space is itself separable.)

Theorem 7. Suppose that X is a metrizable space and Y is a subset of X with the subspace topology. If Y is completely metrizable then Y is a G_δ set in X. If X is completely metrizable and Y is a G_δ set in X then Y is completely metrizable.

Proof. Suppose that Y is completely metrizable. The map $\mathrm{id}_Y : Y \to Y$ is continuous, so Theorem 5 tells us that there is a G_δ set $Y \subset G \subset \overline{Y}$ and a continuous function $g : G \to Y$ whose restriction to Y is equal to id_Y. For $x \in G \subset \overline{Y}$, there are $y_n \in Y$ with $y_n \to x$, and because g is continuous we get $\mathrm{id}_Y(y_n) = g(y_n) \to g(x)$, i.e. $y_n \to g(x)$, hence $g(x) = x$. But $g : G \to Y$ so $x \in Y$, showing that $G = Y$ and hence that Y is a G_δ set.

[7] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 17, Theorem 3.11.

Suppose that X is completely metrizable and that Y is a G_δ subset of X, and let d be a complete metric on X that is compatible with the topology of X; if we restrict this metric to Y then it is a metric on Y that is compatible with the subspace topology on Y inherited from X, but it need not be a complete metric. Let U_n be open sets in X with $Y = \bigcap_{n\in\mathbb{N}} U_n$, let $F_n = X \setminus U_n$, and for $x, y \in Y$ define

$$d_1(x,y) = d(x,y) + \sum_{n\in\mathbb{N}} \min\left\{2^{-n}, \left|\frac{1}{d(x,F_n)} - \frac{1}{d(y,F_n)}\right|\right\}.$$

One proves that d_1 is a metric on Y and that it is compatible with the subspace topology on Y. Suppose that $y_n \in Y$ is Cauchy in (Y, d_1). Because $d \le d_1$, this is also a Cauchy sequence in (X, d), and because (X, d) is complete, there is some $y \in X$ such that $y_n \to y$ in (X, d). Then one proves that $y_n \to y$ in (Y, d_1), from which we have that (Y, d_1) is a complete metric space. □

4 Continuous functions on a compact space

If X and Y are topological spaces, we denote by $C(X,Y)$ the set of continuous functions $X \to Y$. If X is a compact topological space and (Y,ρ) is a metric space, we define

$$d_\rho(f,g) = \sup_{x\in X} \rho(f(x), g(x)), \qquad f, g \in C(X,Y),$$

which is a metric on $C(X,Y)$, which we call the **ρ-supremum metric**. One proves that d_ρ is a complete metric on $C(X,Y)$ if and only if ρ is a complete metric on Y.[8] It follows that if Y is a Banach space then so is $C(X,Y)$ with the supremum norm $\|f\|_\infty = \sup_{x\in X} \|f(x)\|_Y$.

Suppose that X is a compact topological space and that Y is a metrizable space. If ρ_1, ρ_2 are metrics on Y that induce its topology, then d_{ρ_1}, d_{ρ_2} are metrics on $C(X,Y)$, and it can be proved that they induce the same topology,[9] which we call the **topology of uniform convergence**.

Finally, if X is a compact metrizable space and Y is a separable metrizable space, it can be proved that $C(X,Y)$ is separable.[10]

Thus, using what we have stated above, suppose that X is a compact metrizable space and that Y is a Polish space. Because X is a compact metrizable space and Y is a separable metrizable space, $C(X,Y)$ is separable. Because X is a compact topological space and Y is a completely metrizable space, $C(X,Y)$ is completely metrizable, and hence Polish.

[8] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 124, Lemma 3.97.

[9] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 124, Lemma 3.98.

[10] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 125, Lemma 3.99.

5 C([0,1])

$C^1(\mathbb{R})$ consists of those functions $F : \mathbb{R} \to \mathbb{R}$ such that for each $x_0 \in \mathbb{R}$, there is some $F'(x_0) \in \mathbb{R}$ such that

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0},$$

and such that this function F' belongs to $C(\mathbb{R})$. We define $C^1([0,1])$ to be those functions $[0,1] \to \mathbb{R}$ that are the restriction to $[0,1]$ of some element of $C^1(\mathbb{R})$. We shall prove that $C^1([0,1])$ is an $F_{\sigma\delta}$ set in $C([0,1])$.[11]

Suppose that $f \in C^1([0,1])$. For each $x \in [0,1]$,

6 Meager sets and Baire spaces

Let X be a topological space. A subet A of X is called **nowhere dense** if the interior of $\overline{A}$ is $\emptyset$. A subset A of X is called **meager** if it is a countable union of nowhere dense sets. A meager set is also said to be **of first category**, and a nonmeager is said to be **of second category**. Meager is a good name for at least two reasons: it is descriptive and the word is not already used to name anything else. First category and second category are bad names for at least four reasons: the words describe nothing, they are phrases rather than single words, they suggests an ordering, and they conflict with reserving the word "category" for category theory. A complement of a meager is said to be **comeager**.

If X is a set, an **ideal on** X is a collection of subsets of X that includes $\emptyset$ and is closed under subsets and finite unions. A σ**-ideal on** X is an ideal that is closed under countable unions.

Lemma 8. The collection of meager subsets of a topological space is a σ-ideal.

If X is a topological space and $x \in X$, we say that x is **isolated** if $\{x\}$ is open. We say X is **perfect** if it has no isolated points, and a T_1 **space** if $\{x\}$ is closed for each $x \in X$. Suppose that X is a perfect T_1 space and let A be a countable subset of X. For each $x \in A$, because X is T_1, the closure of $\{x\}$ is $\{x\}$, and because X is perfect, the interior of $\{x\}$ is $\emptyset$, and hence $\{x\}$ is nowhere dense. $A = \bigcup_{x \in A} \{x\}$ is a countable union of nowhere dense sets, hence is meager. Thus we have proved that any countable subset of a perfect T_1 space is meager.

Suppose that X is a topological space. If every comeager set in X is dense, we say that X is a **Baire space**.

Lemma 9. A topological space is a Baire space if and only if the intersection of any countable family of dense open sets is dense.

We prove that open subsets of Baire spaces are Baire spaces.[12]

[11] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 70.
[12] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 41, Proposition 8.3.

Theorem 10. If X is a Baire space and U is an open subspace of X, then U is a Baire space.

Proof. Because U is open, an open subset of U is an open subset of X that is contained in U. Suppose that U_n, $n \in \mathbb{N}$, are dense open subsets of U. So they are each open subsets of X, and $U_n \cup (X \setminus \overline{U})$ is a dense open subset of X for each $n \in \mathbb{N}$. Then because X is a Baire space,

$$\bigcap_{n\in\mathbb{N}} (U_n \cup (X \setminus \overline{U})) = \left(\bigcap_{n\in\mathbb{N}} U_n \right) \cup (X \setminus \overline{U})$$

is dense in X. It follows that $\bigcap_{n\in\mathbb{N}} U_n$ is dense in U, showing that U is a Baire space. $\square$

The following is the **Baire category theorem**.[13]

Theorem 11 (Baire category theorem)**.** Every completely metrizable space is a Baire space. Every locally compact Hausdorff space is a Baire space.

Proof. Let X be a completely metrizable space and let d be a complete metric on X compatible with the topology. Suppose that U_n are dense open subsets of X. To show that $\bigcap_{n\in\mathbb{N}} U_n$ is dense it suffices to show that for any nonempty open subset U of X,

$$\bigcap_{n\in\mathbb{N}} (U_n \cap U) = U \cap \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset.$$

Because U is a nonempty open set it contains an open ball B_1 of radius < 1 with $\overline{B_1} \subset U$. Since U_1 is dense and B_1 is open, $B_1 \cap U_1 \neq \emptyset$ and is open because both B_1 and U_1 are open. As $B_1 \cap U_1$ is a nonempty open set it contains an open ball B_2 of radius $< \frac{1}{2}$ with $\overline{B_2} \subset B_1 \cap U_1$. Suppose that $n > 1$ and that B_n is an open ball of radius $< \frac{1}{n}$ with $\overline{B_n} \subset B_{n-1} \cap U_{n-1}$. Since U_n is dense and B_n is open, $B_n \cap U_n \neq \emptyset$ and is open because both B_n and U_n are open. As $B_n \cap U_n$ is a nonempty open set it contains an open ball B_{n+1} of radius $< \frac{1}{n+1}$ with $\overline{B_{n+1}} \subset B_n \cap U_n$. Then, we have $B_{n+1} \subset B_n$ for each $n \in \mathbb{N}$. Letting x_i be the center of B_i, we have $d(x_j, x_i) < \frac{1}{i}$ for $j > i$, and hence x_i is a Cauchy sequence. Since (X, d) is a complete metric space, there is some $x \in X$ such that $x_i \to x$. For any m there is some i_0 such that $i \geq i_0$ implies that $d(x_i, x) < \frac{1}{m}$, and hence $x \in B_m = \bigcap_{n=1}^{m} B_n$. Therefore

$$x \in \bigcap_{n\in\mathbb{N}} B_n \subset \bigcap_{n\in\mathbb{N}} (U_n \cap U),$$

which shows that $\bigcap_{n\in\mathbb{N}} U_n$ is dense and hence that X is a Baire space.

Let X be a locally compact Hausdorff space. Suppose that U_n are dense open subsets of X and that U is a nonempty open set. Let $x_1 \in U$, and because

[13] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 41, Theorem 8.4.

X is a locally compact Hausdorff space there is an open neighborhood V_1 of x_1 with $\overline{V_1}$ compact and $\overline{V_1} \subset U$. Since U_1 is dense and V_1 is open, there is some $x_2 \in V_1 \cap U_1$. As $V_1 \cap U_1$ is open, there is an open neighborhood V_2 of x_2 with $\overline{V_2}$ compact and $\overline{V_2} \subset V_1 \cap U_1$. Thus, $\overline{V_n}$ are compact and satisfy $\overline{V_{n+1}} \subset \overline{V_n}$ for each n, and hence

$$\bigcap_{n\in\mathbb{N}} \overline{V_n} \neq \emptyset.$$

This intersection is contained in $\bigcap_{n\in\mathbb{N}}(U_n \cap U)$ which is therefore nonempty, showing that $\bigcap_{n\in\mathbb{N}} U_n$ is dense and hence that X is a Baire space. $\square$

7 Nowhere differentiable functions

From what we said in §4, because $[0,1]$ is a compact metrizable space and $\mathbb{R}$ is a Polish space, $C([0,1]) = C([0,1],\mathbb{R})$ with the topology of uniform convergence is Polish. This topology is induced by the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$, with which $C([0,1])$ is thus a separable Banach space.

For a function $F : \mathbb{R} \to \mathbb{R}$ to be differentiable at a point x_0 means that there is some $F'(x_0) \in \mathbb{R}$ such that

$$\lim_{x\to x_0} \frac{F(x) - F(x_0)}{x - x_0} = F'(x_0).$$

If $f : [0,1] \to \mathbb{R}$ is a function and $x_0 \in [0,1]$, we say that f is **differentiable at** x_0 if there is some function $F : \mathbb{R} \to \mathbb{R}$ that is differentiable at x_0 and whose restriction to $[0,1]$ is equal to f, and we write $f'(x_0) = F'(x_0)$. The purpose of speaking in this way is to be precise about what we mean by f being differentiable at the endpoints of the interval $[0,1]$.

If $f : [0,1] \to \mathbb{R}$ is differentiable at $x_0 \in [0,1]$, then there is some $\delta > 0$ such that if $0 < |x - x_0| < \delta$ and $x \in [0,1]$, then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < 1,$$

and hence

$$|f(x) - f(x_0)| < (1 + |f'(x_0)|)|x - x_0|.$$

On the other hand, if $f \in C([0,1])$ then $\{x \in [0,1] : |x - x_0| \geq \delta\}$ is a compact set on which $x \mapsto \frac{f(x)-f(x_0)}{x-x_0}$ is continuous, and hence the absolute value of this function is bounded by some M. Thus, if $|x - x_0| \geq \delta$ and $x \in [0,1]$, then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq M,$$

hence

$$|f(x) - f(x_0)| \leq M|x - x_0|.$$

Therefore, if $f \in C([0,1])$ is differentiable at $x_0 \in [0,1]$ then there is some positive integer N such that

$$|f(x) - f(x_0)| \leq N|x - x_0|, \qquad x \in [0,1].$$

For $N \in \mathbb{N}$, let E_N be those $f \in C([0,1])$ for which there is some $x_0 \in [0,1]$ such that

$$|f(x) - f(x_0)| \leq N|x - x_0|, \qquad x \in [0,1].$$

We have established that if $f \in C([0,1])$ and there is some $x_0 \in [0,1]$ such that f is differentiable at x_0, then there is some $N \in \mathbb{N}$ such that $f \in E_N$. Therefore, the set of those $f \in C([0,1])$ that are differentiable at some point in $[0,1]$ is contained in

$$\bigcup_{N \in \mathbb{N}} E_N,$$

and hence to prove that the set of $f \in C([0,1])$ that are nowhere differentiable is comeager in $C([0,1])$, it suffices to prove that each E_N is nowhere dense. To show this we shall follow the proof in Stein and Shakarchi.[14]

Lemma 12. For each $N \in \mathbb{N}$, E_N is a closed subset of the Banach space $C([0,1])$.

Proof. $C([0,1])$ is a metric space, so to show that E_N is closed it suffices to prove that if $f_n \in E_N$ is a sequence tending to $f \in C([0,1])$, then $f \in E_N$. For each n, let $x_n \in [0,1]$ be such that

$$|f_n(x) - f_n(x_n)| \leq N|x - x_n|, \qquad x \in [0,1].$$

Because x_n is a sequence in the compact set $[0,1]$, it has subsequence $x_{a(n)}$ that converges to some $x_0 \in [0,1]$. For all $x \in [0,1]$ we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{a(n)}(x)| + |f_{a(n)}(x) - f_{a(n)}(x_0)| \\ &\quad + |f_{a(n)}(x_0) - f(x_0)|. \end{aligned}$$

Let $\epsilon > 0$. Because $\|f_n - f\|_\infty \to 0$, there is some n_0 such that when $n \geq n_0$, the first and third terms on the right-hand side are each $< \epsilon$. For the second term on the right-hand side, we use

$$|f_{a(n)}(x) - f_{a(n)}(x_0)| \leq |f_{a(n)}(x) - f_{a(n)}(x_{a(n)})| + |f_{a(n)}(x_{a(n)}) - f_{a(n)}(x_0)|.$$

But $f_{a(n)} \in E_N$, so this is $\leq$

$$N|x - x_{a(n)}| + N|x_{a(n)} - x_0|.$$

Putting everything together, for $n \geq n_0$ we have

$$|f(x) - f(x_0)| < 2\epsilon + N|x - x_{a(n)}| + N|x_{a(n)} - x_0|.$$

[14] Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 163, Theorem 1.5.

Because $x_{a(n)} \to x_0$, we get

$$|f(x) - f(x_0)| \leq 2\epsilon + N|x - x_0|.$$

But this is true for any $\epsilon > 0$, so

$$|f(x) - f(x_0)| \leq N|x - x_0|,$$

showing that $f \in E_N$. $\square$

For $M \in \mathbb{N}$ let P_M be the set of those $f \in C([0,1])$ that are piecewise linear and whose line segments have slopes with absolute value $\geq M$. If $M, N \in \mathbb{N}$, $M > N$, and $f \in P_M$, then for any $x_0 \in [0,1]$, this x_0 is the abscissa of a point on at least one line segment whose slope has absolute value $\geq M$ (the point will be on two line segments when it is their common endpoint), and then there is another point on this line segment, with abscissa x, such that $|f(x) - f(x_0)| \geq M|x - x_0| > N|x - x_0|$, and the fact that for every $x_0 \in [0,1]$ there is such $x \in [0,1]$ means that $f \notin E_N$. Therefore, if $M > N$ then $P_M \cap E_N = \emptyset$.

Lemma 13. For each $M \in \mathbb{N}$, P_M is dense in $C([0,1])$.

Proof. Let $f \in C([0,1])$ and $\epsilon > 0$. Because f is continuous on the compact set $[0,1]$ it is uniformly continuous, so there is some positive integer n such that $|x - y| \leq \frac{1}{n}$ implies that $|f(x) - f(y)| \leq \epsilon$. We define $g : [0,1] \to \mathbb{R}$ to be linear on the intervals $[\frac{k}{n}, \frac{k+1}{n}]$, $k = 0, \ldots, n-1$ and to satisfy

$$g\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right), \qquad k = 0, \ldots, n.$$

This nails down g, and for any $x \in [0,1]$ there is some $k = 0, \ldots, n-1$ such that x lies in the interval $[\frac{k}{n}, \frac{k+1}{n}]$. But since g is linear on this interval and we know its values at the endpoints, for any y in this interval we have

$$\begin{aligned} g(y) &= \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{k+1}{n} - \frac{k}{n}} y + f\left(\frac{k}{n}\right) - \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{k+1}{n} - \frac{k}{n}} \cdot \frac{k}{n} \\ &= n\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right) y + f\left(\frac{k}{n}\right) - k\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right), \end{aligned}$$

so

$$\begin{aligned} |g(x) - f(x)| &\leq |g(x) - g(k/n)| + |g(k/n) - f(k/n)| + |f(k/n) - f(x)| \\ &= |g(x) - f(k/n)| + |f(k/n) - f(x)| \\ &= n\left|\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right)\left(x - \frac{k}{n}\right)\right| + |f(k/n) - f(x)| \\ &\leq \left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| + |f(k/n) - f(x)| \\ &\leq 2\epsilon. \end{aligned}$$

This is true for all $x \in [0,1]$, so

$$\|g - f\|_\infty \leq 2\epsilon.$$

Now that we know that we can approximate any $f \in C([0,1])$ with continuous piecewise linear functions, we shall show that we can approximate any continuous piecewise linear function with elements of P_M, from which it will follow that P_M is dense in $C([0,1])$. Let g be a continuous piecewise linear function. We can write g in the following way: there is some positive integer n and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in \mathbb{R}$ such that g is linear on the intervals $[\frac{k}{n}, \frac{k+1}{n}]$, $k = 0, \ldots, n-1$, and satisfies $g(x) = a_k x + b_k$ for $x \in [\frac{k}{n}, \frac{k+1}{n}]$; this can be satisfied precisely when $a_k \frac{k+1}{n} + b_k = a_{k+1} \frac{k+1}{n} + b_{k+1}$ for each $k = 0, \ldots, n-1$. For $\epsilon > 0$, let

$$\phi_\epsilon(x) = g(x) + \epsilon, \qquad \psi_\epsilon(x) = g(x) - \epsilon, \qquad x \in [0,1].$$

We shall define a function $h : [0,1] \to \mathbb{R}$ by describing its graph. We start at $(0, g(0))$, and then the graph of h is a line segment of slope M until it intersects the graph of ϕ_ϵ, at which point the graph of h is a line segment of slope $-M$ until it intersects the graph of ψ_ϵ. We repeat this until we hit the point $(\frac{1}{n}, h(\frac{1}{n}))$; we remark that it need not be the case that $h(\frac{1}{n}) = g(\frac{1}{n})$. If $(\frac{1}{n}, h(\frac{1}{n}))$ lies on the graph of ϕ_ϵ then we start a line segment of slope $-M$, and if it lies on the graph of ψ_ϵ then we start a line segment of slope M, and otherwise we continue the existing line segment until it intersects ϕ_ϵ or ψ_ϵ and we repeat this until the point $(\frac{2}{n}, h(\frac{2}{n}))$, and then repeat this procedure. This constructs a function $h \in P_M$ such that $\|h - g\|_\infty \leq \epsilon$. But for any $f \in C([0,1])$ and $\epsilon > 0$, we have shown that there is some continuous piecewise linear g such that $\|g - f\|_\infty < \epsilon$, and now we know that there is some $h \in P_M$ such that $\|h - g\|_\infty < \epsilon$, so $\|h - f\|_\infty < 2\epsilon$, showing that P_M is dense in $C([0,1])$. $\square$

Let $N \in \mathbb{N}$, suppose that $f \in E_N$, and let $\epsilon > 0$. Let $M > N$, and because P_M is dense in $C([0,1])$, there is some $h \in P_M$ such that $\|f - h\|_\infty < \epsilon$. But $P_M \cap E_N = \emptyset$ because $M > N$, so $h \notin E_N$, showing that there is no open ball with center f that is contained in E_N, which shows that E_N has empty interior. But we have shown that E_N is closed, so the interior of the closure of E_N is empty, namely, E_N is nowhere dense, which completes the proof.

8 The Baire property

Suppose that X is a topological space and that $\mathscr{I}$ is the σ-ideal of meager sets in X. For $A, B \subset X$, write

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

We write $A =^* B$ if $A \triangle B \in \mathscr{I}$. One proves that if $A =^* B$ then $X \setminus A =^* X \setminus B$, and that if $A_n =^* B_n$ then $\bigcap_{n \in \mathbb{N}} A_n =^* \bigcap_{n \in \mathbb{N}} B_n$ and $\bigcup_{n \in \mathbb{N}} A_n =^* \bigcup_{n \in \mathbb{N}} B_n$. A subset A of X is said to have the **Baire property** if there is an open set U

such that $A =^* U$. (It is a common practice to talk about things that are equal to a thing that is somehow easy to work with modulo things that are considered small.) The following theorem characterizes the collection of subsets with the Baire property of a topological space.[15]

Theorem 14. Let X be a topological space and let $\mathscr{B}$ be the collection of subsets of X with the Baire property. Then $\mathscr{B}$ is a σ-algebra on X, and is the algebra generated by all open sets and all meager sets.

Proof. If F is closed, then $F \setminus \mathrm{Int}(F)$ is closed and has empty interior, so is nowhere dense and therefore meager. Thus, if F is closed then $F =^* \mathrm{Int}(F)$.

$\emptyset =^* \emptyset$ and $\emptyset$ is open so $\emptyset$ has the Baire property, and so belongs to $\mathscr{B}$. Suppose that $B \in \mathscr{B}$. This means that there is some open set U such that $B =^* U$, which implies that $X \setminus B =^* X \setminus U$. But $X \setminus U$ is closed, hence $X \setminus U =^* \mathrm{Int}(X \setminus U)$, so $X \setminus B =^* \mathrm{Int}(X \setminus U)$. As $\mathrm{Int}(X \setminus U)$ is open, this shows that $X \setminus B$ has the Baire property, that is, $X \setminus B \in \mathscr{B}$.

Suppose that $B_n \in \mathscr{B}$. So there are open sets U_n such that $B_n =^* U_n$, and it follows that $\bigcup_{n\in\mathbb{N}} B_n =^* \bigcup_{n\in\mathbb{N}} U_n$. The union on the right-hand side is open, so $\bigcup_{n\in\mathbb{N}}$ has the Baire property and thus belongs to $\mathscr{B}$. This shows that $\mathscr{B}$ is a σ-algebra.

Suppose that $\mathscr{A}$ is an algebra containing all open sets and all meager sets, and let $B \in \mathscr{B}$. Because B has the Baire property there is some open set U such that $B =^* U$, which means that $M = B \triangle U = (B \setminus U) \cup (U \setminus B)$ is meager. But $B = M \triangle U = (M \setminus U) \cup (U \setminus M)$, and because $\mathscr{A}$ is an algebra and $U, M \in \mathscr{A}$ we get $B \in \mathscr{A}$, showing that $\mathscr{B} \subset \mathscr{A}$. $\square$

If X_n is a sequence of sets, we call $A \subset \prod_{n\in\mathbb{N}} X_n$ a **tail set** if for all $(x_n) \in A$ and $(y_n) \in \prod_{n\in\mathbb{N}} X_n$, $\{n \in \mathbb{N} : y_n \neq x_n\}$ being finite implies that $(y_n) \in A$. The following theorem states is a **topological zero-one law**,[16] whose proof uses the **Kutatowski-Ulam theorem**,[17] which is about meager sets in a product of two second-countable topological spaces. Since, from the Baire category theorem, any completely metrizable space is a Baire space and a separable metrizable space is second-countable, we can in particular use the following theorem when the X_n are Polish spaces.

Theorem 15. Suppose that X_n is a sequence of second-countable Baire spaces. If $A \subset \prod_{n\in\mathbb{N}} X_n$ has the Baire property and is a tail set, then A is either meager or comeager.

[15] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 47, Proposition 8.22.
[16] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 55, Theorem 8.47.
[17] Alexander S. Kechris, *Classical Descriptive Set Theory*, p. 53, Theorem 8.41.