Proof of the pentagonal number theorem

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Let $A_0 = \prod_{k=1}^{\infty}(1 - z^k)$. We will use the identity

$$\prod_{k=1}^{N}(1-a_k) = 1 - a_1 - \sum_{k=2}^{N} a_k(1-a_1)\cdots(1-a_{k-1}),$$

which is straightforward to prove by induction. We apply the identity with $a_k = z^k$ and $N = \infty$, which gives

$$\begin{aligned} A_0 &= 1 - z - \sum_{k=2}^{\infty} z^k(1-z)\cdots(1-z^{k-1}) \\ &= 1 - z - \sum_{k=0}^{\infty} z^{k+2}(1-z)\cdots(1-z^{k+1}). \end{aligned}$$

For $n \geq 1$ let $A_n = \sum_{k=0}^{\infty} z^{nk}(1-z^n)\cdots(1-z^{n+k})$. We have $A_0 = 1 - z - z^2 A_1$, and for $n \geq 1$ we have

$$\begin{aligned}
A_n &= 1 - z^n + \sum_{k=1}^{\infty} z^{nk}(1-z^n)\cdots(1-z^{n+k}) \\
&= 1 - z^n + \sum_{k=1}^{\infty} z^{nk}(1-z^{n+1})\cdots(1-z^{n+k}) \\
&\quad - \sum_{k=1}^{\infty} z^{n(k+1)}(1-z^{n+1})\cdots(1-z^{n+k}) \\
&= 1 - z^n + z^n(1-z^{n+1}) + \sum_{k=2}^{\infty} z^{nk}(1-z^{n+1})\cdots(1-z^{n+k}) \\
&\quad - \sum_{k=1}^{\infty} z^{n(k+1)}(1-z^{n+1})\cdots(1-z^{n+k}) \\
&= 1 - z^{2n+1} + \sum_{k=0}^{\infty} z^{n(k+2)}(1-z^{n+1})\cdots(1-z^{n+k+2}) \\
&\quad - \sum_{k=0}^{\infty} z^{n(k+2)}(1-z^{n+1})\cdots(1-z^{n+k+1}) \\
&= 1 - z^{2n+1} - \sum_{k=0}^{\infty} z^{n(k+2)+n+k+2}(1-z^{n+1})\cdots(1-z^{n+k+1}) \\
&= 1 - z^{2n+1} - z^{3n+2}\sum_{k=0}^{\infty} z^{(n+1)k}(1-z^{n+1})\cdots(1-z^{n+k+1}) \\
&= 1 - z^{2n+1} - z^{3n+2}A_{n+1}.
\end{aligned}$$

Therefore $A_n = 1 - z^{2n+1} - z^{3n+2}A_{n+1}$ for all $n \geq 0$.

We then check by induction that for all M

$$\begin{aligned}
A_0 &= 1 - z + \sum_{n=1}^{M}(-1)^n\left(z^{n(3n+1)/2} - z^{(n+1)(3n+2)/2}\right) \\
&\quad +(-1)^{M+1}z^{(M+1)(3M+2)/2}A_{M+1},
\end{aligned}$$

and taking $M = \infty$ gives the pentagonal number theorem.