Oscillatory integrals

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1 Oscillatory integrals

Suppose that $\Phi \in C^\infty(\mathbb{R}^d)$, $\psi \in \mathscr{D}(\mathbb{R}^d)$, and that Φ is real-valued. Define $I : (0,\infty) \to \mathbb{C}$ by

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x)}\psi(x)dx, \qquad \lambda > 0.$$

We call Φ a **phase** and ψ an **amplitude**, and $I(\lambda)$ an **oscillatory integral**.

The following proof follows Stein and Shakarchi.[1]

Theorem 1. *If there is some $c > 0$ such that $|(\nabla\Phi)(x)| \geq c$ for all $x \in \operatorname{supp}\psi$, then for each nonnegative integer N there is some $c_N \geq 0$ such that*

$$|I(\lambda)| \leq c_N \lambda^{-N}, \qquad \lambda > 0.$$

Proof. There is some $h \in \mathscr{D}(\mathbb{R}^d)$, $h \geq 0$, such that $h(x) = 1$ for $x \in \operatorname{supp}\psi$.[2] Define $a : \mathbb{R}^d \to \mathbb{R}^d$ by

$$a = h\frac{\nabla\Phi}{|\nabla\Phi|^2},$$

whose entries each belong to $\mathscr{D}(\mathbb{R}^d)$, and define $L : C^\infty(\mathbb{R}^d) \to \mathscr{D}(\mathbb{R}^d)$ by

$$Lf = \frac{1}{i\lambda}\sum_{k=1}^d a_k\partial_k f = \frac{1}{i\lambda}(a\cdot\nabla)f.$$

L satisfies, doing integration by parts and using the fact that a has compact support,

$$\int_{\mathbb{R}^d}(Lf)gdx = \frac{1}{i\lambda}\sum_{k=1}^d \int_{\mathbb{R}^d} a_k(\partial_k f)gdx = \frac{1}{i\lambda}\sum_{k=1}^d - \int_{\mathbb{R}^d} f\partial_k(ga)dx.$$

Thus the **transpose** of L is

$$L^t g = -\frac{1}{i\lambda}\sum_{k=1}^d \partial_k(ga) = -\frac{1}{i\lambda}\nabla\cdot(ga).$$

[1] Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 325, Proposition 2.1.
[2] Walter Rudin, *Functional Analysis*, second ed., p. 162, Theorem 6.20.

Furthermore, in $\operatorname{supp}\psi$,

$$\begin{aligned} L(e^{i\lambda\Phi}) &= e^{i\lambda\Phi}\sum_{k=1}^{d} a_k(\partial_k\Phi) \\ &= e^{i\lambda\Phi}\sum_{k=1}^{d} \frac{\partial_k\Phi}{|\nabla\Phi|^2}\partial_k\Phi \\ &= e^{i\lambda\Phi}. \end{aligned}$$

Thus for any positive integer N and for $x \in \operatorname{supp}\psi$, $L(e^{i\lambda\Phi})(x) = e^{i\lambda\Phi(x)}$, hence

$$I(\lambda) = \int_{\mathbb{R}^d} L^N(e^{i\lambda\Phi})\psi dx = \int_{\mathbb{R}^d} e^{i\lambda\Phi}(L^t)^N\psi dx.$$

But

$$\int_{\mathbb{R}^d} |(L^t)^N\psi| dx = \int_{\mathbb{R}^d} |\lambda^{-N}A_N| dx,$$

where $A_1 = \nabla\cdot(\psi a)$ and $A_n = \nabla\cdot(A_{n-1}a)$. With

$$c_N = \int_{\mathbb{R}^d} |A_N| dx < \infty,$$

we obtain

$$|I(\lambda)| = \left|\int_{\mathbb{R}^d} e^{i\lambda\Phi}(L^t)^N\psi dx\right| \leq \int_{\mathbb{R}^d} |(L^t)^N\psi| dx = c_N\lambda^{-N},$$

completing the proof. □

The following is an estimate for a one-dimensional oscillatory integral without an amplitude term.[3]

Lemma 2. Let $a < b$, and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued, that either $\Phi''(x) \geq 0$ for all $x \in [a,b]$ or $\Phi''(x) \leq 0$ for all $x \in [a,b]$, and that $\Phi'(x) \geq 1$ for all $x \in [a,b]$. Then

$$\left|\int_a^b e^{i\lambda\Phi(x)} dx\right| \leq 3\lambda^{-1}, \qquad \lambda > 0.$$

Proof. Write

$$L = \frac{1}{i\lambda\Phi'}\frac{d}{dx},$$

which satisfies

$$\int_a^b (Lf)g dx = \int_a^b \frac{1}{i\lambda\Phi'}f'g dx = \frac{1}{i\lambda\Phi'}fg\Big|_a^b - \int_a^b f\left(\frac{g}{i\lambda\Phi'}\right)' dx.$$

[3] Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 326, Proposition 2.2.

With $f = e^{i\lambda\Phi}$ and $g = 1$, we have $Lf = e^{i\lambda\Phi}$ and hence

$$\begin{aligned}
\int_a^b e^{i\lambda\Phi} dx &= \frac{e^{i\lambda\Phi}}{i\lambda\Phi'}\bigg|_a^b - \int_a^b e^{i\lambda\Phi}\left(\frac{1}{i\lambda\Phi'}\right)' dx \\
&= \frac{e^{i\lambda\Phi}}{i\lambda\Phi'}\bigg|_a^b + \frac{1}{i\lambda}\int_a^b e^{i\lambda\Phi}(\Phi')^{-2}\Phi'' dx.
\end{aligned}$$

For $\lambda > 0$, using that $\Phi'(x) \geq 1$ for all $x \in [a,b]$ the boundary terms have absolute value

$$\left|\frac{e^{i\lambda\Phi(b)}}{i\lambda\Phi'(b)} - \frac{e^{i\lambda\Phi(a)}}{i\lambda\Phi'(a)}\right| \leq \frac{1}{\lambda|\Phi'(b)|} + \frac{1}{\lambda|\Phi'(a)|} \leq \frac{2}{\lambda}.$$

Because $\Phi'' \geq 0$ or $\Phi'' \leq 0$ on $[a,b]$,

$$\begin{aligned}
\frac{1}{\lambda}\left|\int_a^b e^{i\lambda\Phi}(\Phi')^{-2}\Phi'' dx\right| &\leq \frac{1}{\lambda}\int_a^b |(\Phi')^{-2}\Phi''| dx \\
&= \frac{1}{\lambda}\left|\int_a^b (\Phi')^{-2}\Phi'' dx\right| \\
&= \frac{1}{\lambda}\left|\frac{1}{\Phi'(a)} - \frac{1}{\Phi'(b)}\right| \\
&\leq \frac{1}{\lambda};
\end{aligned}$$

the final inequality uses the fact that the two terms inside the absolute value are both ≥ 1, and thus the absolute value can be bounded by the larger of them. Putting together the two inequalities,

$$\left|\int_a^b e^{i\lambda\Phi} dx\right| \leq \frac{2}{\lambda} + \frac{3}{\lambda} = 3\lambda^{-1}, \qquad \lambda > 0,$$

proving the claim. □

Lemma 3. Let $a < b$, and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued, that either $\Phi''(x) \geq 0$ for all $x \in [a,b]$ or $\Phi''(x) \leq 0$ for all $x \in [a,b]$, and that there is some $\mu > 0$ such that $|\Phi'(x)| \geq \mu$ for all $x \in [a,b]$. Then

$$\left|\int_a^b e^{i\lambda\Phi(x)} dx\right| \leq 3\mu^{-1}\lambda^{-1}, \qquad \lambda > 0.$$

Proof. Φ' is continuous on $[a,b]$, so, by the intermediate value theorem, either $\Phi'(x) \geq \mu$ for all $x \in [a,b]$ or $\Phi'(x) \leq -\mu$ for all $x \in [a,b]$. Let $\epsilon = 1$ in the first case and $\epsilon = -1$ in the second case, and define $\Phi_0 = \epsilon\frac{\Phi}{\mu}$. Then applying Lemma 2, for $\lambda > 0$ we have, writing $\lambda_0 = \mu\lambda$,

$$\left|\int_a^b e^{i\lambda_0\Phi_0(x)} dx\right| \leq 3\lambda_0^{-1},$$

i.e.

$$\left|\int_a^b e^{i\epsilon\lambda\Phi(x)}dx\right| \leq 3(\mu\lambda)^{-1}.$$

If $\epsilon = 1$ this is the claim. If $\epsilon = -1$, then the above integral is the complex conjugate of the integral in the claim, and these have the same absolute values. □

Theorem 4. Let $a < b$, and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued, that either $\Phi''(x) \geq 0$ for all $x \in [a,b]$ or $\Phi''(x) \leq 0$ for all $x \in [a,b]$, and there is some $\mu > 0$ such that $|\Phi'(x)| \geq \mu$ for all $x \in [a,b]$. Suppose also that $\psi \in C^1(\mathbb{R})$. Then with

$$c_\psi = 3\left(|\psi(b)| + \int_a^b |\psi'(x)|dx\right),$$

we have

$$\left|\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx\right| \leq c_\psi\mu^{-1}\lambda^{-1}.$$

Proof. Define $J : [a,b] \to \mathbb{C}$ by

$$J(x) = \int_a^x e^{i\lambda\Phi(u)}du,$$

which satisfies $J'(x) = e^{i\lambda\Phi(x)}$. Integrating by parts,

$$\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx = \int_a^b J'(x)\psi(x)dx = J(x)\psi(x)\Big|_a^b - \int_a^b J(x)\psi'(x)dx,$$

and as $J(a) = 0$ this is equal to

$$J(b)\psi(b) - \int_a^b J(x)\psi'(x)dx.$$

Lemma 3 tells us that $|J(x)| \leq 3\mu^{-1}\lambda^{-1}$ for all $x \in [a,b]$, so

$$\left|J(b)\psi(b) - \int_a^b J(x)\psi'(x)dx\right| \leq 3\mu^{-1}\lambda^{-1}|\psi(b)| + 3\mu^{-1}\lambda^{-1}\int_a^b |\psi'(x)|dx,$$

proving the claim. □

The following is **van der Corput's lemma**.[4]

Lemma 5 (van der Corput's lemma)**.** Let $a < b$ and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued and satisfies $\Phi''(x) \geq 1$ for all $x \in [a,b]$. Then

$$\left|\int_a^b e^{i\lambda\Phi(x)}dx\right| \leq 8\lambda^{-1/2}, \qquad \lambda > 0.$$

[4]Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 328, Proposition 2.3.

Proof. Because Φ' is strictly increasing on $[a,b]$, Φ' has at most one zero in this interval. If $\Phi'(x_0) = 0$, then for $x \geq x_0 + \lambda^{-1/2}$ we have $\Phi'(x) \geq \lambda^{-1/2}$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$,

$$\left| \int_{[x_0+\lambda^{-1/2},b]} e^{i\lambda\Phi(x)} dx \right| \leq 3\mu^{-1}\lambda^{-1} = 3\lambda^{-1/2}.$$

For $x \leq x_0 - \lambda^{-1/2}$ we have $\Phi'(x) \leq -\lambda^{-1/2}$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$,

$$\left| \int_{[a,x_0-\lambda^{-1/2}]} e^{i\lambda\Phi(x)} dx \right| \leq 3\mu^{-1}\lambda^{-1} = 3\lambda^{-1/2}.$$

But

$$\left| \int_{[x_0-\lambda^{-1/2},x_0+\lambda^{-1/2}]\cap[a,b]} e^{i\lambda\Phi(x)} dx \right| \leq \int_{[x_0-\lambda^{-1/2},x_0+\lambda^{-1/2}]\cap[a,b]} dx \leq 2\lambda^{-1/2},$$

and

$$\int_a^b = \int_{[a,x_0-\lambda^{-1/2}]} + \int_{[x_0-\lambda^{-1/2},x_0+\lambda^{-1/2}]\cap[a,b]} + \int_{[x_0+\lambda^{-1/2},b]},$$

so

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 3\lambda^{-1/2} + 2\lambda^{-1/2} + 3\lambda^{-1/2} = 8\lambda^{-1/2}.$$

If there is no $x_0 \in [a,b]$ such that $\Phi'(x_0) = 0$, then either $\Phi' > 0$ on $[a,b]$ or $\Phi' < 0$ on $[a,b]$. In the first case, because Φ' is strictly increasing on $[a,b]$, $\Phi'(x) > \lambda^{-1/2}$ for $x \in [a+\lambda^{-1/2}, b]$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$ gives

$$\begin{aligned} \left| \int_a^b e^{i\lambda\Phi(x)} dx \right| &\leq \left| \int_{[a,a+\lambda^{-1/2}]\cap[a,b]} e^{i\lambda\Phi(x)} dx \right| + \left| \int_{[a+\lambda^{-1/2},b]} e^{i\lambda\Phi(x)} dx \right| \\ &\leq \lambda^{-1/2} + 3\mu^{-1}\lambda^{-1} \\ &= 4\lambda^{-1/2}. \end{aligned}$$

In the second case, $\Phi'(x) < -\lambda^{-1/2}$ for $x \in [a, b-\lambda^{-1/2}]$, and applying Lemma 3 with $\mu = \lambda^{-1/2}$ also gives

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 4\lambda^{-1/2}.$$

Therefore, if Φ' does not have a zero on $[a,b]$ then

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 4\lambda^{-1/2} < 8\lambda^{-1/2}.$$

□

Lemma 6. Let $a < b$ and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued and that there is some $\mu > 0$ such that $|\Phi''(x)| \geq \mu$ for all $x \in [a, b]$. Then

$$\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq 8\mu^{-1/2}\lambda^{-1/2}, \qquad \lambda > 0.$$

Proof. Φ'' is continuous on $[a, b]$, so by the intermediate value theorem either $\Phi''(x) \geq \mu$ for all $x \in [a, b]$ or $\Phi''(x) \leq -\mu$ for all $x \in [a, b]$. Let $\epsilon = 1$ in the first case and $\epsilon = -1$ in the second case, and define $\Phi_0 = \epsilon \frac{\Phi}{\mu}$. Then $\Phi_0''(x) \geq 1$ for all $x \in [a, b]$, and applying Lemma 5,

$$\left| \int_a^b e^{i\mu\lambda\Phi_0(x)} dx \right| \leq 8(\mu\lambda)^{-1/2}, \qquad \lambda > 0,$$

i.e.

$$\left| \int_a^b e^{i\epsilon\lambda\Phi(x)} dx \right| \leq 8(\mu\lambda)^{-1/2}, \qquad \lambda > 0.$$

If $\epsilon = 1$ this is the inequality in the claim. If $\epsilon = -1$, then the above integral is the complex conjugate of the integral in the claim, and these have the same absolute values. □

We use the above to prove the following estimate which involves an amplitude.[5]

Theorem 7. Let $a < b$ and suppose that $\Phi \in C^2(\mathbb{R})$ is real-valued and that there is some $\mu > 0$ such that $|\Phi''(x)| \geq \mu$ for all $x \in [a, b]$. Suppose also that $\psi \in C^1(\mathbb{R})$. Then with

$$c_\psi = 8\left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right),$$

we have

$$\left| \int_a^b e^{i\lambda\Phi(x)} \psi(x) dx \right| \leq c_\psi \mu^{-1/2}\lambda^{-1/2}, \qquad \lambda > 0.$$

Proof. Define $J : [a, b] \to \mathbb{C}$ by

$$J(x) = \int_a^x e^{i\lambda\Phi(u)} du,$$

which satisfies $J'(x) = e^{i\lambda\Phi(x)}$. Integrating by parts,

$$\int_a^b e^{i\lambda\Phi(x)}\psi(x)dx = \int_a^b J'(x)\psi(x)dx = J(x)\psi(x)\Big|_a^b - \int_a^b J(x)\psi'(x)dx.$$

[5]Elias M. Stein and Rami Shakarchi, *Functional Analysis*, p. 328, Corollary 2.4.

and as $J(a) = 0$ this is equal to

$$J(b)\psi(b) - \int_a^b J(x)\psi'(x)dx.$$

But for each $x \in [a, b]$ we have by Lemma 6 that $|J(x)| \le 8\mu^{-1/2}\lambda^{-1/2}$, so

$$\left| J(b)\psi(b) - \int_a^b J(x)\psi'(x)dx \right| \le 8\mu^{-1/2}\lambda^{-1/2}|\psi(b)| + 8\mu^{-1/2}\lambda^{-1/2}\int_a^b |\psi'(x)|dx,$$

completing the proof. □

2 Bessel functions

For $n \in \mathbb{Z}$, the n**th Bessel function of the first kind** $J_n : \mathbb{R} \to \mathbb{R}$ is

$$J_n(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda \sin x} e^{-inx} dx, \qquad \lambda \in \mathbb{R}.$$

Let

$$I_1 = \left[0, \frac{\pi}{4}\right], \quad I_2 = \left[\frac{3\pi}{4}, \pi\right], \quad I_3 = \left[\pi, \frac{5\pi}{4}\right], \quad I_4 = \left[\frac{7\pi}{4}, 2\pi\right],$$

on which $|\cos x| \ge \frac{1}{\sqrt{2}}$, and

$$I_5 = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right], \quad I_6 = \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right],$$

on which $|\sin x| \ge \frac{1}{\sqrt{2}}$. Write $\Phi(x) = \sin x$ and $\psi(x) = e^{-inx}$. $\Phi'(x) = \cos(x)$ and $\Phi''(x) = -\sin(x)$, and for I_1, I_2, I_3, I_4 we apply Theorem 4 with $\mu = \frac{1}{\sqrt{2}}$. For each of I_1, I_2, I_3, I_4 we compute $c_\psi = 3\left(1 + \frac{\pi n}{4}\right)$, which gives us

$$\left| \int_{I_k} e^{i\lambda\Phi(x)}\psi(x)dx \right| \le c_\psi \mu^{-1}\lambda^{-1} = 3\left(1 + \frac{\pi n}{4}\right) \cdot \sqrt{2} \cdot \lambda^{-1}.$$

For I_5 and I_6, we apply Theorem 7 with $\mu = \frac{1}{\sqrt{2}}$. For each of I_5 and I_6 we compute $c_\psi = 8\left(1 + \frac{\pi n}{2}\right)$, which gives us

$$\left| \int_{I_k} e^{i\lambda\Phi(x)}\psi(x)dx \right| \le c_\psi \mu^{-1/2}\lambda^{-1/2} = 8\left(1 + \frac{\pi n}{2}\right) \cdot 2^{1/4} \cdot \lambda^{-1/2}.$$

Therefore

$$|J_n(\lambda)| \le 4 \cdot \frac{1}{2\pi} \cdot 3\left(1 + \frac{\pi n}{4}\right) \cdot \sqrt{2} \cdot \lambda^{-1} + 2 \cdot \frac{1}{2\pi} \cdot 8\left(1 + \frac{\pi n}{2}\right) \cdot 2^{1/4} \cdot \lambda^{-1/2},$$

which shows that for each $n \in \mathbb{Z}$,

$$J_n(\lambda) = O_n(\lambda^{-1/2})$$

as $\lambda \to \infty$.