Kronecker's theorem

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1 Equivalent statements of Kronecker's theorem

We shall now give two statements of **Kronecker's theorem**, and prove that they are equivalent before proving that they are true.

Theorem 1. *If $\theta_1, \ldots, \theta_k, 1$ are real numbers that are linearly independent over $\mathbb{Z}$, $\alpha_1, \ldots, \alpha_k$ are real numbers, and N and ϵ are positive real numbers, then there are integers $n > N$ and $p_1, \ldots, p_k$ such that for $m = 1, \ldots, k$,*

$$|n\theta_m - p_m - \alpha_m| < \epsilon.$$

Theorem 2. *If $\theta_1, \ldots, \theta_k$ are real numbers that are linearly independent over $\mathbb{Z}$, $\alpha_1, \ldots, \alpha_k$ are real numbers, and T and ϵ are positive real numbers, then there is a real number $t > T$ and integers $p_1, \ldots, p_k$ such that for $m = 1, \ldots, k$,*

$$|t\theta_m - p_m - \alpha_m| < \epsilon.$$

We now prove that the above two statements are equivalent.[1]

Lemma 3. *Theorem 1 is true if and only if Theorem 2 is true.*

Proof. Assume that Theorem 2 is true and let $\theta'_1, \ldots, \theta'_k, 1$ be real numbers that are linearly independent over $\mathbb{Z}$, let $\alpha_1, \ldots, \alpha_k$ be real numbers, let $N > 0$ and let $0 < \epsilon < 1$. Let $\theta_m = \theta'_m - q_m$ with $0 < \theta_m \leq 1$. Because $\theta'_1, \ldots, \theta'_k, 1$ are linearly independent over $\mathbb{Z}$, so are $\theta_1, \ldots, \theta_k, 1$. Using Theorem 2 with $k+1$ instead of k, $N+1$ instead of T, $\frac{1}{2}\epsilon$ instead of ϵ, applied with

$$\theta_1, \ldots, \theta_k, 1, \qquad \alpha_1, \ldots, \alpha_k, 0,$$

there is a real number $t > N+1$ and integers $p_1, \ldots, p_k, p_{k+1}$ such that for $m = 1, \ldots, k$,

$$|t\theta_m - p_m - \alpha_m| < \frac{1}{2}\epsilon,$$

[1] K. Chandrasekharan, *Introduction to Analytic Number Theory*, pp. 92–93, Chapter VIII, §5.

and

$$|t - p_{k+1}| < \frac{1}{2}\epsilon.$$

Then $p_{k+1} > t - \frac{1}{2}\epsilon > t - \frac{1}{2} > N$, and for $m = 1, \dots, k$, because $0 < \theta_m \leq 1$,

$$\begin{aligned}
|p_{k+1}\theta_m - p_m - \alpha_m| &= |p_{k+1}\theta_m - p_m + t\theta_m - t\theta_m - \alpha_m| \\
&\leq |t\theta_m - p_m - \alpha_m| + |(p_{k+1} - t)\theta_m| \\
&\leq |t\theta_m - p_m - \alpha_m| + |p_{k+1} - t| \\
&< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon.
\end{aligned}$$

Thus for $n = p_{k+1}$, we have $n > N$, and for $m = 1, \dots, k$,

$$|n\theta'_m - (nq_m + p_m) - \alpha| = |n\theta_m - p_m - \alpha_m| < \epsilon,$$

proving Theorem 1.

Assume that Theorem 1 is true. The claim of Theorem 2 is immediate when $k = 1$. For $k > 1$, let $\theta'_1, \dots, \theta'_k$ be linearly independent over $\mathbb{Z}$, let $\alpha_1, \dots, \alpha_k$ be real numbers, and let T and ϵ be positive real numbers. Let $\theta_m = |\theta'_m| > 0$, and because $\theta'_1, \dots, \theta'_k$ are linearly independent over $\mathbb{Z}$, so are $\theta_1, \dots, \theta_k$, and then

$$\frac{\theta_1}{\theta_k}, \frac{\theta_2}{\theta_k}, \dots, \frac{\theta_{k-1}}{\theta_k}, 1$$

are linearly independent over $\mathbb{Z}$. Applying Theorem 1 with $N = T\theta_k$ and

$$\frac{\theta_1}{\theta_k}, \frac{\theta_2}{\theta_k}, \dots, \frac{\theta_{k-1}}{\theta_k}, \qquad \operatorname{sgn}\theta'_1 \cdot \alpha_1, \dots, \operatorname{sgn}\theta'_{k-1} \cdot \alpha_{k-1},$$

we get that there are integers $n > T\theta_k$ and $p_1, \dots, p_{k-1}$ such that for $m = 1, \dots, k-1$,

$$\left| n\frac{\theta_m}{\theta_k} - p_m - \operatorname{sgn}\theta'_m \cdot \alpha_m \right| < \frac{1}{2}\epsilon.$$

Let $t = \frac{n}{\theta_k}$. Then $t > T$ and for $m = 1, \dots, k-1$,

$$|t\theta_m - p_m - \operatorname{sgn}\theta'_m \cdot \alpha_m| = \left| n\frac{\theta_m}{\theta_k} - p_m - \operatorname{sgn}\theta'_m \cdot \alpha_m \right| < \frac{1}{2}\epsilon,$$

and

$$|t\theta_k - n| = 0 < \frac{1}{2}\epsilon.$$

On the other hand, applying Theorem 1 with $N = T$ and

$$\theta_1, \dots, \theta_k, \qquad 0, \dots, 0, \operatorname{sgn}\theta'_k \cdot \alpha_k,$$

we get that there are integers $\nu > T$ and $q_1, \dots, q_k$ such that for $m = 1, \dots, k-1$,

$$|\nu\theta_m - q_m| < \frac{1}{2}\epsilon$$

and

$$|\nu\theta_k - q_k - \operatorname{sgn}\theta'_k \cdot \alpha_k| < \frac{1}{2}\epsilon.$$

For $m = 1, \ldots, k-1$,

$$\begin{aligned}|(t+\nu)\theta_m - (p_m + q_m) - \operatorname{sgn}\theta'_m \cdot \alpha_m| &\leq |t\theta_m - p_m - \operatorname{sgn}\theta'_m \cdot \alpha_m| + |\nu\theta_m - q_m| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon\end{aligned}$$

and

$$\begin{aligned}|(t+\nu)\theta_k - (p_k + q_k) - \operatorname{sgn}\theta'_k \cdot \alpha_k| &\leq |t\theta_k - p_k| + |\nu\theta_k - q_k - \operatorname{sgn}\theta'_k \cdot \alpha_k| \\ &< \frac{1}{2}\epsilon + \frac{1}{2}.\end{aligned}$$

Therefore for $m = 1, \ldots, k$,

$$\begin{aligned}&|(t+\nu)\theta'_m - \operatorname{sgn}\theta'_m \cdot (p_m + q_m) - \alpha_m| \\ =&|\operatorname{sgn}\theta'_m \cdot (t+\nu)\theta_m - \operatorname{sgn}\theta'_m \cdot (p_m + q_m) - \alpha_m| \\ =&|(t+\nu)\theta_m - (p_m + q_m) - \operatorname{sgn}\theta'_m \cdot \alpha_m| \\ <&\epsilon,\end{aligned}$$

which proves Theorem 2. □

2 Proof of Kronecker's theorem

We now prove Theorem 2.[2]

Proof of Theorem 2. Let $\theta_1, \ldots, \theta_k$ be real numbers that are linearly independent over $\mathbb{Z}$, let $\alpha_1, \ldots, \alpha_k$ be real numbers, and let T and ϵ be positive real numbers.

For real c and $\tau > 0$,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau e^{ict} dt = \begin{cases} 0 & c \neq 0 \\ 1 & c = 0. \end{cases}$$

For $c_1, \ldots, c_r \in \mathbb{R}$ with $c_m \neq c_n$ for $m \neq n$, and for $b_\nu \in \mathbb{C}$, let

$$\chi(t) = \sum_{\nu=1}^{r} b_\nu e^{ic_\nu t}.$$

Then for $1 \leq \mu \leq r$,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \chi(t) e^{-ic_\mu t} dt = \sum_{\nu=1}^{r} b_\nu \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau e^{i(c_\nu - c_\mu)t} dt = b_\mu.$$

[2] K. Chandrasekharan, *Introduction to Analytic Number Theory*, pp. 93–96, Chapter VIII, §5.

Let

$$F(t) = 1 + \sum_{m=1}^{k} e^{2\pi i(t\theta_m - \alpha_m)} = 1 + \sum_{m=1}^{k} e^{-2\pi i\alpha_m} e^{2\pi i t\theta_m}$$

and let

$$\phi(t) = |F(t)|,$$

which satisfies $0 \leq \phi(t) \leq k+1$.

Define $\phi : \mathbb{R}^k \to \mathbb{R}$ by

$$\psi(x_1, \ldots, x_k) = 1 + x_1 + \cdots + x_k$$

and let p be a positive integer. By the multinomial theorem,

$$\begin{aligned} \psi^p &= (1 + x_1 + \cdots + x_k)^p \\ &= \sum_{\nu_0+\nu_1+\cdots+\nu_k=p} \binom{p}{\nu_0, \nu_1, \ldots, \nu_k} x_1^{\nu_1} \cdots x_k^{\nu_k} \\ &= \sum_{\nu} a_{\nu_1,\ldots,\nu_k} x_1^{\nu_1} \cdots x_k^{\nu_k}, \end{aligned}$$

for which

$$\sum_{\nu} a_{\nu_1,\ldots,\nu_k} = (k+1)^p$$

and the number of terms in the above sum is $\binom{p+k}{k}$. We can write $F(t)$ as

$$F(t) = \psi(e^{2\pi i(t\theta_1 - \alpha_1)}, \ldots, e^{2\pi i(t\theta_k - \alpha_k)}).$$

Then

$$F(t)^p = \sum a_{\nu_1,\ldots,\nu_k} \exp\left(\sum_{m=1}^{k} \nu_m \cdot 2\pi i(t\theta_m - \alpha_m)\right).$$

Because $\theta_1, \ldots, \theta_k$ are linearly independent over $\mathbb{Z}$, for $\nu \neq \mu$ it is the case that $2\pi \sum_{m=1}^{k} \nu_m\theta_m \neq 2\pi \sum_{m=1}^{k} \mu_m\theta_m$. Write $c_\nu = 2\pi\nu \cdot \theta$ and

$$b_\nu = a_{\nu_1,\ldots,\nu_k} \exp\left(-2\pi i \sum_{m=1}^{k} \nu_m\alpha_m\right),$$

with which

$$F(t)^p = \sum b_\nu e^{ic_\nu t}.$$

Then for each multi-index μ,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau F(t)^p e^{-ic_\mu t} dt = b_\mu. \tag{1}$$

Suppose by contradiction that

$$\limsup_{t\to\infty} \phi(t) < k+1.$$

Then there is some $\lambda < k+1$ and some t_0 such that when $t \geq t_0$,

$$|F(t)| = \phi(t) \leq \lambda.$$

Thus for p a positive integer,

$$\begin{aligned}
\limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau |F(t)|^p dt &\leq \limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^{t_0} |F(t)|^p dt + \limsup_{\tau\to\infty} \frac{1}{\tau}\int_{t_0}^\tau |F(t)|^p dt \\
&= \limsup_{\tau\to\infty} \frac{1}{\tau}\int_{t_0}^\tau |F(t)|^p dt \\
&\leq \limsup_{\tau\to\infty} \frac{1}{\tau}\lambda^p(\tau - t_0) \\
&= \lambda^p.
\end{aligned}$$

But then by (1),

$$|b_\mu| \leq \limsup_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau |F(t)|^p dt \leq \lambda^p,$$

and then

$$\begin{aligned}
(k+1)^p &= \sum_\nu a_{\nu_1,\ldots,\nu_k} \\
&= \sum_\nu |b_\nu| \\
&\leq \sum_\nu \lambda^p \\
&\leq \lambda^p \cdot \binom{p+k}{k}.
\end{aligned}$$

Let $r = \frac{\lambda}{k+1}$, for which $0 < r < 1$, and so for each positive integer p it holds that

$$1 \leq r^p \cdot \binom{p+k}{k}. \tag{2}$$

Now,

$$\binom{p+k}{k} = \binom{p+k}{p} = \frac{p^k}{\Gamma(k+1)}\left(1 + \frac{k(k+1)}{2p} + O(p^{-2})\right), \qquad p \to \infty.$$

In particular,

$$r^p \cdot \binom{p+k}{k} = O(r^p \cdot p^k), \qquad p \to \infty,$$

and because $0 < r < 1$, $r^p \cdot p^k \to 0$ as $p \to \infty$, contradicting (2) being true for all positive integers p. This contradiction shows that in fact

$$\limsup_{t\to\infty} \phi(t) \geq k+1,$$

and because $\phi(t) \le k+1$,

$$\limsup_{t\to\infty} \phi(t) = k+1. \tag{3}$$

Now let $0 < \eta < 1$. By (3) there is some $t \ge T$ for which $\phi(t) \ge k+1-\eta$. For $1 \le m \le k$, write

$$z_m = e^{2\pi i (t\theta_m - \alpha_m)} = x_m + iy_m.$$

It is straightforward from the definition of $\phi(t)$ that

$$k+1-\eta \le \phi(t) \le (k-1) + |1 + e^{2\pi i(t\theta_m-\alpha_m)}|,$$

which yields

$$2 \ge |1 + e^{2\pi i(t\theta_m-\alpha_m)}| \ge 2-\eta.$$

Because $|z_m| = 1$,

$$|1+z_m|^2 = (1+x_m)^2 + y_m^2 = (1+x_m)^2 + (1-x_m^2) = 2+2x_m,$$

hence

$$2+2x_m \ge (2-\eta)^2 = 4 - 4\eta + \eta^2 > 4-4\eta,$$

so

$$1-2\eta < x_m \le 2.$$

Furthermore,

$$y_m^2 = 1 - x_m^2 = (1-x_m)(1+x_m) \le 2(1-x_m) < 2\cdot 2\eta = 4\eta.$$

Therefore

$$|z_m - 1|^2 = (x_m-1)^2 + y_m^2 < 4\eta^2 + 4\eta < 8\eta,$$

hence

$$2|\sin \pi(t\theta_m - \alpha_m)| = |e^{2\pi i(t\theta_m-\alpha_m)} - 1| < 8^{1/2}\eta^{1/2} < 4\eta^{1/2}.$$

For $x \in \mathbb{R}$, denote by $\|x\|$ the distance from x to the nearest integer. We check that

$$|\sin(\pi x)| = \sin(\pi \|x\|) \ge \frac{2}{\pi}\cdot \pi \|x\| = 2\|x\|.$$

Thus, for each $m = 1, \ldots, k$,

$$\|t\theta_m - \alpha_m\| < \eta^{1/2}.$$

We have taken $t \ge T$ .Take $\eta^{1/2} = \epsilon$, i.e. $\eta = \epsilon^2$, and take p_m to be the nearest integer to $t\theta_m - \alpha_m$, for which $|t\theta_m - p_m - \alpha_m| < \epsilon$, proving the claim. $\square$

3 Uniform distribution modulo 1

For $x \in \mathbb{R}$ let $[x]$ be the greatest integer $\leq x$, and let $\{x\} = x - [x]$, called the fractional part of x. For $P = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let $\{P\} = (\{x_1\}, \ldots, \{x_d\})$, which belongs to the set $Q = [0,1)^d$. Let $P_j = (x_{j,1}, \ldots, x_{j,d})$, $j \geq 1$, be a sequence in $\mathbb{R}^d$, and for $A \subset Q$ let

$$\phi_n(A) = \{k : 1 \leq k \leq n, \{P_j\} \in A\}.$$

We say that (P_j) is **uniformly distributed modulo** 1 if for each closed rectangle V contained in Q,

$$\lim_{n \to \infty} \frac{\phi_n(V)}{n} = \lambda(V),$$

where λ is Lebesgue measure on $\mathbb{R}^d$: for $V = [a_1, b_1] \times \cdots [a_d, b_d]$, $\lambda(V) = \prod_{j=1}^d (b_j - a_j)$.

We have proved that if $\theta_1, \ldots, \theta_k, 1$ are linearly independent over $\mathbb{Z}$, then the sequence $\{n\theta\} = (\{n\theta_1\}, \ldots, \{n\theta_k\})$ is dense in Q.a It can in fact be proved that $(n\theta)$ is uniformly distributed modulo 1.[3]

4 Unique ergodicity

Let X be a compact metric space, let $C(X)$ be the Banach space of continuous functions $X \to \mathbb{R}$, and let $\mathscr{M}(X)$ be the space of Borel probability measures on X, with the subspace topology inherited from $C(X)^*$ with the weak-* topology.[4] One proves that μ and ν in $\mathscr{M}(X)$ are equal if and only if $\int_X f d\mu = \int_X f d\nu$ for all $f \in C(X)$. $\mathscr{M}(X)$ is a closed set in $C(X)^*$ that is contained in the closed unit ball, and by the Banach-Alaoglu theorem that closed unit ball is compact, so $\mathscr{M}(X)$ is itself compact. $C(X)^*$, with the weak-* topology, is not metrizable, but it is the case that $\mathscr{M}(X)$ with the subspace topology inherited from $C(X)^*$ is metrizable.

For a continuous map $T : X \to X$, define $T_* : \mathscr{M}(X) \to \mathscr{M}(X)$ by

$$(T_*\mu)(A) = \mu(T^{-1}A)$$

for Borel sets A in X. For $\mu_n \to \mu$ in $\mathscr{M}(X)$ and $f \in C(X)$, by the change of variables theorem we have

$$\int_X f d(T_*\mu_n) = \int_X f \circ T d\mu_n \to \int_X f \circ T d\mu = \int_X f d(T_*\mu),$$

which means that $T_*\mu_n \to T_*\mu$, and therefore the map T_* is continuous. We say that $\mu \in \mathscr{M}(X)$ is T**-invariant** if $T_*\mu = \mu$. Equivalently, $T : (X, \mathscr{B}_X, \mu) \to (X, \mathscr{B}_X, \mu)$ is **measure-preserving**. We denote by $\mathscr{M}^T(X)$ the set of T-invariant $\mu \in \mathscr{M}(X)$. The **Kryloff-Bogoliouboff theorem** states that $\mathscr{M}^T(X)$

[3] Giancarlo Travaglini, *Number Theory, Fourier Analysis and Geometric Discrepancy*, p. 108, Theorem 6.3.

[4] This is the same as the narrow topology on $\mathscr{M}(X)$.

is nonempty. It is immediate that $\mathscr{M}^T(X)$ is a convex subset of $C(X)^*$. Let $\mu_n \in \mathscr{M}^T(X)$ converge to some $\mu \in \mathscr{M}(X)$. For $f \in C(X)$ we have, because T_* is continuous,

$$\int_X f d(T_*\mu) = \lim_{n\to\infty} \int_X f d(T_*\mu_n) = \lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu,$$

which shows that μ is T-invariant. Therefore $\mathscr{M}^T(X)$ is a closed set in $\mathscr{M}(X)$, and we have thus established that $\mathscr{M}^T(X)$ is a nonempty compact convex set.

A measure $\mu \in \mathscr{M}^T(X)$ is called **ergodic** if for any $A \in \mathscr{B}_X$ with $T^{-1}A = A$ it holds that $\mu(A) = 0$ or $\mu(A) = 1$. It is proved that $\mu \in \mathscr{M}^T(X)$ is ergodic if and only if μ is an extreme point of $\mathscr{M}^T(X)$.[5] The **Krein-Milman theorem** states that if S is a nonempty compact convex set in a locally convex space, then S is equal to the closed convex hull of the set of extreme points of S.[6] In particular this shows us that there exist extreme points of S. Let $\mathscr{E}^T(X)$ be the set of extreme points of $\mathscr{M}^T(X)$, and applying the Krein-Milman theorem with $\mathscr{M}^T(X)$, which is a nonempty compact convex set in the locally convex space $C(X)^*$, we have that $\mathscr{M}^T(X)$ is equal to the closed convex hull $\mathscr{E}^T$. That is, $\mathscr{M}^T(X)$ is equal to the closed convex hull of the set of ergodic $\mu \in \mathscr{M}^T(X)$.

Choquet's theorem[7] tells us that for each $\mu \in \mathscr{M}^T(X)$ there is a unique Borel probability measure λ on the compact metrizable space $\mathscr{M}^T(X)$ such that

$$\lambda(\mathscr{E}^T(X)) = 1$$

and for all $f \in C(X)$,

$$\int_X f d\mu = \int_{\mathscr{E}^T(X)} \left(\int_X f d\nu \right) d\lambda(\nu).$$

We have established that $\mathscr{M}^T(X)$ contains at least one element. T is called **uniquely ergodic** if $\mathscr{M}^T(X)$ is a singleton. If $\mathscr{M}^T(X) = \{\mu_0\}$ then μ_0 is an extreme point of $\mathscr{M}^T(X)$, hence is ergodic. If $\mathscr{E}^T(X) = \{\mu_0\}$, then for $\mu \in \mathscr{M}^T(X)$, by Choquet's theorem there is a unique Borel probability measure λ on $\mathscr{M}^T(X)$ satisfying $\lambda = \delta_{\mu_0}$ and

$$\int_X f d\mu = \int_{\{\mu_0\}} \left(\int_X f d\nu \right) d\lambda(\nu),$$

i.e.

$$\int_X f d\mu = \int_X f d\mu_0,$$

which means that $\mu = \mu_0$. Therefore, T is uniquely ergodic if and only if $\mathscr{E}^T(X)$ is a singleton. It can be proved that T is uniquely ergodic if and only if for each

[5] Manfred Einsiedler and Thomas Ward, *Ergodic Theory with a view towards Number Theory*, p. 99, Theorem 4.4.

[6] Walter Rudin, *Functional Analysis*, second ed., p. 75, Theorem 3.23.

[7] Manfred Einsiedler and Thomas Ward, *Ergodic Theory with a view towards Number Theory*, p. 103, Theorem 4.8.

$f \in C(X)$ there is some C_f such that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to C_f$$

uniformly on X.[8] This constant C_f is equal to $\int_X f d\mu$, where $\mathscr{M}^T(X) = \{\mu\}$.

For a topological group X and for $g \in X$, define $R_g(x) = gx$, which is continuous $X \to X$. For a compact metrizable group, there is a unique Borel probability measure m_X on X that is R_g-invariant for every $g \in X$, called the **Haar measure on** X. Thus for each $g \in X$, the Haar measure m_X belongs to $\mathscr{M}^{R_g}(X)$, and for R_g to be uniquely ergodic means that m_X is the only element of $\mathscr{M}^{R_g}(X)$. For a locally compact abelian group X, let $\widehat{X}$ be its Pontryagin dual. The following theorem gives a condition that is equivalent to a translation being uniquely ergodic.[9]

Theorem 4. *Let X be a compact metrizable group and let $g \in X$. R_g is uniquely ergodic if and only if X is abelian and $\chi(g) \neq 1$ for all nontrivial $\chi \in \widehat{X}$.*

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, let $X = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, which is a compact abelian group, and let $g = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$. For $\chi \in \widehat{X} = \mathbb{Z}^d$, $\chi = (k_1, \dots, k_d)$,

$$\chi(g) = \exp\left(2\pi i \sum_{j=1}^{d} k_j \alpha_j\right).$$

$\chi(g) = 1$ if and only if $\sum_{j=1}^{d} k_j \alpha_j \in \mathbb{Z}$ if and only if there is some $k_{d+1} \in \mathbb{Z}$ such that $k_1\alpha_1 + \cdots + k_d\alpha_d + k_{d+1} = 0$. Therefore for $\alpha_1, \dots, \alpha_d \in \mathbb{R}$, the set $\{\alpha_1, \dots, \alpha_d, 1\}$ is linearly independent over $\mathbb{Z}$ if and only if for $g = (\alpha_1, \dots, \alpha_d)$, the map $R_g(x) = x + g$, $\mathbb{T}^d \to \mathbb{T}^d$, is uniquely ergodic.

[8] Manfred Einsiedler and Thomas Ward, *Ergodic Theory with a view towards Number Theory*, p. 105, Theorem 4.10.

[9] Manfred Einsiedler and Thomas Ward, *Ergodic Theory with a view towards Number Theory*, p. 108, Theorem 4.14.