The Kolmogorov extension theorem

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1 σ-algebras and semirings

If X is a nonempty set, an **algebra** of sets on X is a collection $\mathscr{A}$ of subsets of X such that if $\{A_i\} \subset \mathscr{A}$ is finite then $\bigcup_i A_i \in \mathscr{A}$, and if $A \in \mathscr{A}$ then $X \setminus A \in \mathscr{A}$. An algebra $\mathscr{A}$ is called a **σ-algebra** if $\{A_i\} \subset \mathscr{A}$ being countable implies that $\bigcup_i A_i \in \mathscr{A}$.

If X is a set and $\mathscr{G}$ is a collection of subsets of X, we denote by $\sigma(\mathscr{G})$ the smallest σ-algebra containing $\mathscr{G}$, and we say that $\sigma(\mathscr{G})$ is the **σ-algebra generated by** $\mathscr{G}$.

Later we will also use the following notion. If X is a nonempty set, a **semiring** of sets on X is a collection $\mathscr{S}$ of subsets of X such that (i) $\emptyset \in \mathscr{S}$, (ii) if $A, B \in \mathscr{S}$ then $A \cap B \in \mathscr{S}$, and (iii) if $A, B \in \mathscr{S}$ then there are pairwise disjoint $S_1, \ldots, S_n \in \mathscr{S}$ such that $A \setminus B = \bigcup_{i=1}^n S_i$; we do not demand that this union itself belong to $\mathscr{S}$. We remark that a semiring on X need not include X.

If $\mathscr{S}$ is a semiring of sets and $\mu_0 : \mathscr{S} \to [0, \infty]$, we say that μ_0 is **finitely additive** if $\{S_i\} \subset \mathscr{S}$ being finite, pairwise disjoint, and satisfying $\bigcup_i S_i \in \mathscr{S}$ implies that $\mu_0 (\bigcup_i S_i) = \sum_i \mu_0(S_i)$, and **countably additive** if $\{S_i\} \subset \mathscr{S}$ being countable, pairwise disjoint, and satisfying $\bigcup_i S_i \in \mathscr{S}$ implies that $\mu_0 (\bigcup_i S_i) = \sum_i \mu_0(S_i)$. If $\mathscr{G}$ is a collection of subsets of X, the **algebra generated by** $\mathscr{G}$ is the smallest algebra containing $\mathscr{G}$. We shall use the following lemma in the proof of Lemma 10.

Lemma 1. *If $\mathscr{S}$ is a semiring on a set X and $X \in \mathscr{S}$, then the algebra generated by $\mathscr{S}$ is equal to the collection of finite unions of members of $\mathscr{S}$.*

For a bounded countably additive function, the **Carathéodory extension theorem** states the following.[1]

Theorem 2 (Carathéodory extension theorem)**.** *Suppose that X is a nonempty set, that $\mathscr{S}$ is a semiring on X, and that $\mu_0 : \mathscr{S} \to [0, 1]$ is countably additive. Then there is one and only one measure on $\sigma(\mathscr{S})$ whose restriction to $\mathscr{S}$ is equal to μ_0.*

[1] René L. Schilling, *Measures, Integrals and Martingales*, p. 37, Theorem 6.1. If we had not specified that $\mu_0 : \mathscr{S} \to [0, 1]$ but rather had talked about $\mu_0 : \mathscr{S} \to [0, \infty]$, then the Carathéodory extension theorem shows that there is some extension of μ_0 to $\sigma(\mathscr{S})$, but this extension need not be unique.

2 Product σ-algebras

Suppose that X is a set, that $\{(Y_i, \mathscr{M}_i) : i \in I\}$ is a family of measurable spaces, and that $f_i : X \to Y_i$ are functions. The smallest σ-algebra on X such that each f_i is measurable is called the **σ-algebra generated by** $\{f_i : i \in I\}$. This is analogous to the initial topology induced by a family of functions on a set. Calling this σ-algebra $\mathscr{M}$ and supposing that $\sigma(\mathscr{G}_i) = \mathscr{M}_i$ for each $i \in I$, we check then that

$$\mathscr{M} = \sigma\left(\{f_i^{-1}(A) : i \in I, A \in \mathscr{G}_i\}\right). \tag{1}$$

Suppose that $\{(X_i, \mathscr{M}_i) : i \in I\}$ is a family of measurable spaces. Let

$$X = \prod_{i \in I} X_i,$$

the cartesian product of the sets X_i, and let $\pi_i : X \to X_i$ be the projection maps. The **product σ-algebra** on X is the σ-algebra $\mathscr{M}$ generated by $\{\pi_i : i \in I\}$, and is denoted

$$\mathscr{M} = \bigotimes_{i \in I} \mathscr{M}_i.$$

This is analogous to the product topology on a cartesian product of topological spaces, which has the initial topology induced by the family of projection maps.

For $H \subset I$, we define

$$X_H = \prod_{i \in H} X_i.$$

Thus, $X_I = X$ and $X_\emptyset = \{\emptyset\}$, and for $G \subset H$,

$$X_H = X_G \times X_{H \setminus G}.$$

For $H \subset I$, let

$$\mathscr{M}_H = \bigotimes_{i \in H} \mathscr{M}_i,$$

the product σ-algebra on X_H. Thus, $\mathscr{M}_I = \mathscr{M}$ and $\mathscr{M}_\emptyset = \{\emptyset, \{\emptyset\}\}$, and for $G \subset H$ we have

$$\mathscr{M}_H = \mathscr{M}_G \otimes \mathscr{M}_{H \setminus G}.$$

For $G \subset H$, we define $P_{H,G} : X_H \to X_G$ to be the projection map: an element of X_H is a function x on H such that $x(i) \in X_i$ for all $i \in H$, and $P_{H,G}(x)$ is the restriction of x to G.

Lemma 3. *For $G \subset H$, $P_{H,G} : (X_H, \mathscr{M}_H) \to (X_G, \mathscr{M}_G)$ is measurable.*

If F is a finite subset of I and $A \in \mathscr{M}_F$, we call $A \times X_{I \setminus F} \in \mathscr{M}$ an **F-cylinder set**. Cylinder sets for the product σ-algebra are analogous to the usual basic open sets for the product topology.

Lemma 4. *The collection of all cylinder sets is an algebra of sets on $\prod_{i \in I} X_i$, and this collection generates the product σ-algebra $\bigotimes_{I \in I} \mathscr{M}_i$.*

The product σ-algebra can in fact be generated by a smaller collection of sets. (The following collection of sets is not a minimal collection of sets that generates the product σ-algebra, but it is smaller than the collection of all cylinder sets and it has the structure of a semiring, which will turn out to be useful.) An intersection of finitely many sets of the form $A \times \mathscr{M}_{I\setminus\{t\}}$, $A \in \mathscr{M}_t$, is called a **product cylinder**.

Lemma 5. *The collection of all product cylinders is a semiring of sets on $\prod_{i\in I} X_i$, and this collection generates the product σ-algebra $\bigotimes_{i\in I} \mathscr{M}_i$.*

3 Borel σ-algebras

If (X, τ) is a topogical space, the **Borel σ-algebra** on X is $\sigma(\tau)$, and is denoted $\mathscr{B}_X$. A member of $\mathscr{B}_X$ is called a **Borel set**.

Lemma 6. *If X is a topological space and $\mathscr{G}$ is a countable subbasis for the topology of X, then*

$$\mathscr{B}_X = \sigma(\mathscr{G}).$$

A separable metrizable space is second-countable, so we can apply the following theorem to such spaces.

Theorem 7. *Suppose that X_i, $i \in \mathbb{N}$, are second-countable topological spaces and let $X = \prod_{i\in\mathbb{N}} X_i$, with the product topology. Then*

$$\mathscr{B}_X = \bigotimes_{i\in\mathbb{N}} \mathscr{B}_{X_i}.$$

Proof. For each $i \in \mathbb{N}$, let $\mathscr{G}_i$ be a countable subbasis for the topology of X_i. Because $\mathscr{G}_i$ is a subbasis for the topology of X_i for each i, we check that $\mathscr{G}$ is a subbasis for the product topology of X, where

$$\mathscr{G} = \{\pi_i^{-1}(A) : i \in \mathbb{N}, A \in \mathscr{G}_i\}.$$

Because each $\mathscr{G}_i$ is countable and $\mathbb{N}$ is countable, $\mathscr{G}$ is countable. Hence by Lemma 6,

$$\mathscr{B}_X = \sigma(\mathscr{G}).$$

On the other hand, for each $i \in \mathbb{N}$ we have by Lemma 6 that $\mathscr{B}_{X_i} = \sigma(\mathscr{G}_i)$, and so by (1),

$$\bigotimes_{i\in\mathbb{N}} \mathscr{B}_{X_i} = \sigma(\mathscr{G}).$$

□

4 Product measures

If $\{(X_i, \mathscr{M}_i, \mu_i) : 1 \leq i \leq n\}$ are σ-finite measure spaces, let $X = \prod_{i=1}^n X_i$ and $\mathscr{M} = \bigotimes_{i=1}^n \mathscr{M}_i$. It is a fact that there is a unique measure μ on $\mathscr{M}$ such that for $A_i \in \mathscr{M}_i$,

$$\mu\left(\prod_{i=1}^n A_i\right) = \prod_{i=1}^n \mu_i(A_i),$$

and μ is a σ-finite measure. We write $\mu = \prod_{i=1}^n \mu_i$ and call μ the **product measure**.[2]

5 Compact classes

If X is a set and $\mathscr{C}$ is a collection of subsets of X, we say that $\mathscr{C}$ is a **compact class** if every countable subset of $\mathscr{C}$ with the finite intersection property has nonempty intersection. We remind ourselves that a collection $\mathscr{E}$ of sets is said to have the **finite intersection property** if for any finite subset $\mathscr{F}$ of $\mathscr{E}$ we have $\bigcap_{A \in \mathscr{F}} A \neq \emptyset$. Usually one speaks about a collection of sets having the finite intersection property in the following setting: A topological space Y is compact if and only if every collection of closed sets that has the finite intersection property has nonempty intersection.

We will employ the following lemma in the proof of the Kolmogorov extension theorem.[3]

Lemma 8. *Suppose that $\mathscr{C}^0$ is a compact class of subsets of a set X and let $\mathscr{C}$ be the collection of countable intersections of finite unions of members of $\mathscr{C}^0$. $\mathscr{C}$ is the smallest collection of subsets of X containing $\mathscr{C}^0$ that is closed under finite unions and countable intersections, and $\mathscr{C}$ is itself a compact class.*

We state the following result that gives conditions under which a finitely additive functions on an algebra of sets is in fact countably additive,[4] and then use it to prove an analogous result for semirings.

Lemma 9. *Suppose that $\mathscr{A}$ is an algebra of sets on a set X, and that $\mu_0 : \mathscr{A} \to [0, \infty)$ is finitely additive and $\mu_0(X) < \infty$. If there is a compact class $\mathscr{C} \subset \mathscr{A}$ such that*

$$\mu_0(A) = \sup\{\mu_0(C) : C \in \mathscr{C} \text{ and } C \subset A\}, \qquad A \in \mathscr{A},$$

then μ_0 is countably additive.

The following lemma gives conditions under which a finitely additive function on a semiring of sets is in fact countably additive.[5]

[2] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., pp. 64–65, and p. 31, Theorem 1.14.

[3] V. I. Bogachev, *Measure Theory*, volume I, p. 50, Proposition 1.12.4.

[4] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 378, Theorem 10.13.

[5] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 521, Lemma 15.25.

Lemma 10. *Suppose that $\mathscr{S}$ is a semiring of sets on X with $X \in \mathscr{S}$ and that $\mu_0 : \mathscr{S} \to [0, \infty)$ is finitely additive and $\mu_0(X) < \infty$. If there is a compact class $\mathscr{C} \subset \mathscr{S}$ such that*

$$\mu_0(A) = \sup\{\mu_0(C) : C \in \mathscr{C} \text{ and } C \subset A\}, \qquad A \in \mathscr{S},$$

then μ_0 is a countably additive.

Proof. Let $\mathscr{C}_u$ be the collection of finite unions of members of $\mathscr{C}$. $\mathscr{C}_u$ is a subset of the compact class produced in Lemma 8, hence is itself a compact class. Let $\mathscr{A}$ be the collection of finite unions of members of $\mathscr{S}$, which by Lemma 1 is the algebra generated by $\mathscr{S}$. Because $\mathscr{C} \subset \mathscr{S} \subset \mathscr{A}$ and $\mathscr{A}$ is closed under finite unions, it follows that $\mathscr{C}_u \subset \mathscr{A}$.

Because $\mathscr{S}$ is a semiring, it is a fact that if $A_1, \ldots, A_n, A \in \mathscr{S}$, then there are pairwise disjoint $S_1, \ldots, S_m \in \mathscr{S}$ such that $A \setminus \bigcup_{i=1}^n A_i = \bigcup_{i=1}^m S_i$.[6] Thus, if $A_1, \ldots, A_n \in \mathscr{S}$, defining $E_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j$, with $E_1 = A_1 \setminus \emptyset = A_1$, the sets $E_1, \ldots, E_n$ are pairwise disjoint, and for each i there are pairwise disjoint $S_{i,1}, \ldots, S_{i,a_i} \in \mathscr{S}$ such that $E_i = \bigcup_{j=1}^{a_i} S_{i,j}$. Then the sets $S_{i,j}$, $1 \le i \le n$, $1 \le j \le a_i$ are pairwise disjoint and their union is equal to $\bigcup_{i=1}^n A_i$. This shows that any element of $\mathscr{A}$ can be written as a union of pairwise disjoint elements of $\mathscr{S}$.

Furthermore, because $\mathscr{S}$ is a semiring, if $A_1, \ldots, A_N \in \mathscr{S}$, there are pairwise disjoint $S_1, \ldots, S_k \in \mathscr{S}$ such that for each $1 \le i \le k$ there is some $1 \le n \le N$ such that $S_i \in A_n$, and for each $1 \le n \le N$, there is a subset $F \subset \{1, \ldots, k\}$ such that $A_n = \bigcup_{i \in F} S_i$.[7]

Let $E \in \mathscr{A}$ and suppose that $E = \bigcup_{n=1}^N A_n$, where $A_1, \ldots, A_N \in \mathscr{S}$ are pairwise disjoint, and that $E = \bigcup_{m=1}^M B_m$, where $B_1, \ldots, B_M \in \mathscr{S}$ are pairwise disjoint. There are pairwise disjoint $S_1, \ldots, S_k \in \mathscr{S}$ such that for each $1 \le i \le k$ there is some $1 \le n \le N$ or $1 \le m \le M$ such that, respectively, $S_i \in A_n$ or $S_i \in B_m$, and for each $1 \le n \le N$ there is some subset $F \subset \{1, \ldots, k\}$ such that $A_n = \bigcup_{i \in F} S_i$, and for each $1 \le m \le M$ there is some subset $F \subset \{1, \ldots, k\}$ such that $B_m = \bigcup_{i \in F} S_i$. It follows that $E = \bigcup_{i=1}^k S_i$, and because μ_0 is finitely additive,

$$\sum_{n=1}^N \mu_0(A_n) = \sum_{i=1}^k \mu_0(S_i) = \sum_{m=1}^M \mu_0(B_m).$$

Therefore, for $E \in \mathscr{A}$ it makes sense to define

$$\mu(E) = \sum_{n=1}^n \mu_0(A_i),$$

where $A_1, \ldots, A_n$ are pairwise disjoint elements of $\mathscr{S}$ whose union is equal to E. Also, $\mu(X) = \mu_0(X) < \infty$.

[6] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 134, Lemma 4.7.

[7] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 134, Lemma 4.8.

We shall now show that the function $\mu : \mathscr{A} \to [0,\infty)$ is finitely additive. If $E_1,\dots,E_N \in \mathscr{A}$ are pairwise disjoint, for each n there are pairwise disjoint $A_{n,1},\dots,A_{n,a_n} \in \mathscr{S}$ such that $E_n = \bigcup_{j=1}^{a_n} A_{n,j}$, and there are pairwise disjoint $S_1,\dots,S_k \in \mathscr{S}$ such that for each $1 \le i \le k$, there is some $1 \le n \le N$ and some $1 \le j \le a_n$ such that $S_i \in A_{n,j}$, and for each $1 \le n \le N$ and each $1 \le j \le a_n$ there is some subset $F \subset \{1,\dots,k\}$ such that $A_{n,j} = \bigcup_{i\in F} S_i$. It follows that $\bigcup_{n=1}^N E_n = \bigcup_{i=1}^k S_i$, and

$$\mu\left(\bigcup_{n=1}^N E_n\right) = \mu\left(\bigcup_{i=1}^k S_i\right) = \sum_{i=1}^k \mu_0(S_i) = \sum_{n=1}^N \sum_{j=1}^{a_n} \mu_0(A_{n,j}) = \sum_{n=1}^N \mu(E_n),$$

showing that μ is finitely additive.

For $E = \bigcup_{i=1}^n A_i \in \mathscr{A}$ with pairwise disjoint $A_1,\dots,A_n \in \mathscr{S}$, let $\epsilon > 0$, and for each $1 \le i \le n$ let $C_i \in \mathscr{C}$ with $\mu_0(C_i) > \mu_0(A_i) + \frac{\epsilon}{n}$ and $C_i \subset A_i$. Then $C = \bigcup_{i=1}^n C_i \in \mathscr{C}_u$. As $A_1,\dots,A_n$ are pairwise disjoint and $C_i \subset A_i$, $C_1,\dots,C_n$ are pairwise disjoint, so because μ is finitely additive on $\mathscr{A}$,

$$\mu(C) = \sum_{i=1}^n \mu(C_i) = \sum_{i=1}^n \mu_0(C_i) > \sum_{i=1}^n \left(\mu_0(A_i) + \frac{\epsilon}{n}\right) = \mu(E) + \epsilon.$$

Lemma 9 tells us now that $\mu : \mathscr{A} \to [0,\infty)$ is countably additive, and therefore μ_0, its restriction to the semiring $\mathscr{S}$, is countably additive. $\square$

6 Kolmogorov consistent families

Suppose that $\{(X_i,\mathscr{M}_i) : i \in I\}$ is a family of measurable spaces. The collection D of all finite subsets of I, ordered by set inclusion, is a directed set. Suppose that for each $F \in D$, μ_F is a probability measure on $\mathscr{M}_F$; we defined the notation $\mathscr{M}_F$ in §2 and we use that here. We say that the family of measures $\{\mu_F : F \in D\}$ is **Kolmogorov consistent** if whenever $F, G \in D$ with $F \subset G$, it happens that $P_{G,F*}\mu_G = \mu_F$, where $f_*\mu$ denotes the **pushforward of** μ **by** f, i.e. $f_*\mu = \mu \circ f^{-1}$. It makes sense to talk about $P_{G,F*}\mu_G$ because $P_{G,F} : (X_G,\mathscr{M}_G) \to (X_F,\mathscr{M}_F)$ is measurable, as stated in Lemma 3.

We are now prepared to prove the **Kolmogorov extension theorem**.[8]

Theorem 11 (Kolmogorov extension theorem)**.** *Suppose that $\{(X_i,\mathscr{M}_i) : i \in I\}$ is a family of measurable spaces and suppose that for each $F \in D$, μ_F is a probability measure on $\mathscr{M}_F$. If the family of probability measures $\{\mu_F : F \in D\}$ is Kolmogorov consistent and if for each $i \in I$ there is a compact class $\mathscr{C}_i \subset \mathscr{M}_i$ satisfying*

$$\mu_i(A) = \sup\{\mu_i(C) : C \in \mathscr{C}_i \text{ and } C \subset A\}, \qquad A \in \mathscr{M}_i, \tag{2}$$

then there is a unique probability measure on $\mathscr{M}_I$ such that for each $F \in D$, the pushforward of μ by the projection map $P_{I,F} : X_I \to X_F$ is equal to μ_F.

[8] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 522, Theorem 15.26.

Proof. Define

$$\mathscr{C}^0 = \{C \times X_{I\setminus\{i\}} : i \in I, C \in \mathscr{C}_i\}.$$

We shall show that $\mathscr{C}^0$ is a compact class. Suppose

$$\{C_n \times X_{I\setminus\{i_n\}} : n \in \mathbb{N}, C_n \in \mathscr{C}_{i_n}\} \subset \mathscr{C}^0$$

has empty intersection. For each $i \in I$, let

$$Q_i = \bigcap_{i_n = i} C_n,$$

and if there are no such i_n, then $Q_i = X_i$. Then

$$\bigcap_{n\in\mathbb{N}} C_n \times X_{I\setminus\{i_n\}} = \prod_{i\in I} Q_i.$$

Because this intersection is equal to $\emptyset$, one of the factors in the product is equal to $\emptyset$. (For some purposes one wants to keep track of where the axiom of choice is used, so we mention that concluding that some factor of any empty cartesian product is itself empty is equivalent to the axiom of choice). No X_i is empty, so this empty Q_i must be of the form $\bigcap_{i_n=i} C_n$ for which at least one i_n is equal to i. But if $i_n = i$ then $C_n \in \mathscr{C}_i$, and because $\mathscr{C}_i$ is a compact class, $\bigcap_{i_n=i} C_n = \emptyset$ implies that there are finitely many $a_1, \ldots, a_N$ such that $\bigcap_{n=1}^N C_{a_n} = \emptyset$, and this yields $\bigcap_{n=1}^N C_{a_n} \times X_{I\setminus\{i_{a_n}\}} = \emptyset$. We have thus proved that if an intersection of countably many members of $\mathscr{C}^0$ is empty then some intersection of finitely many of these is empty, showing that $\mathscr{C}^0$ is a compact class. Let $\mathscr{C}^1$ be the smallest collection of subsets of X_I containing $\mathscr{C}$ that is closed under finite unions and countable intersections, and by Lemma 8 we know that $\mathscr{C}^1$ is a compact class; we use the notation $\mathscr{C}^1$ because presently we will use a subset of $\mathscr{C}^1$.

Let $\mathscr{A}$ be the collection of all cylinder sets of the product σ-algebra $\mathscr{M}_I$. Explicitly,

$$\mathscr{A} = \{A \times X_{I\setminus F} : F \in D, A \in \mathscr{M}_F\}.$$

Suppose $F, G \in D$, $F \subset G$, $A \in \mathscr{M}_F$, $B \in \mathscr{M}_G$, and that $A \times X_{I\setminus F} = B \times X_{I\setminus G}$. It follows that $B = A \times X_{G\setminus F}$, and then using $P_{G,F*}\mu_G = \mu_F$ we get

$$\mu_G(B) = \mu_G(A \times X_{G\setminus F}) = \mu_G(P_{G,F}^{-1}(A)) = \mu_F(A).$$

Therefore it makes sense to define $\mu_0 : \mathscr{A} \to [0,1]$ as follows: for $A \times X_{I\setminus F} \in \mathscr{A}$,

$$\mu_0(A \times X_{I\setminus F}) = \mu_F(A).$$

Let $F_1, \ldots, F_n \in D$ and $A_1 \in \mathscr{M}_{F_1}, \ldots, A_n \in \mathscr{M}_{F_n}$, and suppose that $A_1 \times X_{I\setminus F_1}, \ldots, A_n \times X_{I\setminus F_n} \in \mathscr{G}$ are pairwise disjoint. With $F = \bigcup_{j=1}^n F_j \in D$,

$$\bigcup_{j=1}^n A_j \times X_{I\setminus F_j} = \left(\bigcup_{j=1}^n A_j \times X_{F\setminus F_j}\right) \times X_{I\setminus F},$$

which is an F-cylinder set. Then,

$$\begin{aligned}
\mu_0\left(\bigcup_{j=1}^n A_j \times X_{I\setminus F_j}\right) &= \mu_F\left(\bigcup_{j=1}^n A_j \times X_{F\setminus F_j}\right)\\
&= \sum_{j=1}^n \mu_F(A_j \times X_{F\setminus F_j})\\
&= \sum_{j=1}^n \mu_0(A_j \times X_{I\setminus F_j}),
\end{aligned}$$

showing that $\mu_0 : \mathscr{A} \to [0,1]$ is finitely additive.

Let $\mathscr{G}$ be the collection of all product cylinder sets of the product σ-algebra $\mathscr{M}_I$. Explicitly,

$$\mathscr{G} = \left\{\bigcap_{i\in F} A_i \times X_{I\setminus\{i\}} : F \in D, \text{ and } A_i \in \mathscr{M}_i \text{ for } i \in F\right\}.$$

It is apparent that $\mathscr{C}^0 \subset \mathscr{G}$. Let $\mathscr{C}$ be the intersection of $\mathscr{C}^1$ and $\mathscr{G}$. A subset of a compact class is a compact class, so $\mathscr{C}$ is a compact class, and $\mathscr{C}^0 \subset \mathscr{C}$.

Suppose that $E \in \mathscr{G}$: there is some $F \in D$ and $A_i \in \mathscr{M}_i$ for each $i \in F$ such that

$$E = \bigcap_{i\in F} A_i \times X_{I\setminus\{i\}} = \left(\prod_{i\in F} A_i\right) \times X_{I\setminus F}.$$

Take $n = |F|$, and let $\epsilon > 0$. Then, for each $i \in F$, by (2) there is some $C_i \in \mathscr{C}_i$ such that $C_i \subset A_i$ and $\mu_i(A_i) < \mu_i(C_i) + \frac{\epsilon}{n}$, and we set

$$C = \bigcap_{i\in F} C_i \times X_{I\setminus\{i\}} = \left(\prod_{i\in F} C_i\right) \times X_{I\setminus F},$$

which is a finite intersection of members of $\mathscr{C}^0$ and hence belongs to $\mathscr{C}^1$, and which visibly belongs to $\mathscr{G}$, and hence belongs to $\mathscr{C}$. We have

$$\begin{aligned}
E \setminus C &= \left(\bigcup_{i\in F} (A_i \setminus C_i) \times \prod_{j\in F\setminus\{i\}} A_j\right) \times X_{I\setminus F}\\
&\subset \bigcup_{i\in F} (A_i \setminus C_i) \times X_{I\setminus\{i\}}.
\end{aligned}$$

Both $E \setminus C$ and the above union are cylinder sets so it makes sense to apply μ_0

to them, and because μ_0 is finitely additive,

$$\begin{aligned}\mu_0(E \setminus C) &\leq \sum_{i \in F} \mu_0((A_i \setminus C_i) \times X_{I \setminus \{i\}}) \\ &= \sum_{i \in F} \mu_i(A_i \setminus C_i) \\ &= \sum_{i \in F} \mu_i(A_i) - \mu_i(C_i) \\ &< \sum_{i \in F} \frac{\epsilon}{n} \\ &= \epsilon.\end{aligned}$$

Hence $\mu_0(E) - \mu_0(C) = \mu_0(E \setminus C) < \epsilon$, i.e. $\mu_0(E) < \mu_0(C) + \epsilon$. Thus, we have proved that for each $\epsilon > 0$, there is some $C \in \mathscr{C}$ such that $C \subset E$ and $\mu_0(E) < \mu_0(C) + \epsilon$, which means that

$$\mu_0(E) = \sup\{\mu_0(C) : C \in \mathscr{C} \text{ and } C \subset E\}.$$

Using the restriction of $\mu_0 : \mathscr{A} \to [0, 1]$ to the semiring $\mathscr{G}$ and the compact class $\mathscr{C}$, the conditions of Lemma 10 are satisfied, and therefore the restriction of μ_0 to $\mathscr{G}$ is countably additive.

By Lemma 5, $\mathscr{M}_I = \sigma(\mathscr{G})$. Because the restriction of μ_0 to the semiring $\mathscr{G}$ is countably additive, we can apply the Carathéodory extension theorem, which tells us that there is a unique measure μ on $\sigma(\mathscr{G}) = \mathscr{M}_I$ whose restriction to $\mathscr{G}$ is equal to the restriction of μ_0 to $\mathscr{G}$. Check that the restriction of μ to $\mathscr{A}$ is equal to μ_0. For $F \in D$ and $A \in \mathscr{M}_F$,

$$P_{I,F*}\mu(A) = \mu(P_{I,F}^{-1}(A)) = \mu(A \times X_{I \setminus F}) = \mu_0(A \times X_{I \setminus F}) = \mu_F(A),$$

showing that $P_{I,F*}\mu = \mu_F$. Certainly $\mu(X_I) = 1$, namely, μ is a probability measure. If ν is a probability measure on X_I whose pushforward by $P_{I,F}$ is equal to μ_F for each $F \in D$, then check that the restriction of ν to $\mathscr{G}$ is equal to the restriction of μ_0 to $\mathscr{G}$, and then by the assertion of uniqueness in Carathéodory's theorem, $\nu = \mu$, completing the proof. □

If X is a Hausdorff space, we say that a Borel measure μ on X is **tight** if for every $A \in \mathscr{B}_X$,

$$\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}.$$

A **Polish space** is a topological space that is homeomorphic to a complete separable metric space, and it is a fact that a finite Borel measure on a Polish space is tight.[9] In particular, any Borel probability measure on a Polish space is tight. We use this in the proof of the following version of the Kolmogorov extension theorem, which applies for instance to the case where $X_i = \mathbb{R}$ for each $i \in I$, with I any index set.

[9] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 438, Theorem 12.7.

Corollary 12. *Suppose that $\{X_i : i \in I\}$ is a family of Polish spaces and suppose that for each $F \in D$, μ_F is a Borel probability measure on X_F. If the family of measures $\{\mu_F : F \in D\}$ is Kolmogorov consistent, then there is a unique probability measure on $\mathscr{M}_I = \bigotimes_{i \in I} \mathscr{B}_{X_i}$ such that for each $F \in D$, the pushforward of μ by the projection map $P_{I,F} : X_I \to X_F$ is equal to μ_F.*

Proof. For each $i \in I$, let $\mathscr{C}_i$ be the collection of all compact subsets of X_i. In any topological space, check that a collection of compact sets is a compact class. The fact that μ_i is a Borel probability measure on a Polish space then implies that it is tight, which we can write as

$$\mu_i(A) = \sup\{\mu_i(C) : K \in \mathscr{C}_i \text{ and } K \subset A\}, \qquad A \in \mathscr{M}_i.$$

Therefore the conditions of Theorem 11 are satisfied, so the claim follows. $\square$

If the index set I in the above corollary is countable, then by Theorem 7 the product σ-algebra $\bigotimes_{i \in I} \mathscr{B}_{X_i}$ is equal to the Borel σ-algebra of the product $\prod_{t \in T} X_i$, so that the probability measure μ on the product σ-algebra is in this case a Borel measure.