

Tauber's theorem and Karamata's proof of the Hardy-Littlewood tauberian theorem

Jordan Bell

November 11, 2017

The following lemma is attributed to Kronecker by Knopp.¹

Lemma 1 (Kronecker's lemma). If $b_n \rightarrow 0$ then

$$\frac{b_0 + b_1 + \cdots + b_n}{n+1} \rightarrow 0.$$

Proof. Suppose that $|b_n| \leq K$ for all n , and let $\epsilon > 0$. As $b_n \rightarrow 0$ there is some n_0 such that $n \geq n_0$ implies that $|b_n| < \epsilon$. If $n \geq \frac{(n_0+1)K}{\epsilon}$, then

$$\begin{aligned} \left| \frac{b_0 + b_1 + \cdots + b_n}{n+1} \right| &\leq \left| \frac{b_0 + b_1 + \cdots + b_{n_0}}{n+1} \right| + \left| \frac{b_{n_0} + \cdots + b_n}{n+1} \right| \\ &\leq \frac{(n_0+1)K}{n+1} + \frac{(n-n_0)\epsilon}{n+1} \\ &\leq \epsilon + \epsilon. \end{aligned}$$

□

We now use the above lemma to prove Tauber's theorem.²

Theorem 2 (Tauber's theorem). If $a_n = o(1/n)$ and $\sum_{n=0}^{\infty} a_n x^n \rightarrow s$ as $x \rightarrow 1^-$, then

$$\sum_{n=0}^{\infty} a_n = s.$$

Proof. Let $\epsilon > 0$. Because $\sum_{n=0}^{\infty} a_n x^n \rightarrow s$ as $x \rightarrow 1^-$, there is some $\delta > 0$ such that $x > 1 - \delta$ implies that

$$\left| \sum_{n=0}^{\infty} a_n x^n - s \right| < \epsilon.$$

¹Konrad Knopp, *Theory and Application of Infinite Series*, p. 129, Theorem 3.

²cf. E. C. Titchmarsh, *The Theory of Functions*, second ed., p. 10, §1.23.

Next, because $n|a_n| \rightarrow 0$, there is some $N > \frac{1}{\delta}$ such that (i) if $n \geq N$ then $n|a_n| < \epsilon$ and by Lemma 1, (ii) $\frac{1}{N+1} \sum_{n=0}^N n|a_n| < \epsilon$.

Take $x = 1 - \frac{1}{N}$, so $N = \frac{1}{1-x}$ and $1-x = \frac{1}{N}$. We have

$$\begin{aligned} \left| \sum_{n=N+1}^{\infty} a_n x^n \right| &= \left| \sum_{n=N+1}^{\infty} n a_n \cdot \frac{x^n}{n} \right| \\ &< \sum_{n=N+1}^{\infty} \epsilon \cdot \frac{x^n}{N+1} \\ &< \frac{\epsilon}{N+1} \cdot \frac{1}{1-x} \\ &= \epsilon \cdot \frac{N}{N+1} \\ &< \epsilon. \end{aligned}$$

Also, using

$$1 - x^n = (1-x)(1+x+\cdots+x^{n-1}) < (1-x)n$$

we have

$$\begin{aligned} \left| \sum_{n=0}^N a_n (1-x^n) \right| &\leq \sum_{n=0}^N |a_n| (1-x^n) \\ &< \sum_{n=0}^N |a_n| (1-x)n \\ &= \sum_{n=0}^N \frac{|a_n|n}{N} \\ &= \frac{N+1}{N} \cdot \frac{1}{N+1} \sum_{n=0}^N n|a_n| \\ &< \frac{N+1}{N} \cdot \epsilon \\ &< 2\epsilon. \end{aligned}$$

Now,

$$\begin{aligned}
\sum_{n=0}^N a_n - s &= \sum_{n=0}^N a_n - \sum_{n=0}^N a_n x^n + \sum_{n=0}^N a_n x^n - s \\
&= \sum_{n=0}^N a_n (1 - x^n) + \sum_{n=0}^N a_n x^n - s \\
&= \sum_{n=0}^N a_n (1 - x^n) + \sum_{n=0}^N a_n x^n + \sum_{n=N+1}^{\infty} a_n x^n - \sum_{n=N+1}^{\infty} a_n x^n - s \\
&= \sum_{n=0}^N a_n (1 - x^n) + \sum_{n=0}^{\infty} a_n x^n - s - \sum_{n=N+1}^{\infty} a_n x^n
\end{aligned}$$

and then

$$\begin{aligned}
\left| \sum_{n=0}^N a_n - s \right| &\leq \left| \sum_{n=0}^N a_n (1 - x^n) \right| + \left| \sum_{n=0}^{\infty} a_n x^n - s \right| + \left| \sum_{n=N+1}^{\infty} a_n x^n \right| \\
&< 2\epsilon + \epsilon + \epsilon,
\end{aligned}$$

proving the claim. \square

Lemma 3. Let $g : [0, 1] \rightarrow \mathbb{R}$ and $0 < c < 1$. Suppose that the restrictions of g to $[0, c)$ and $[c, 1]$ are continuous and that

$$g(c-0) = \lim_{x \rightarrow c^-} g(x) \leq g(c).$$

For $\epsilon > 0$, there are polynomials $p(x)$ and $P(x)$ such that

$$p(x) \leq g(x) \leq P(x), \quad 0 \leq x \leq 1$$

and

$$\|g - p\|_1 \leq \epsilon, \quad \|g - P\|_1 \leq \epsilon.$$

Proof. There is some $\delta > 0$ such that $c - \delta \leq x < c$ implies that

$$g(c-0) - \frac{\epsilon}{2} \leq g(x) \leq g(c-0) + \frac{\epsilon}{2};$$

further, take $\delta < \frac{\epsilon}{g(c) - g(c-0)}$ and $\delta < \frac{1}{2}$.

Take L to be the linear function satisfying

$$L(c - \delta) = g(c - \delta) + \frac{\epsilon}{2}, \quad L(c) = g(c) + \frac{\epsilon}{2}.$$

For $c - \delta \leq x < c$,

$$\begin{aligned}
L(x) - g(x) &= L(x) - g(c - \delta) + g(c - \delta) - g(c - 0) + g(c - 0) - g(x) \\
&= L(x) - L(c - \delta) + \frac{\epsilon}{2} + g(c - \delta) - g(c - 0) + g(c - 0) - g(x) \\
&\leq L(c) - L(c - \delta) + \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= g(c) - g(c - \delta) + \frac{3\epsilon}{2} \\
&= g(c) - g(c - 0) + g(c - 0) - g(c - \delta) + \frac{3\epsilon}{2} \\
&< \frac{\epsilon}{\delta} + \frac{\epsilon}{2} + \frac{3\epsilon}{2} \\
&< \frac{2\epsilon}{\delta}.
\end{aligned}$$

Define $\Phi : [0, 1] \rightarrow \mathbb{R}$ by

$$\Phi(x) = \begin{cases} g(x) + \frac{\epsilon}{2} & 0 \leq x < c - \delta \\ \max\{L(x), g(x) + \frac{\epsilon}{2}\} & c - \delta \leq x \leq c \\ g(x) + \frac{\epsilon}{2} & c < x \leq 1. \end{cases}$$

Φ is continuous and $\Phi \geq g + \frac{\epsilon}{2}$. We have

$$\begin{aligned}
\|g - \Phi\|_1 &= \int_0^1 (\Phi(x) - g(x)) dx \\
&= \int_0^{c-\delta} \frac{\epsilon}{2} dx + \int_{c-\delta}^c (\Phi(x) - g(x)) dx + \int_c^1 \frac{\epsilon}{2} dx \\
&< \frac{\epsilon}{2} + \int_{c-\delta}^c (\Phi(x) - g(x)) dx \\
&\leq \frac{\epsilon}{2} + \int_{c-\delta}^c \max\left\{L(x) - g(x), \frac{\epsilon}{2}\right\} dx \\
&\leq \frac{\epsilon}{2} + \int_{c-\delta}^c \max\left\{\frac{2\epsilon}{\delta}, \frac{\epsilon}{2}\right\} dx \\
&= \frac{\epsilon}{2} + \delta \cdot \frac{2\epsilon}{\delta} \\
&= \frac{5\epsilon}{2}.
\end{aligned}$$

Because Φ is continuous, by the Weierstrass approximation theorem there is a polynomial $P(x)$ such that $\|\Phi - P\|_\infty \leq \frac{\epsilon}{2}$. Then,

$$g(x) \leq P(x), \quad 0 \leq x \leq 1,$$

and

$$\|g - P\|_1 \leq \|g - \Phi\|_1 + \|\Phi - P\|_1 < \frac{5\epsilon}{2} + \|\Phi - P\|_\infty \leq \frac{5\epsilon}{2} + \frac{\epsilon}{2} = 3\epsilon.$$

On the other hand, take l to be the linear function satisfying

$$l(c - \delta) = g(c - \delta) - \frac{\epsilon}{2}, \quad l(c) = g(c) - \frac{\epsilon}{2}.$$

One checks that for $c - \delta \leq x < c$,

$$g(x) - l(x) < \frac{2\epsilon}{\delta},$$

Define $\phi : [0, 1] \rightarrow \mathbb{R}$ by

$$\phi(x) = \begin{cases} g(x) - \frac{\epsilon}{2} & 0 \leq x < c - \delta \\ \min\{l(x), g(x) - \frac{\epsilon}{2}\} & c - \delta \leq x \leq c \\ g(x) - \frac{\epsilon}{2} & c < x \leq 1, \end{cases}$$

which is continuous and satisfies $\phi \leq g - \frac{\epsilon}{2}$. One checks that

$$\|g - \phi\|_1 < \frac{5\epsilon}{2}.$$

Because ϕ is continuous, there is a polynomial $p(x)$ such that $\|\phi - p\|_\infty \leq \frac{\epsilon}{2}$. Then,

$$p(x) \leq g(x), \quad 0 \leq x \leq 1,$$

and

$$\|g - p\|_1 \leq \|g - \phi\|_1 + \|\phi - p\|_1 < \frac{5\epsilon}{2} + \|\phi - p\|_\infty \leq \frac{5\epsilon}{2} + \frac{\epsilon}{2} = 3\epsilon.$$

□

The following is the Hardy-Littlewood tauberian theorem.³

Theorem 4 (Hardy-Littlewood tauberian theorem). If $a_n \geq 0$ for all n and

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x}, \quad x \rightarrow 1^-,$$

then

$$s_n = \sum_{\nu=0}^n a_\nu \sim n.$$

³E. C. Titchmarsh, *The Theory of Functions*, second ed., p. 227, §7.53, attributed to Karamata.

Proof. For any $k \geq 0$,

$$\begin{aligned}
(1-x) \sum_{n=0}^{\infty} a_n x^n (x^n)^k &= \frac{1-x}{1-x^{k+1}} (1-x^{k+1}) \sum_{n=0}^{\infty} a_n (x^{k+1})^n \\
&= \frac{1}{1+x+\dots+x^k} (1-x^{k+1}) \sum_{n=0}^{\infty} a_n (x^{k+1})^n \\
&\rightarrow \frac{1}{k+1} \cdot 1 \\
&= \int_0^1 t^k dt,
\end{aligned}$$

as $x \rightarrow 1^-$. Hence if $P(x)$ is a polynomial, then

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n P(x^n) = \int_0^1 P(t) dt. \quad (1)$$

Define $g : [0, 1] \rightarrow \mathbb{R}$ by

$$g(t) = \begin{cases} 0 & 0 \leq t < e^{-1} \\ t^{-1} & e^{-1} \leq t \leq 1. \end{cases}$$

Let $\epsilon > 0$. By Lemma 3, there are polynomials $p(x), P(x)$ such that

$$p(x) \leq g(x) \leq P(x), \quad 0 \leq x \leq 1$$

and

$$\|g - p\|_1 \leq \epsilon, \quad \|P - g\|_1 \leq \epsilon.$$

Because the coefficients a_n are nonnegative, taking upper limits and then using (1) we obtain

$$\begin{aligned}
\limsup_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) &\leq \limsup_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n P(x^n) \\
&= \lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n P(x^n) \\
&= \int_0^1 P(t) dt \\
&< \int_0^1 g(t) dt + \epsilon.
\end{aligned}$$

Taking lower limits and then using (1) we obtain

$$\begin{aligned}
\liminf_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) &\geq \liminf_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n p(x^n) \\
&= \lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n p(x^n) \\
&= \int_0^1 p(t) dt \\
&> \int_0^1 g(t) dt - \epsilon.
\end{aligned}$$

The above two inequalities do not depend on the polynomials $p(x), P(x)$ but only on ϵ , and taking $\epsilon \rightarrow 0$ yields

$$\limsup_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) \leq \int_0^1 g(t) dt$$

and

$$\liminf_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) \geq \int_0^1 g(t) dt.$$

Thus

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) = \int_0^1 g(t) dt = \int_{e^{-1}}^1 t^{-1} dt = 1. \quad (2)$$

For $x = e^{-1/N}$ we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n g(x^n) &= \sum_{n=0}^{\infty} a_n e^{-n/N} g(e^{-n/N}) \\
&= \sum_{n=0}^N a_n e^{-n/N} e^{n/N} \\
&= s_N.
\end{aligned}$$

Thus, (2) tells us that

$$\lim_{N \rightarrow \infty} (1 - e^{-1/N}) s_N = 1.$$

That is,

$$s_N \sim \frac{1}{1 - e^{-1/N}},$$

and using

$$\frac{1}{1 - e^{-1/N}} = N + \frac{1}{2} + O(N^{-1})$$

we get

$$s_N \sim N,$$

completing the proof. □