Integral operators

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1 Product measures

Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. Then with $\mathscr{A} \otimes \mathscr{A}$ the product σ-algebra and $\mu \otimes \mu$ the product measure on $\mathscr{A} \otimes \mathscr{A}$, $(X \times X, \mathscr{A} \otimes \mathscr{A}, \mu \otimes \mu)$ is itself a σ-finite measure space.

Write $F_x(y) = F(x, y)$ and $F^y(x) = F(x, y)$. For any measurable space $(X', \mathscr{A}')$, it is a fact that if $F : X \times X \to X'$ is measurable then F_x is measurable for each $x \in X$ and F^y is measurable for each $y \in X$.[1]

Suppose that $F \in \mathscr{L}^1(X \times X)$, $F : X \to \mathbb{C}$. Fubini's theorem tells us the following.[2] [3] There are sets $N_1, N_2 \in \mathscr{A}$ with $\mu(N_1) = 0$ and $\mu(N_2) = 0$ such that if $x \in N_1^c$ then $F_x \in \mathscr{L}^1(X)$ and if $y \in N_2^c$ then $F^y \in \mathscr{L}^1(X)$. Define

$$I_1(x) = \begin{cases} \int_X F_x(y) d\mu(y) & x \in N_1^c \\ 0 & x \in N_1 \end{cases}$$

and

$$I_2(y) = \begin{cases} \int_X F^y(x) d\mu(x) & y \in N_2^c \\ 0 & y \in N_2. \end{cases}$$

$I_1 \in \mathscr{L}^1(X)$ and $I_2 \in \mathscr{L}^1(X)$, and

$$\int_{X \times X} F d(\mu \otimes \mu) = \int_X I_2(y) d\mu(y) = \int_X I_1(x) d\mu(x).$$

[1]Heinz Bauer, *Measure and Integration Theory*, p. 138, Lemma 23.5.

[2]Heinz Bauer, *Measure and Integration Theory*, p. 139, Corollary 23.7.

[3]Suppose that $F : X \times X \to [0, \infty]$ is measurable. Tonelli's theorem, Heinz Bauer, *Measure and Integration Theory*, p. 138, Theorem 23.6, tells us that the functions

$$x \mapsto \int_X F_x d\mu, \qquad y \mapsto \int_X F^y d\mu$$

are measurable $X \to [0, \infty]$, and that

$$\int_{X \times X} F d(\mu \otimes \mu) = \int_X \left(\int_X F^y d\mu \right) d\mu(y) = \int_X \left(\int_X F_x d\mu \right) d\mu(x).$$

2 Integral operators in L^2

Let $k \in \mathscr{L}^2(X \times X)$ and let $g \in \mathscr{L}^2(X)$. By Fubini's theorem, there is a set $Z \in \mathscr{A}$ with $\mu(Z) = 0$ such that if $x \in Z^c$ then $k_x \in \mathscr{L}^2(X)$. For $x \in Z_n^c$, by the Cauchy-Schwarz inequality,

$$\int_X |k_x g| d\mu \leq \left(\int_X |k_x|^2 d\mu \right)^{1/2} \left(\int_X |g|^2 d\mu \right)^{1/2} = \|k_x\|_{L^2} \|g\|_{L^2},$$

so $k_x g \in \mathscr{L}^1(X)$.

Since μ is σ-finite, there are $A_n \in \mathscr{A}$, $\mu(A_n) < \infty$, with $A_n \uparrow X$. For each n, the function $(x, y) \mapsto 1_{A_n}(x) g(y)$ belongs to $\mathscr{L}^2(X \times X)$ and hence, by the Cauchy-Schwarz inequality, $(x, y) \mapsto k(x, y) 1_{A_n}(x) g(y)$ belongs to $\mathscr{L}^1(X \times X)$. Applying Fubini's theorem, there is a set $N_n \in \mathscr{A}$ with $\mu(N_n) = 0$ such that if $x \in N_n^c$ then $y \mapsto k(x, y) 1_{A_n}(x) g(y)$ belongs to $\mathscr{L}^1(X)$, and the function $I_n : X \to \mathbb{C}$ defined by

$$I_n(x) = \begin{cases} \int_X k_x(y) 1_{A_n}(x) g(y) d\mu(y) & x \in N_n^c \\ 0 & x \in N_n \end{cases}$$

belongs to $\mathscr{L}^1(X)$.

Let $M = \bigcup_n (Z \cup N_n)$, for which

$$\mu(M) \leq \sum_n \mu(Z \cup N_n) \leq \sum_n (\mu(Z) + \mu(N_n)) = 0.$$

We note

$$M^c = \bigcap_n (Z^c \cap N_n^c).$$

For $g \in \mathscr{L}^2(X)$, define $K_M g : X \to \mathbb{C}$ by

$$K_M g(x) = \begin{cases} \int_X k_x(y) g(y) d\mu(y) & x \in M^c \\ 0 & x \in M. \end{cases} \tag{1}$$

For $x \in M^c$,

$$I_n(x) = \int_X k_x(y) 1_{A_n}(x) g(y) d\mu(y) = 1_{A_n}(x) \int_X k_x(y) g(y) d\mu(y) = 1_{A_n}(x) \cdot K_M g(x).$$

Then

$$1_{A_n} \cdot K_M g = 1_{M^c} \cdot 1_{A_n} \cdot K g = 1_{M^c} \cdot I_n,$$

which shows that $f_n = 1_{A_n} \cdot K_M g$ is measurable $X \to \mathbb{C}$. For any $x \in X$, for sufficiently large n we have $f_n(x) = K_M g(x)$, thus $f_n \to K_M g$ pointwise, which implies that $K_M g : X \to \mathbb{C}$ is measurable.[4]

[4] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 142, Lemma 4.29.

Using the Cauchy-Schwarz inequality and then Fubini's theorem,

$$\begin{aligned}\int_X |K_M g(x)|^2 d\mu(x) &= \int_{M^c} \left| \int_X k_x(y) g(y) d\mu(y) \right|^2 d\mu(x) \\ &\leq \|g\|_{L^2}^2 \cdot \int_{M^c} \left(\int_X |k_x(y)|^2 d\mu(y) \right) d\mu(x) \\ &= \|g\|_{L^2}^2 \cdot \|k\|_{L^2}^2 .\end{aligned}$$

This shows that $K_M g \in \mathscr{L}^2(X)$, with

$$\|K_M g\|_{L^2} \leq \|k\|_{L^2} \cdot \|g\|_{L^2} .$$

Recapitulating, for $g \in \mathscr{L}^2(X)$ there is some $M \in \mathscr{A}$ with $\mu(M) = 0$ such that for $x \in M^c$, $k_x \in \mathscr{L}^2(X)$, and such that $K_M g : X \to \mathbb{C}$ defined by (1) belongs to $\mathscr{L}^2(X)$. If N is any set satisfying these conditions, then for $x \in M^c \cap N^c$,

$$K_M g(x) = \int_X k_x(y) g(y) d\mu(y) = K_N g(x),$$

and $\mu((M^c \cap N^c)^c) = \mu(M \cup N) = 0$. Therefore, for $g \in \mathscr{L}^2(X)$ it makes sense to define $Kg \in L^2(X)$ by $Kg = K_M g$.

If $f, g \in \mathscr{L}^2(X)$ and $f = g$ in $L^2(X)$, check that $Kf = Kg$ in $L^2(X)$. We thus define $K : L^2(X) \to L^2(X)$ for $g \in L^2(X)$ as

$$Kg(x) = \int_X k_x(y) g(y) d\mu(y) = \left\langle g, \overline{k_x} \right\rangle ,$$

where

$$\langle f, g \rangle = \int_X f \cdot \overline{g} d\mu.$$

Theorem 1. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. For $k \in L^2(X \times X)$, it makes sense to define $Kg \in L^2(X)$ by*

$$Kg(x) = \int_X k_x(y) g(y) d\mu(y) = \left\langle g, \overline{k_x} \right\rangle .$$

$K : L^2(X) \to L^2(X)$ is a bounded linear operator with $\|K\| \leq \|k\|_{L^2}$.

3 Integrals of functions

Suppose that $f : X \to \mathbb{C}$ is a function, which we do not ask to be measurable, and that $Z_1, Z_2 \in \mathscr{A}$, $\mu(Z_1) = 0$, $\mu(Z_2) = 0$, satisfy $1_{Z_1^c} \cdot f, 1_{Z_2^c} \cdot f \in \mathscr{L}^1(X)$.

We have

$$\begin{aligned}\int_X 1_{Z_1^c} \cdot f d\mu &= \int_X 1_{Z_1^c} \cdot (1_{Z_2} + 1_{Z_2^c}) \cdot f d\mu \\ &= \int_X 1_{Z_1^c \cap Z_2} \cdot f d\mu + \int_X 1_{Z_1^c \cap Z_2^c} \cdot f d\mu \\ &= \int_X 1_{Z_1^c \cap Z_2^c} \cdot f d\mu \\ &= \int_X 1_{Z_2^c \cap Z_1^c} \cdot f d\mu \\ &= \int_X 1_{Z_2^c} \cdot f d\mu.\end{aligned}$$

Therefore if there is some $Z \in \mathscr{A}$ with $\mu(Z) = 0$ and $1_Z \cdot f \in \mathscr{L}^1(X)$, it makes sense to define

$$\int_X f d\mu = \int_X 1_Z \cdot f d\mu.$$

However, only if f is itself measurable do we write $f \in \mathscr{L}^1(X)$.

4 Self-adjoint operators

Theorem 2. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. For $k \in L^2(X \times X)$ satisfying $k_x = \overline{k^x}$, $K : L^2(X) \to L^2(X)$ is self-adjoint.*

Proof. For $f, g \in L^2(X)$,

$$\begin{aligned}\langle Kf, g\rangle &= \int_X Kf(x) \cdot \overline{g(x)} d\mu(x) \\ &= \int_X \left(\int_X k_x(y) f(y) d\mu(y) \right) \overline{g(x)} d\mu(x) \\ &= \int_X \left(\int_X k^y(x) \cdot \overline{g(x)} d\mu(x) \right) f(y) d\mu(y) \\ &= \int_X \left(\int_X \overline{k_y(x) g(x)} d\mu(x) \right) f(y) d\mu(y) \\ &= \int_X \overline{Kg(y)} \cdot f(y) d\mu(y) \\ &= \langle f, Kg\rangle .\end{aligned}$$

It follows that $K : L^2(X) \to L^2(X)$ is self-adjoint. □

5 Hilbert-Schmidt operators

Let $(X, \mathscr{A}, \mu)$ be a measure space and let $1 \le p < \infty$. It is a fact that if μ is σ-finite and $\mathscr{A}$ is countably generated, then the Banach space $L^p(X)$ is separable.[5]

Theorem 3. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite countably generated measure space. For $k \in L^2(X \times X)$, $K : L^2(X) \to L^2(X)$ is a Hilbert-Schmidt operator with*

$$\|K\|_{\mathrm{HS}} = \|k\|_{L^2}.$$

Proof. $L^2(X)$ is separable, so there is an orthonormal basis $\{e_n\}$ for $L^2(X)$. Using Parseval's formula and then Fubini's theorem,

$$\begin{aligned}
\sum_n \langle Ke_n, Ke_n \rangle &= \sum_n \int_X |Ke_n(x)|^2 d\mu(x) \\
&= \sum_n \int_X |\langle e_n, \overline{k_x} \rangle|^2 d\mu(x) \\
&= \int_X \left(\sum_n |\langle e_n, \overline{k_x} \rangle|^2 \right) d\mu(x) \\
&= \int_X \langle \overline{k_x}, \overline{k_x} \rangle \, d\mu(x) \\
&= \int_X \left(\int_X |k_x|^2 d\mu(y) \right) d\mu(x) \\
&= \int_{X \times X} |k|^2 d(\mu \otimes \mu) \\
&= \|k\|_{L^2}^2.
\end{aligned}$$

This shows that

$$\|K\|_{\mathrm{HS}} = \left(\sum_n \langle Ke_n, Ke_n \rangle \right)^{1/2} = \|k\|_{L^2}.$$

□

If T is a compact linear operator on $L^2(X)$, then T^*T is a positive compact operator on $L^2(X)$. Then $|T| = \sqrt{T^*T}$ is a positive compact operator.[6] Let s_j be the nonzero eigenvalues of $|T|$ repeated according to geometric multiplicity, with $s_{j+1} \leq s_j$, $j \geq 1$, called the **singular values of** T. By the spectral theorem, there is an orthonormal basis for $\{e_j : j \geq 1\}$ for $L^2(X)$ such that

[5] Donald L. Cohn, *Measure Theory*, second ed., p. 102, Proposition 3.4.5.

[6] See Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 109, Theorem 5.1.3

$|T|e_j = s_j e_j$ for each $j \geq 1$. Then

$$\begin{aligned}
\|T\|_{\mathrm{HS}}^2 &= \sum_{j\geq 1} \langle Te_j, Te_j\rangle \\
&= \sum_{j\geq 1} \langle T^*Te_j, e_j\rangle \\
&= \sum_{j\geq 1} \left\langle |T|^2 e_j, e_j\right\rangle \\
&= \sum_{j\geq 1} \langle |T|e_j, |T|e_j\rangle \\
&= \sum_{j\geq 1} \langle s_j, s_j\rangle \\
&= \sum_{j\geq 1} |s_j|^2.
\end{aligned}$$

Summarizing,

$$\|k\|_{L^2}^2 = \|K\|_{\mathrm{HS}}^2 = \sum_{j\geq 1} |s_j(T)|^2.$$

6 Trace class operators

A compact operator T on $L^2(X)$ is called **trace class** if $\|T\|_{\mathrm{tr}} < \infty$, where

$$\|T\|_{\mathrm{tr}} = \sum_{j\geq 1} s_j(T).$$

For a trace class operator it makes sense to define

$$\mathrm{tr}\,(T) = \sum_n \langle Te_n, e_n\rangle\,,$$

which does not depend on the orthonormal basis $\{e_n\}$ of $L^2(X)$.

Let X be a locally compact Hausdorff space and let $\mathscr{B}$ be the Borel σ-algebra of X. A **Borel measure** on X is a measure on $\mathscr{B}$. We say that a Borel measure μ on X is **locally finite** if for each $x \in X$ there is an open set U_x with $x \in U_x$ and $\mu(U_x) < \infty$. A **Radon measure** on X is a locally finite Borel measure μ on X such that for each $A \in \mathscr{B}$ and for any $\epsilon > 0$ there is an open set U_ϵ with $A \subset U_\epsilon$ and

$$\mu(A) > \mu(U_\epsilon) - \epsilon$$

and for each open set U and for any $\epsilon > 0$ there is a compact set K_ϵ with $K_\epsilon \subset U$ and

$$\mu(U) < \mu(K_\epsilon) + \epsilon.$$

By definition, if μ is a Radon measure then $\mu(U)$ can be approximated by $\mu(K)$ for compact sets K contained in U. We prove that this holds for $\mu(A)$ if $\mu(A) < \infty$.[7]

Lemma 4. *Let X be a locally compact Hausdorff space and let μ be a Radon measure on X. If $A \in \mathscr{B}$ with $\mu(A) < \infty$, there for any $\epsilon > 0$ there is a compact set K_ϵ, $K_\epsilon \subset A$, such that*

$$\mu(A) < \mu(K_\epsilon) + \epsilon.$$

Proof. If L is a compact set, $B \in \mathscr{B}$, and $B \subset L$, let $T = L \setminus B$. For $\delta > 0$ there is an open set W_δ, $T \subset W_\delta$, such that $\mu(W_\delta) < \mu(T) + \delta$. Let $K_\delta = L \setminus W_\delta$, and because X is Hausdorff, L is closed and hence K_δ is closed and therefore compact. Now, as $B \subset L$,

$$L \setminus W_\delta \subset L \setminus T = L \setminus (L \setminus B) = B$$

and

$$\mu(B \setminus K_\delta) = \mu(B \setminus (L \setminus W_\delta)) \leq \mu(W_\delta \setminus (L \setminus B)) = \mu(W_\delta \setminus T) < \delta.$$

We have proved that if L is a compact set and B is a Borel set contained in L, then for any $\delta > 0$ then there is a compact set K_δ with $K_\delta \subset B$ and

$$\mu(B \setminus K_\delta) < \delta.$$

Now let U be an open set with $A \subset U$ and $\mu(U) < \infty$, say $\mu(U) < \mu(A) + 1$. Let L be a compact set with $L \subset U$ and

$$\mu(U) < \mu(L) + \epsilon.$$

$A = (A \cap L) \cup (A \setminus L)$, so

$$\mu(A) = \mu(A \cap L) + \mu(A \setminus L),$$

and

$$\mu(A \setminus L) \leq \mu(U \setminus L) < \epsilon.$$

Let $B = A \cap L$. Because B is a Borel set contained in a compact set L, there is a compact set K contained in B such that

$$\mu(B \setminus K) < \epsilon.$$

As $A = B \cup (A \setminus L)$ and $K \subset B$,

$$\mu(A \setminus K) = \mu((B \setminus K) \cup (A \setminus L)) = \mu(B \setminus K) + \mu(A \setminus L) < 2\epsilon.$$

□

[7] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 291, Lemma B.2.1.

Let X be a locally compact Hausdorff space and let μ be a Radon measure on X. An **admissible kernel** is a function $k \in C(X \times X) \cap \mathscr{L}^2(X \times X)$ for which there is some $g \in C(X) \cap \mathscr{L}^2(X)$ such that $|k(x,y)| \leq g(x)g(y)$ for all $(x,y) \in X \times X$. We call $S : L^2(X) \to L^2(X)$ an **admissible integral operator** if there is an admissible kernel k such that

$$Sg(x) = \int_X k_x(y)g(y)d\mu(y).$$

The following gives conditions under which we can calculate the trace of an integral operator.[8]

Theorem 5. *Let X be a first-countable locally compact Hausdorff space and let μ be a Radon measure on X. Let $k \in C(X \times X) \cap \mathscr{L}^2(X)$ and let*

$$Kg(x) = \int_X k_x(y)g(y)d\mu(y).$$

If there are admissible integral operators S_1 and S_2 such that $K = S_1 S_2$, then K is of trace class and

$$\operatorname{tr}(K) = \int_X k(x,x)d\mu(x).$$

The following is **Mercer's theorem**.[9]

Theorem 6 (Mercer's theorem). *If $k \in C(X \times X) \cap \mathscr{L}^2(X \times X)$ and $K : L^2(X) \to L^2(X)$ is a positive operator, then*

$$\operatorname{tr}(K) = \int_X k(x,x)d\mu(x).$$

[8] Anton Deitmar and Siegfried Echterhoff, *Principles of Harmonic Analysis*, second ed., p. 172, Proposition 9.3.1.

[9] E. Brian Davies, *Linear Operators and their Spectra*, p. 156, Proposition 5.6.9.