The infinite-dimensional torus

Jordan Bell

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1 Locally compact abelian groups

Let $\mathbb{N}$ denote the positive integers.

If G_i, $i \in I$, are compact abelian groups, we define their **direct product** to be the cartesian product

$$\prod_{i \in I} G_i$$

with the coarsest topology such that the projection maps $\pi_i : \prod_{j \in I} G_j \to G_i$ are continuous (namely the product topology), with which the direct product is a compact abelian group. We write

$$G^\omega = \prod_{\mathbb{N}} G.$$

We shall be interested especially in the compact abelian group $\mathbb{T} = S^1$, and we call $\mathbb{T}^\omega$ the **infinite-dimensional torus**.

If Γ_i, $i \in I$, are discrete abelian groups, their **direct sum**, denoted by

$$\bigoplus_{i \in I} \Gamma_i,$$

consists of those elements x of the cartesian product $\prod_{i \in I} \Gamma_i$ such that the set $\{i \in I : \pi_i(x) \neq 0\}$ is finite. Let $p_i : \bigoplus_{j \in I} \Gamma_j \to \Gamma_i$ be the restriction of π_i to $\bigoplus_{j \in I} \Gamma_j$. We give the direct sum the finest topology such that the inclusion maps $q_i : \Gamma_i \to \bigoplus_{j \in I} \Gamma_j$, defined by

$$(p_j \circ q_i)(x) = \begin{cases} x & j = i \\ 0 & j \neq i \end{cases}, \qquad x \in \Gamma_i,$$

are continuous. With this topology, the direct sum is a discrete abelian group. We write

$$\Gamma^\infty = \bigoplus_{\mathbb{N}} \Gamma.$$

We shall be interested especially in the discrete abelian group $\mathbb{Z}$, and in the infinite direct sum $\mathbb{Z}^\infty$. (I don't know how significant an object it is, but I mention that the abelian group $\prod_{\mathbb{N}} \mathbb{Z}$ is called the Baer-Specker group.)

When speaking about 0 or 1 in a locally compact abelian group, it is unambiguous that this symbol denotes the identity element of the group, because there is only one distinguished element in a locally compact abelian group. Often we denote the identity element of a compact abelian group by 1 and the identity element of a discrete abelian group by 0.

If $G_1, \dots, G_n$ are locally compact abelian groups, it is straightforward to check that the cartesian product

$$\prod_{k=1}^{n} G_k$$

with the product topology is a locally compact abelian group. We call this both the direct product and the direct sum and write

$$G_1 \oplus \cdots \oplus G_n = \bigoplus_{k=1}^{n} G_k = \prod_{k=1}^{n} G_k = G_1 \times \cdots \times G_n.$$

2 Dual groups

If G is a locally compact abelian group, denote by $\widehat{G}$ its **dual group**, that is, the set of continuous group homomorphisms $G \to S^1$. For $g \in G$ and $\phi \in \widehat{G}$we write

$$\langle x, \phi \rangle = \phi(x).$$

$\widehat{G}$ has the initial topology induced by $\{\phi \mapsto \langle x, \phi \rangle : x \in G\}$, with which it is a locally compact abelian group. If G is compact then $\widehat{G}$ is discrete, and if G is discrete then $\widehat{G}$ is compact.

Theorem 1. *Suppose that $G_1, \dots, G_n$ are locally compact abelian groups. Then the dual group of $G_1 \oplus \cdots \oplus G_n$ is isomorphic as a topological group to $\widehat{G}_1 \oplus \cdots \oplus \widehat{G}_n$.*

We prove in the following theorem that for discrete abelian groups, the dual group of a direct sum is the direct product of the dual groups.[1] In particular, this shows that the dual group of $\mathbb{Z}^\infty$ is $\mathbb{T}^\omega$. Then by the **Pontryagin duality theorem**[2] we get that the dual group of $\mathbb{T}^\omega$ is $\mathbb{Z}^\infty$.

Theorem 2. *Suppose that Γ_i, $i \in I$, are discrete abelian groups and let*

$$\Gamma = \bigoplus_{i \in I} \Gamma_i, \qquad G = \prod_{i \in I} \widehat{\Gamma}_i.$$

Then $\Phi : G \to \widehat{\Gamma}$, defined by

$$(\Phi g)(\gamma) = \prod_{i \in I} \langle p_i(\gamma), \pi_i(g) \rangle, \qquad g \in G, \gamma \in \Gamma,$$

[1] Karl H. Hofmann and Sidney A. Morris, *The Structure of Compact Groups*, second ed., p. 12, Proposition 1.17. Cf. Walter Rudin, *Fourier Analysis on Groups*, p. 37, §2.2.3.

[2] Walter Rudin, *Fourier Analysis on Groups*, p. 28, Theorem 1.7.2.

is an isomorphism of topological groups. Here, $\pi_i : G \to \widehat{\Gamma}_i$ *and* $p_i : \Gamma \to \Gamma_i$ *are the projection maps.*

Proof. The definition of $(\Phi g)(\gamma)$ makes sense because $\{i \in I : p_i(\gamma) \neq 0\}$ is finite and hence $\{i \in I : \langle p_i(\gamma), \pi_i(g)\rangle \neq 1\}$ is finite. For $g, h \in G$ and $\gamma \in \Gamma$,

$$\begin{aligned} (\Phi(gh))(\gamma) &= \prod_{i\in I} \langle p_i(\gamma), \pi_i(gh)\rangle \\ &= \prod_{i\in I} \langle p_i(\gamma), \pi_i(g)\rangle \langle p_i(\gamma), \pi_i(h)\rangle \\ &= (\Phi g)(\gamma)(\Phi h)(\gamma) \\ &= ((\Phi g)(\Phi h))(\gamma), \end{aligned}$$

showing that $\Phi(gh) = \Phi(g)\Phi(h)$ and hence that Φ is a homomorphism. Suppose that $g \in \ker \Phi$. For each $i \in I$ and each $\gamma \in \Gamma_i$,

$$((\Phi g) \circ q_i)(\gamma) = (\Phi g)(q_i(\gamma)) = 1,$$

where $q_i : \Gamma_i \to$
Gamma is the inclusion map. This is true for all $\gamma \in \Gamma_i$, so $(\Phi g) \circ q_i$ is the identity element of $\widehat{\Gamma}_i$. And this is true for all $i \in I$, so Φg is the identity element of G. Therefore Φ is one-to-one. Suppose that $\alpha \in \widehat{\Gamma}$. Define $g \in G$ as follows: for each $i \in I$, take $\pi_i(g) = \alpha \circ q_i \in \widehat{\Gamma}_i$. Then g satisfies $\Phi g = \alpha$, hence Φ is onto and is therefore a group isomorphism.

A continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism, so to prove that Φ is a homeomorphism it suffices to prove that Φ is continuous. $\widehat{\Gamma}$ has the initial topology induced by $\{\alpha \mapsto \langle \gamma, \alpha\rangle : \gamma \in \Gamma\}$, which are maps $\widehat{\Gamma} \to S^1$, so by the **universal property** of the initial topology, to prove that Φ is continuous it suffices to prove that for each $\gamma \in \Gamma$,

$$g \mapsto \langle \gamma, \Phi g\rangle$$

is continuous $G \to S^1$. For $\gamma \in \Gamma$, let $J_\gamma = \{i \in I : p_i(\gamma) \neq 0\}$, which is a finite set. For each $i \in J_\gamma$, it is straightforward to check that the map $g \mapsto \langle p_i(\gamma), \pi_i(g)\rangle$ is continuous $G \to S^1$. Hence the map

$$g \mapsto (\Phi g)(\gamma) = \prod_{i\in J_\gamma} \langle p_i(\gamma), \pi_i(g)\rangle$$

is continuous $G \to S^1$, being a product of finitely many continuous functions $G \to S^1$, and this completes the proof. □

Let G be a locally compact abelian group. If Γ_0 is a finite subset of $\widehat{G}$ and $a_\gamma \in \mathbb{C}$ for each $\gamma \in \Gamma_0$, we call the function $G \to \mathbb{C}$ defined by

$$x \mapsto \sum_{\gamma\in\Gamma_0} a_\gamma \langle x, \gamma\rangle$$

a **trigonometric polynomial** on G. Suppose that G is a compact abelian group. Its dual group $\widehat{G}$ separates points in G; this is not immediate and is proved using the inversion theorem for the Fourier transform.[3] The set of trigonometric polynomials on G is a self-adjoint algebra that contains the constant functions, so the Stone-Weierstrass theorem then tells us that it is dense in the Banach algebra $C(G)$. Because $\mathbb{C}$ is separable, it follows that if $\widehat{G}$ is countable then $C(G)$ is separable. In particular, any closed subgroup G of $\mathbb{T}^\omega$ is a compact abelian group whose dual group one checks to be countable, so $C(G)$ is separable.

A compact Hausdorff space X is metrizable if and only if the Banach algebra $C(X)$ is separable.[4] We established in the previous paragraph that if G is a compact abelian group with countable dual group then the trigonometric polynomials are dense in the Banach algebra $C(G)$. Therefore, every compact abelian group with countable dual group is metrizable. In particular, $\mathbb{T}^\omega$ and all its closed subgroups are metrizable. In fact, it is proved in Rudin that for a compact abelian group, (i) being metrizable, (ii) having a countable dual group, and (iii) being isomorphic as a topological group to a closed subgroup of $\mathbb{T}^\omega$ are equivalent.[5]

3 $\mathbb{T}^\omega$ and $\mathbb{Z}^\infty$

Let $\pi_n : \mathbb{T}^\omega \to S^1$ and $p_n : \mathbb{Z}^\infty \to \mathbb{Z}$ be the projection maps and let $q_n : \mathbb{Z} \to \mathbb{Z}^\infty$ be the inclusion map.

For $x \in \mathbb{T}^\omega$ and $\gamma \in \mathbb{Z}^\infty$,

$$\langle x, \gamma \rangle = \prod_{n\in\mathbb{N}} \langle \pi_n(x), p_n(\gamma) \rangle = \prod_{n\in\mathbb{N}} \pi_n(x)^{p_n(\gamma)},$$

where for each n, $\pi_n(x) \in S^1$ and $p_n(\gamma) \in \mathbb{Z}$.

Let m be the Haar measure on $\mathbb{T}^\omega$ such that $m(\mathbb{T}^\omega) = 1$. Because the dual group of $\mathbb{T}^\omega$ is $\mathbb{Z}^\infty$, for any $f \in L^1(m)$ the Fourier transform of f is the function $\hat{f} \in C_0(\mathbb{Z}^\infty)$ defined by

$$\hat{f}(\gamma) = \int_{\mathbb{T}^\omega} f(x) \langle -x, \gamma \rangle dm(x) = \int_{\mathbb{T}^\omega} f(x) \prod_{n\in\mathbb{N}} \pi_n(\gamma)^{-p_n(x)} dm(x), \qquad \gamma \in \mathbb{Z}^\infty.$$

4 Kronecker sets

Suppose that G is a locally compact abelian group and that E is a subset of G, which we give the subspace topology. E is called a **Kronecker set** if for every

[3] Walter Rudin, *Fourier Analysis on Groups*, p. 24, §1.5.2.

[4] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 353, Theorem 9.14.

[5] Walter Rudin, *Fourier Analysis on Groups*, p. 38, §2.2.6.

continuous $f : E \to S^1$ and every $\epsilon > 0$, there is some $\gamma \in \widehat{G}$ such that

$$\sup_{x \in E} |f(x) - \langle x, \gamma \rangle| < \epsilon.$$

We first prove the following lemma from Rudin.[6]

Lemma 3. *If $0 < \alpha < \beta < 1$, then the set of polynomials with integer coefficients and 0 constant term is dense in the real Banach algebra $C([\alpha, \beta])$ of continuous functions $[\alpha, \beta] \to \mathbb{R}$.*

Proof. Let R be the closure in $C([\alpha, \beta])$ of the set of polynomials with integer coefficients and 0 constant term. Because $x \in R$, R separates points in $[\alpha, \beta]$ and for every $a \in [\alpha, \beta]$ there is some $f \in R$ such that $f(a) \neq 0$. It is straightforward to check that R is closed under addition and multiplication. If we show that $\mathbb{R} \subset R$, it will follow that R is an algebra over $\mathbb{R}$, and then by the Stone-Weierstrass theorem we will get that R is dense in $C([\alpha, \beta])$, and hence equal to $C([\alpha, \beta])$ as R is closed.

Let $c \in \mathbb{R}$, let p be prime, and define

$$S_p(x) = \frac{1 - x^p - (1 - x)^p}{p}, \qquad x \in [\alpha, \beta].$$

Using that p is prime, by the binomial theorem it follows that S_p is a polynomial with integer coefficients and 0 constant term. Partitioning $\mathbb{R}$ into intervals of length p, c lies in one of these intervals and hence there is some integer q_p such that $\left| c - \frac{q_p}{p} \right| < \frac{1}{p}$. For $x \in [\alpha, \beta]$,

$$\begin{aligned} |q_p S_p(x) - c| &\leq \left| c - \frac{q_p}{p} \right| + \frac{|q_p|}{p} (\beta^p + (1 - \alpha)^p) \\ &< \frac{1}{p} + \left(|c| + \frac{1}{p} \right) (\beta^p + (1 - \alpha)^p). \end{aligned}$$

Hence $\|q_p S_p - c\|_\infty \to 0$ as $p \to \infty$. q_p is an integer so for each p, $q_p S_p$ is a polynomial with integer coefficients and 0 constant term, so this shows that $c \in R$, completing the proof. □

An **arc** in a topological space is a homeomorphic image of a compact subset of $\mathbb{R}$ of nonzero length. The following theorem shows that there is an arc in $\mathbb{T}^\omega$ that is a Kronecker set.[7]

Theorem 4. *$\mathbb{T}^\omega$ contains an arc that is a Kronecker set.*

Proof. Let $0 < \alpha < \beta < 1$, define $x : [\alpha, \beta] \to \mathbb{T}^\omega$ by

$$(\pi_n \circ x)(t) = \exp(2\pi i t^n), \qquad t \in [\alpha, \beta], \quad n \in \mathbb{N},$$

[6] Walter Rudin, *Fourier Analysis on Groups*, p. 104, Lemma 5.2.8.

[7] Walter Rudin, *Fourier Analysis on Groups*, p. 103, Theorem 5.2.7.

and let L be the image of $[\alpha, \beta]$ under x. Assign L the subspace topology inherited from $\mathbb{T}^\omega$, and suppose that $f : L \to S^1$ is continuous. One proves that there is a continuous function $h : [\alpha, \beta] \to \mathbb{R}$ that satisfies

$$(f \circ x)(t) = \exp(2\pi i h(t)), \qquad \alpha \leq t \leq \beta.$$

Let $\epsilon > 0$, and by Lemma 3, let $S_m(x) = \sum_{j=1}^m a_j x^j$ be a polynomial with integer coefficients such that $\|S_m - h\|_\infty < \epsilon$. Define $\gamma \in \mathbb{Z}^\infty$ by $p_j(\gamma) = a_j$ for $1 \leq j \leq m$ and $p_j(\gamma) = 0$ otherwise. For $t \in [\alpha, \beta]$,

$$\begin{aligned}
|f(x(t)) - \langle x(t), \gamma\rangle| &= \left|\exp(2\pi i h(t)) - \prod_{n\in\mathbb{N}} \langle \pi_n(x(t)), p_n(\gamma)\rangle\right| \\
&= \left|\exp(2\pi i h(t)) - \prod_{n=1}^m \langle \pi_n(x(t)), a_n\rangle\right| \\
&= \left|\exp(2\pi i h(t)) - \prod_{n=1}^m \exp(2\pi i a_n t^n)\right| \\
&= \left|\exp(2\pi i h(t)) - \exp\left(\sum_{n=1}^m 2\pi i a_n t^n\right)\right| \\
&\leq \left|2\pi h(t) - \sum_{n=1}^m 2\pi a_n t^n\right| \\
&= 2\pi|h(t) - S_m(t)| \\
&< 2\pi\epsilon,
\end{aligned}$$

using the fact that $|\exp(iA) - \exp(iB)| \leq |A - B|$ for $A, B \in \mathbb{R}$. Hence, for every $\epsilon > 0$ there is some $\gamma \in \mathbb{Z}^\infty$ such that

$$\sup_{y\in L} |f(y) - \langle y, \gamma\rangle| < \epsilon,$$

showing that L is a Kronecker set. □

5 Subgroups

Suppose that G is a locally compact abelian group. For each $x \in G$, let $t_x : G \to G$ be defined by $t_x(y) = x + y$, which is a homeomorphism, and let $\sigma : G \to G$ be defined by $\sigma(x) = -x$, which is also a homeomorphism. If A is an open set in G and B is a subset of G, then

$$A + B = \bigcup_{x\in B} t_x(A),$$

which is open because $t_x(A)$ is open for each $x \in B$. Furthermore, if A and B are both compact sets in G then $A \times B$ is compact in $G \times G$ and $A + B$ is the image of $A \times B$ under the continuous map $(x, y) \mapsto x + y$ hence is compact.

By a **neighborhood** of a point x in a topological space we mean a set such that x lies in the interior of the set, in other words, a set that contains an open neighborhood of the point. The collection of all neighborhoods of a point x is a filter, and a **neighborhood base at** x is a filter base for the neighborhood filter of x. In a locally compact Hausdorff space, every point x has a neighborhood base consisting of compact neighborhoods of x.

Let $A : G \times A \to G$ be $A(x, y) = x + y$, which is continuous. If W is a neighborhood of 0 in G, then $A^{-1}(W)$ is a neighborhood of $(0, 0)$ in $G \times G$. A base for the product topology on $G \times G$ consists of sets of the form $U_1 \times U_2$ where U_1, U_2 are open sets in G, so there are open sets U_1, U_2 in G such that $(0, 0) \in U_1 \times U_2 \subset A^{-1}(W)$. Each of U_1 and U_2 are then open neighborhoods of 0 in G, so $V = U_1 \cap U_2$ is also an open neighborhood of 0 in G, and then $V \times V$ is open in $G \times G$ and

$$(0, 0) \in V \times V \subset U_1 \times U_2 \subset A^{-1}(W).$$

Hence $A(0, 0) \subset A(V \times V) \subset W$, i.e. $0 \in V + V \subset W$, and $V + V$ is open because V is open. Therefore, for every neighborhood W of 0 in a locally compact abelian group, there is some V that is an open neigborhood of 0 and that satisfies $V + V \subset W$.

Suppose that G is a locally compact abelian group. A subset E of G is called **symmetric** if $E = -E$. If N is a compact neighborhood of 0 then N contains an open neighborhood U of 0. The set $U \cap \sigma(U)$ is an open neighborhood of 0 and the set $N \cap \sigma(N)$ is compact (an intersection of compact sets in a Hausdorff space is compact) and contains $U \cap \sigma(U)$, hence $N \cap \sigma(N)$ is a compact symmetric neighborhood of 0 that is contained in N. It follows that in a locally compact abelian group, there is a neighborhood base at 0 consisting of compact symmetric neighborhoods of 0.

Suppose that G is an abelian group and that H is a subgroup of G. We define the **quotient group** G/H be the collection of cosets of H, which is an abelian group where we define

$$(x + H) + (y + H) = (x + y) + H, \qquad x, y \in G.$$

Let $\pi : G \to G/H$ be the projection map, which is a homomorphism with $\ker \pi = H$.

We are now equipped to define quotient groups in the category of locally compact abelian groups. Suppose that G is a locally compact abelian group and that H is a closed subgroup of G. We assign G/H the final topology induced by the projection map π (namely, the quotient topology). For $x + H \in G/H$, there is a compact neighborhood N of x in G; that is, there is a compact set N and an open set U such that $x \in U \subset N$. Because π is continuous, $\pi(N)$ is compact, and because π is open, $\pi(U)$ is open, so $\pi(N)$ is a compact neighborhood of $x + H$ in G/H. Therefore G/H is locally compact. It remains to prove that G/H is Hausdorff and that addition and negation are continuous to prove that G/H is a locally compact abelian group. Suppose that $x + H, y + H$ are distinct elements of G/H, i.e. $x - y \notin H$. The set $y + H = t_y(H)$ is closed because

H is closed, and $x \notin y + H$ so $G \setminus t_y(H)$ is an open neighborhood of x, and hence $W = t_{-x}(G \setminus t_y(H))$ is an open neighborhood of 0 such that $x + W$ is disjoint from $y + H$. Because W is an open neighborhood of 0 there is an open neighborhood V of 0 such that $V + V \subset W$. Furthermore, there is a compact symmetric neighborhood of 0, N, contained in V. If $(x+H+N)\cap(y+H+N) \neq \emptyset$ then there are $h_1, h_2 \in H$ and $n_1, n_2 \in N$ such that $x + h_1 + n_1 = y + h_2 + n_2$, and then $x + (n_1 - n_2) = y + (h_2 - h_1)$. But $-n_2 \in N$ because N is symmetric and so $n_1 - n_2 \in N + N \subset V + V \subset W$, so $x + (n_1 - n_2) \in x + W$, and $h_2 - h_1 \in H$, so $y + (h_2 - h_1) \in y + H$, contradicting that $x + W$ and $y + H$ are disjoint. Therefore $x+H+N$ and $y+H+N$ are disjoint, and their images under π are then disjoint neighborhoods of $x + H$ and $y + H$ in G/H, showing that G/H is Hausdorff. It is straightforward to prove that addition and negation are continuous in G/H, and therefore G/H is a locally compact abelian group.

If H is a closed subgroup of a locally compact abelian group G, the **annihilator of** H, denoted Λ_H, is the set of all $\gamma \in \widehat{G}$ such that

$$\langle x, \gamma \rangle = 1, \qquad x \in H.$$

For each $x \in H$, the map $\gamma \mapsto \langle x, \gamma \rangle$ is continuous $\widehat{G} \to S^1$ so the inverse image of $\{1\}$ under this map is closed. Λ_H is the intersection of all these inverse images hence is closed, and is a closed subgroup because it is apparent that Λ_H is a subgroup of $\widehat{G}$. It can be proved that Λ_H is the dual of the quotient group G/H and that the quotient group $\widehat{G}/\Lambda_H$ is the dual of H.[8]

The following lemma shows that we can extend continuous characters on a closed subgroup to the entire group.[9]

Lemma 5. *Suppose that H is a closed subgroup of a locally compact abelian group G. If $\phi \in \widehat{H}$, then there is some $\gamma \in \widehat{G}$ whose restriction to H is equal to ϕ.*

Proof. $\phi \in \widehat{H} = \widehat{G}/\Lambda_H$, so there is some $\gamma \in \widehat{G}$ such that for all $x \in H$, $\gamma(x) = \phi(x)$. □

Suppose that G is a locally compact abelian group. It can be proved that if E is a compact open set in G and $0 \in E$, then E contains a compact open subgroup of G.[10]

We are now equipped to prove the following theorem.[11]

Theorem 6. *Suppose that G is a compact group. G is connected if and only if $\gamma \in \widehat{G}$ having finite order implies that $\gamma = 0$.*

Proof. Assume that G is not connected. Then there is a clopen subset A that is neither G nor $\emptyset$. Because G is compact, both A and $G \setminus A$ are compact and open, and one of them, call it E, contains 0. Because E is a compact open set

[8] Walter Rudin, *Fourier Analysis on Groups*, p. 35, Theorem 2.1.2.
[9] Walter Rudin, *Fourier Analysis on Groups*, p. 36, Theorem 2.1.4.
[10] Walter Rudin, *Fourier Analysis on Groups*, p. 41, Lemma 2.4.3.
[11] Walter Rudin, *Fourier Analysis on Groups*, p. 47, Theorem 2.5.6.

containing 0, E contains a compact open subgroup H of G, and $H \neq G$ because $E \neq G$. Because H is open, the singleton $\{0+H\}$ in the quotient group G/H is an open set, and therefore G/H is discrete. But G is compact and G/H is the image of G under the projection map, so G/H is compact. Hence G/H is finite. The dual of G/H is Λ_H, which is a subgroup of $\widehat{G}$. Because G/H contains more than one element (as $H \neq G$), Λ_H contains some $\gamma \neq 0$, and γ has finite order because it is contained in the finite subgroup Λ_H.

Assume that $\gamma \in \widehat{G}$ has order finite order and that $\gamma \neq 0$. Every element of $\gamma(G)$ has finite order and $\gamma(G) \neq \{1\}$, so $\gamma(G)$ is not connected. But if G were connected then $\gamma(G)$, a continuous image of G, would be connected, hence G is not connected. □

Lemma 7. *Suppose that G is a locally compact abelian group. If A is an open subgroup of G, then A is closed.*

Proof. A is a subgroup of G, which gives us

$$A = G \setminus \bigcup_{x \in G \setminus A} (x + A).$$

Because each set $x + A$ is open, this shows that A is closed. □

6 Measures

Suppose that $\mathscr{M}$ is a σ-algebra on a set X. If μ is a complex measure on $\mathscr{M}$ we denote by $|\mu|$ its **total variation**, which is a finite positive measure on $\mathscr{M}$.[12] The **total variation norm** of μ is $\|\mu\| = |\mu|(X)$.

Suppose that X is a Hausdorff space with Borel σ-algebra $\mathscr{B}_X$ and that μ is a complex Borel measure on X. We say that μ is **outer regular** if for each $E \in \mathscr{B}_X$,

$$|\mu|(E) = \inf\{|\mu|(V) : E \subset V \text{ and } V \text{ is open}\}$$

inner regular if for each $E \in \mathscr{B}_X$,

$$|\mu|(E) = \sup\{|\mu|(F) : F \subset E \text{ and } F \text{ is closed}\},$$

and **tight** if for each $E \in \mathscr{B}_X$,

$$|\mu|(E) = \sup\{|\mu|(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

(Because we demand that X be Hausdorff, a compact set is closed and hence belongs to the Borel σ-algebra of X; compact sets need not belong to the Borel σ-algebra of a topological space that is not Hausdorff.) We remark that the words "inner regular" often means what we call tight. We say that μ is **regular** if it is both outer regular and tight, and we also remark that calling a measure

[12] Walter Rudin, *Real and Complex Analysis*, third ed., p. 117, Theorem 6.2 and p. 118, Theorem 6.4.

regular often means being outer regular and what we call inner regular. What we call a regular complex Borel measure means precisely what Rudin means by these words in *Fourier Analysis on Groups*, and using Rudin's notation we define

$$M(X) = \{\mu : \mu \text{ is a regular complex Borel measure on } X\}.$$

It is a fact that a complex Borel measure on a metrizable space is outer regular and inner regular,[13] and that a complex Borel measure on a Polish space is regular.[14]

Suppose that X and Y are locally compact Hausdorff spaces and that $\mu \in M(X)$ and $\lambda \in M(Y)$. It is a fact that there is a unique element of $M(X \times Y)$, denoted $\mu \times \lambda$, such that for any $A \in \mathscr{B}_X$ and $B \in \mathscr{B}_Y$,

$$(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B).$$

We call $\mu \times \lambda$ the **product measure** of μ and λ.

Suppose that G is a locally compact abelian group with addition $A : G \times G \to G$. For $\mu, \lambda \in M(G)$, we define the **convolution** of μ and λ to be the pushforward of the product $\mu \times \lambda$ by A,

$$\mu * \lambda = A_*(\mu \times \lambda),$$

and it can be proved that $\mu * \lambda \in M(G)$, that convolution is commutative and associative, and that $\|\mu * \lambda\| \leq \|\mu\|\|\lambda\|$.[15] Then, with convolution as multiplication and using the total variation norm, $M(G)$ is a unital commutative Banach algebra, with unity δ_0.

For $\mu \in M(G)$, the **Fourier transform of** μ is the function $\hat{\mu} : \widehat{G} \to \mathbb{C}$ defined by

$$\hat{\mu}(\gamma) = \int_G \langle -x, \gamma \rangle d\mu(x), \qquad \gamma \in \widehat{G}.$$

One proves that $\hat{\mu}$ is bounded and uniformly continuous, and we define

$$B(\widehat{G}) = \{\hat{\mu} : \mu \in M(G)\}.$$

7 Idempotent measures

If G is a locally compact abelian group and $\mu \in M(G)$, we say that μ is **idempotent** if $\mu * \mu = \mu$, and we denote the set of idempotent elements of $M(G)$ by $J(G)$. Because the Fourier transform of a convolution is the product of the Fourier transforms, for $\mu \in M(G)$ we have $\mu * \mu = \mu$ if and only if $\hat{\mu}^2 = \hat{\mu}$. But

[13] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 436, Theorem 12.5.

[14] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 438, Theorem 12.7.

[15] Walter Rudin, *Fourier Analysis on Groups*, p. 13, Theorem 1.3.2; Karl Stromberg, *A note on the convolution of regular measures*, Math. Scand. **7** (1959), 347–352.

$\hat{\mu}^2 = \hat{\mu}$ is equivalent to $\hat{\mu}$ having range contained in $\{0,1\}$, so for $\mu \in M(G)$, we have that $\mu \in J(G)$ if and only if $\hat{\mu}$ is the characteristic function of some subset of $\widehat{G}$. For $\mu \in J(G)$, we write

$$S(\mu) = \{\gamma \in \widehat{G} : \hat{\mu}(\gamma) = 1\}.$$

Suppose that Λ is an open subgroup of $\widehat{G}$. Then Λ is closed, and the fact that Λ is open implies that the singleton containing the identity in $\widehat{G}/\Lambda$ is open and hence that $\widehat{G}/\Lambda$ is a discrete abelian group. Denoting the annihilator of Λ by H, which is a closed subgroup of G, the quotient group $\widehat{G}/\Lambda$ is the dual group of H and hence H is compact. Let m_H be the Haar measure on H such that $m_H(H) = 1$. Taking $m_H(E) = m_H(E \cap H)$, $m_H \in M(G)$. If $\gamma \in \Lambda$ then

$$\hat{m}_H(\gamma) = \int_G \langle -x, \gamma \rangle dm_H(x) = \int_H \langle -x, \gamma \rangle dm_H(x) = \int_H dm_H(x) = m_H(H) = 1.$$

If $\gamma \in \widehat{G} \setminus \Lambda$ then there is some $x_0 \in H$ such that $\langle x_0, \gamma \rangle \neq 1$, and then

$$\int_H \langle -x, \gamma \rangle dm_H(x) = \langle x_0, \gamma \rangle \int_H \langle -x_0 - x, \gamma \rangle dm_H(x) = \langle x_0, \gamma \rangle \int_H \langle -x, \gamma \rangle dm_H(x),$$

showing that $\hat{m}_H(\gamma) = \langle x_0, \gamma \rangle \hat{m}_H(\gamma)$, and because $\langle x_0, \gamma \rangle \neq 1$ this implies that $\hat{m}_H(\gamma) = 0$. Therefore, $\Lambda = S(m_H)$.

If $E = \gamma_0 + \Lambda$, then with

$$d\mu(x) = \langle x, \gamma_0 \rangle dm_(H)$$

we have $\mu \in J(G)$ and $E = S(\mu)$.

8 Sidon sets

Let G be a compact abelian group and let $E \subset \widehat{G}$. A function $f \in L^1(G)$ is called an **E-function** if $\gamma \in \widehat{G} \setminus E$ implies that $\hat{f}(\gamma) = 0$. An **E-polynomial** is a trigonometric polynomial f on G that is an E-function.

We call a subset E of $\widehat{G}$ a **Sidon set** if there is some $B_E \geq 0$ such that for every E-polynomial f on G,

$$\sum_{\gamma \in E} |\hat{f}(\gamma)| \leq B_E \|f\|_\infty.$$

We shall use the following lemma later.[16]

Lemma 8. *Suppose that Γ is a discrete abelian group that is the dual group of a compact abelian group G. If $E \subset \Gamma$ is a Sidon set with constant B_E, then every bounded E-function f on G satisfies*

$$\sum_{\gamma \in E} |\hat{f}(\gamma)| \leq B_E \|f\|_\infty.$$

[16] Walter Rudin, *Fourier Analysis on Groups*, p. 121, Theorem 5.7.3.

9 Dirichlet series

Define $\sigma : \mathbb{Z}^\infty \to \mathbb{Z}$ by $\sigma(\gamma) = \sum_{n\in\mathbb{N}} p_n(\gamma)$, i.e. the sum of the entries of γ, which makes sense because any element of $\mathbb{Z}^\infty$ has only finitely many nonzero entries.

Let Y be those $\gamma \in \mathbb{Z}^\infty$ such that $p_n(\gamma) \geq 0$ for all $n \in \mathbb{N}$, and let $E = Y \cap \sigma^{-1}(1)$. In other words, the elements of E are those $\gamma \in \mathbb{Z}^\infty$ one coordinate of which is 1 and all other coordinates of which are 0. The proof of the following theorem is from Rudin.[17]

Theorem 9. *If $f \in L^\infty(\mathbb{T}^\omega)$ and $\hat{f}(\gamma) = 0$ for all $\gamma \in X \setminus Y$, then*

$$\sum_{\gamma\in E} |\hat{f}(\gamma)| \leq \|f\|_\infty.$$

Proof. $\sigma : \mathbb{Z}^\infty \to \mathbb{Z}$ is a continuous group homomorphism, and $\ker\sigma$ is an open subgroup of $\mathbb{Z}^\infty$, because $\mathbb{Z}^\infty$ is discrete. Because $\sigma^{-1}(1)$ is a coset of this open subgroup, there is some $\mu \in J(\mathbb{T}^\omega)$ such that $\hat{\mu}$ is the characteristic function of $\sigma^{-1}(1)$, and this μ satisfies $\|\mu\| = 1$. Define $g : \mathbb{T}^\omega \to \mathbb{C}$ by

$$g(x) = (f * \mu)(x) = \int_{\mathbb{T}^\omega} f(x-y)d\mu(y), \qquad x \in \mathbb{T}^\omega,$$

whose Fourier transform is $\hat{g}(\gamma) = \hat{f}(\gamma)\hat{\mu}(\gamma)$. If $\gamma \notin E$ then $\gamma \notin Y$ or $\gamma \notin \sigma^{-1}(1)$. In the first case $\hat{f}(\gamma) = 0$ and in the second case $\hat{\mu}(\gamma) = 0$, and hence $\gamma \notin E$ implies that $\hat{g}(\gamma) = 0$, namely, g is an E-function. Also, it is apparent from the definition of g that $\|g\|_\infty \leq \|f\|_\infty$.

Suppose that P is an E-polynomial. Hence there is a finite subset E_0 of E such that $\gamma \notin E_0$ implies that $\hat{P}(\gamma) = 0$, and thus there are $c_\gamma \in \mathbb{C}$, $\gamma \in E_0$, such that

$$P(x) = \sum_{\gamma\in E_0} c_\gamma \langle x, \gamma\rangle = \sum_{\gamma\in E_0} c_\gamma \prod_{n\in\mathbb{N}} \langle \pi_n(x), p_n(\gamma)\rangle, \qquad x \in \mathbb{T}^\omega.$$

$E_0 \subset E$, so any element of E_0 has one entry 1, say $p_{n_\gamma}(\gamma) = 1$, and all other entries 0, so

$$P(x) = \sum_{\gamma\in E_0} c_\gamma \pi_{n_\gamma}(x).$$

Define $x \in \mathbb{T}^\omega$ by taking $c_\gamma \cdot \pi_{n_\gamma}(x) = |c_\gamma|$ for each $\gamma \in E_0$, and all other entries of x to be $1 \in S^1$; this makes sense because if $\gamma_1, \gamma_2 \in E_0$ and $n_{\gamma_1} = n_{\gamma_2}$ then $\gamma_1 = \gamma_2$. For this x, $P(x) = \sum_{\gamma\in E_0} |c_\gamma|$. But it is apparent that $\|P\|_\infty \leq \sum_{\gamma\in E_0} |c_\gamma|$, so

$$\|P\|_\infty = \sum_{\gamma\in E_0} |c_\gamma|.$$

[17]Walter Rudin, *Fourier Analysis on Groups*, p. 224, Theorem 8.7.9.

This shows that E is a Sidon set with $B_E = 1$. Therefore by Lemma 8, because g is a bounded E-function on $\mathbb{T}^\omega$ we get $\sum_{\gamma \in E} |\hat{g}(\gamma)| \leq \|g\|_\infty$. But $\hat{\mu}$ is the characteristic function of $\sigma^{-1}(1)$ and $E = Y \cap \sigma^{-1}(1)$, so

$$\sum_{\gamma \in E} \hat{f}(\gamma) = \sum_{\gamma \in E} \hat{f}(\gamma)\hat{\mu}(\gamma) = \sum_{\gamma \in E} \hat{g}(\gamma) \leq \|g\|_\infty \leq \|f\|_\infty,$$

proving the claim. □

Following Rudin, we use the above theorem to prove a theorem about Dirichlet series due to Bohr.[18]

Theorem 10 (Bohr)**.** *If*

$$\phi(s) = \sum_{k=1}^\infty \frac{c_k}{k^s}$$

and $|\phi(s)| \leq 1$ *for all* s *such that* $\operatorname{Re} s > 0$*, then*

$$\sum_p |c_p| \leq 1.$$

Proof. For $k \in \mathbb{N}$, let $\gamma(k) \in Y$ such that $k = \prod_{n=1}^\infty p_n^{h_n(\gamma(k))}$, where p_n are the primes and where $h_n : \mathbb{Z}^\infty \to \mathbb{Z}$ are the projection maps; so far we have denoted these projection maps by p_n, rather than using h_n, but the symbol p_n has such a strong association with the primes that we change notation here. The map $k \mapsto \gamma(k)$ is a bijection $\mathbb{N} \to Y$, and we write $c_\gamma = c_k$. We shall use the fact that the image of the primes under this bijection is E.

Let s be a complex number in the half-plane of convergence of ϕ and write $z_n(s) = p_n^{-s} = \exp(-s \log p_n)$. Then,

$$\begin{aligned}
\phi(s) &= \sum_{k=1}^\infty c_k k^{-s} \\
&= \sum_{\gamma \in Y} c_\gamma \left(\prod_{n=1}^\infty p_n^{h_n(\gamma)} \right)^{-s} \\
&= \sum_{\gamma \in Y} c_\gamma \prod_{n=1}^\infty p_n^{-s h_n(\gamma)} \\
&= \sum_{\gamma \in Y} c_\gamma \prod_{n=1}^\infty z_n(s)^{h_n(\gamma)}
\end{aligned}$$

Defining $T : \mathbb{R} \to \mathbb{T}^\omega$ by

$$(\pi_n \circ T)(\sigma) = \exp(-i\sigma \log p_n), \qquad n \in \mathbb{N}, \sigma \in \mathbb{R},$$

[18] Walter Rudin, *Fourier Analysis on Groups*, pp. 224–225. See also Maxime Bailleul and Pascal Lefèvre, *Some Banach spaces of Dirichlet series*, `arxiv.org/abs/1311.3845`

we have, as $z_n(i\sigma) = \exp(-i\sigma \log p_n)$,

$$\phi(i\sigma) = \sum_{\gamma \in Y} c_\gamma \prod_{n=1}^{\infty} \langle \pi_n(T(\sigma)), h_n(\gamma) \rangle = \sum_{\gamma \in Y} c_\gamma \langle T(\sigma), \gamma \rangle.$$

One checks that the function $f : \mathbb{T}^\omega \to \mathbb{C}$ defined by $f(x) = \sum_{\gamma \in Y} c_\gamma \langle x, \gamma \rangle$ satisfies the conditions of Theorem 9, and thus gets

$$\sum_p |c_p| = \sum_{\gamma \in E} |c_\gamma| = \sum_{\gamma \in E} |\hat{f}(\gamma)| \leq \|f\|_\infty$$

I do not see why $\|f\|_\infty \leq 1$. However, granted this, the claim follows. □

10 Descriptive set theory

If (X, d) is a compact metric space, $C(X, X)$ is a Polish space with the **uniform metric** $(f, g) \mapsto \sup_{x \in X} d(f(x), g(x))$. We denote by $H(X)$ the group of homeomorphisms of X, which one proves is a G_δ set in $C(X, X)$. Because $H(X)$ is a G_δ set in a Polish space, it is a Polish space with the subspace topology. A homeomorphism h of X is said to be **minimal** if there is no proper closed subset of X that is invariant under h, and is called **distal** if $x \neq y$ implies that there is some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, $d(h^n(x), h^n(y)) > \epsilon$. It has been proved (Beleznay-Foreman) that the collection of minimal distal homeomorphisms of $\mathbb{T}^\omega$ is a Borel $\mathbf{\Sigma}_1^1$-complete set in $H(\mathbb{T}^\omega)$.[19]

11 Further reading

Albeverio, Sergio, Daletskii, Alexei and Kondratiev, Yuri, *Stochastic analysis on (infinite-dimensional) product manifolds*, Stochastic dynamics (Bremen, 1997), 339–369, Springer, New York, 1999.

Aoki, Nobuo and Totoki, Haruo, *Ergodic automorphisms of T^∞ are Bernoulli automorphisms*, Publ. RIMS, Kyoto Univ. **10** (1975), 535–544.

Balasubramanian, R., Calado, B. and Queffélec, H., *The Bohr inequality for ordinary Dirichlet series*, Studia Math. **175** (2006), no. 3, 285–304.

Bendikov, A. and Saloff-Coste, L., *On the sample paths of diagonal Brownian motions on the infinite dimensional torus*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), no. 2, 227–254.

Bendikov, A., *Lévy measures for symmetric stable semigroups on torus $\mathbb{T}^\infty$*. Functional analysis, 173–181, Narosa, New Delhi, 1998.

Bendikov, A. and Saloff-Coste, L., *Spaces of smooth functions and distributions on infinite-dimensional compact groups*, J. Funct. Anal. **218** (2005), no. 1, 168–218.

Bendikov, A. D., *Symmetric stable semigroups on the infinite-dimensional torus*, Exposition. Math. **13** (1995), no. 1, 39–79.

[19] Alexander S. Kechris, *Classical Descriptive Set Theory*, p 262, Theorem 33.22.

Berg, Christian, *On Brownian and Poissonian convolution semigroups on the infinite dimensional torus*, Invent. Math. **38** (1976/77), no. 3, 227–235.

Berg, Christian, *Potential theory on the infinite dimensional torus*, Invent. Math. **32** (1976), no. 1, 49–100.

Biroli, Marco and Maheux, Patrick, *Logarithmic Sobolev inequalities and Nash-type inequalities for sub-markovian symmetric semigroups*, `https://hal.inria.fr/hal-00465177v1/document`

Fournier, John, *Extensions of a Fourier multiplier theorem of Paley*, Pacific J. Math. **30** (1969), 415–431.

Haezendonck, J., *Les groupes commutatifs de Lebesgue-Rohlin*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **58** (1972), 344–353.

Hastings, H. M., *On expansive homeomorphisms of the infinite torus*, The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ.)

Hedenmalm, Håkan and Saksman, Eero, *Carleson's convergence theorem for Dirichlet series*, Pacific Journal of Mathematics **208** (2003), no. 1, 85–109.

Helson, Henry, *Hankel forms and sums of random variables*, Studia Math. **176** (2006), no. 1, 85–92.

Holley, R. and Stroock, D., *Diffusions on an infinite-dimensional torus*, J. Funct. Anal. **42** (1981), no. 1, 29–63.

Jessen, B., *The theory of integration in a space of an infinite number of dimensions*, Acta Math. **63** (1934), no. 1, 249–323.

Kholshchevnikova, N. N., *Uniqueness for trigonometric series with respect to an increasing number of variables*, Proc. Steklov Inst. Math. 2005, Function Theory, suppl. 2, S160–S166.

Lind, D. A., *Ergodic automorphisms of the infinite torus are Bernoulli*, Israel J. Math. **17** (1974), 162–168.

Olsen, Jan-Fredrik, *Local properties of Hilbert spaces of Dirichlet series*, J. Funct. Anal. **261** (2011), no. 9, 2669–2696.

Saksman, Eero and Seip, Kristian, *Integral means and boundary limits of Dirichlet series*, Bull. Lond. Math. Soc. **41** (2009), no. 3, 411–422.

Saloff-Coste, Laurent, *Probability on groups: random walks and invariant diffusions*, Notices Amer. Math. Soc. **48** (2001), no. 9, 968–977.

Skorikov, A. V., *Bessel potentials in spaces with a mixed norm on the group* $\mathbb{T}^\infty$, Moscow Univ. Math. Bull. **48** (1993), no. 6, 1–4

Stackelberg, Olaf P., *Metric theorems related to the Kronecker-Weyl theorem mod m*, Monatsh. Math. **82** (1976), no. 1, 57–69.

Sugita, Hiroshi and Takanobu, Satoshi, *A limit theorem for Weyl transformation in infinite-dimensional torus and central limit theorem for correlated multiple Wiener integrals*, J. Math. Sci. Univ. Tokyo **7** (2000), no. 1, 99–146.

Tanaka, Jun-ichi, *Dirichlet series induced by the Riemann zeta-function*, Studia Math. **187** (2008), no. 2, 157–184.

Taylor, Thomas J. S., *Applications of harmonic analysis on the infinite-dimensional torus to the theory of the Feynman integral*, Functional integration with emphasis on the Feynman integral (Sherbrooke, PQ, 1986), Rend. Circ. Mat. Palermo (2) Suppl. No. 17 (1987), 349–362 (1988).