The Fréchet space of holomorphic functions on the unit disc

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1 Introduction

The goal of this note is to develop all the machinery necessary to understand what it means to say that the set $H(D)$ of holomorphic functions on the unit disc is a separable and reflexive Fréchet space that has the Heine-Borel property and is not normable.

2 Topological vector spaces

If X is a topological space and $p \in X$, a *local basis at* p is a set $\mathscr{B}$ of open neighborhoods of p such that if U is an open neighborhood of p then there is some $U_0 \in \mathscr{B}$ that is contained in U. We emphasize that to say that a topological vector space (X, τ) is normable is to say not just that there is a norm on the vector space X, but moreover that the topology τ is induced by the norm.

A *topological vector space* over $\mathbb{C}$ is a vector space X over $\mathbb{C}$ that is a topological space such that singletons are closed sets and such that vector addition $X \times X \to X$ and scalar multiplication $\mathbb{C} \times X \to X$ are continuous. It is not true that a topological space in which singletons are closed need be Hausdorff, but one can prove that every topological vector space is a Hausdorff space.[1] For any $a \in X$, we check that the map $x \mapsto a + x$ is a homeomorphism. Therefore, a subset U of X is open if and only if $a + U$ is open for all $a \in X$. It follows that if X is a vector space and $\mathscr{B}$ is a set of subsets of X each of which contains 0, then there is at most one topology for X such that X is a topological vector space for which $\mathscr{B}$ is a local basis at 0. In other words, the topology of a topological vector space is determined by specifying a local basis at 0. A topological vector space X is said to be *locally convex* if there is a local basis at 0 whose elements are convex sets.

If X is a vector space and $\mathscr{F}$ is a set of seminorms on X, we say that $\mathscr{F}$ is a *separating family* if $x \neq 0$ implies that there is some $m \in \mathscr{F}$ with $m(x) \neq 0$. (Thus, if m is a seminorm on X, the singleton $\{m\}$ is a separating family if and only if m is a norm.) The following theorem presents a local basis at 0

[1] Walter Rudin, *Functional Analysis*, second ed., p. 11, Theorem 1.12.

for a topology and shows that there is a topology for which the vector space is a locally convex space and for which this is a local basis at 0.[2] We call this topology the *seminorm topology induced by* $\mathscr{F}$.

Theorem 1 (Seminorm topology)**.** *If X is a vector space and $\mathscr{F}$ is a separating family of seminorms on X, then there is a topology τ on X such that (X,τ) is a locally convex space and the collection $\mathscr{B}$ of finite intersections of sets of the form*

$$B_{m,\epsilon} = \{x \in X : m(x) < \epsilon\}, \qquad m \in \mathscr{F}, \epsilon > 0$$

is a local basis at 0*.*

Proof. We define τ to be those subsets U of X such that for all $x \in U$ there is some $N \in \mathscr{B}$ satisfying $x + N \subseteq U$. If $\mathscr{U}$ is a subset of τ and $x \in \bigcup_{U \in \mathscr{U}} U$, then there is some $U_0 \in \mathscr{U}$ with $x \in U_0$, and there is some $N_0 \in \mathscr{B}$ satisfying $x + N_0 \subseteq U_0$. We have

$$x + N_0 \subseteq U_0 \subseteq \bigcup_{U \in \mathscr{U}} U,$$

which tells us that $\bigcup_{U \in \mathscr{U}} U \in \tau$. If $U_1, \ldots, U_n \in \tau$ and $x \in \bigcap_{k=1}^n U_k$, then there are $N_1, \ldots, N_n \in \mathscr{B}$ satisfying $x + N_k \in U_k$ for $1 \leq k \leq n$. But the intersection of finitely many elements of $\mathscr{B}$ is itself an element of $\mathscr{B}$, so $N = \bigcap_{k=1}^n N_k \in \mathscr{B}$, and

$$x + N \subseteq \bigcap_{k=1}^n U_k,$$

showing that $\bigcap_{k=1}^n U_k \in \tau$. Therefore, τ is a topology.

Suppose that $x \in X$. For $y \neq x$, let $m_y \in \mathscr{F}$ with $\epsilon_y = m_y(x - y) \neq 0$; there is such a seminorm because $\mathscr{F}$ is a separating family. Then $U_y = y + B_{m_y,\epsilon_y}$ is an open set that contains y and does not contain x. Therefore $X \setminus U_y$ is a closed set that contains x and does not contain y, and

$$\bigcap_{y \neq x} X \setminus U_y = \{x\}$$

is a closed set, showing that singletons are closed.

Let $x, y \in X$ and $N \in \mathscr{B}$. There are $m_k \in \mathscr{F}$ and $\epsilon_k > 0$, $1 \leq k \leq n$, such that $N = \bigcap_{k=1}^n B_{m_k,\epsilon_k}$. Let $U = \bigcap_{k=1}^n B_{m_k,\epsilon_k/2}$. If $v \in (x + U) + (y + U)$ and $1 \leq k \leq n$, then there are $x_k \in B_{m_k,\epsilon_k/2}$ and $y_k \in B_{m_k,\epsilon_k/2}$ such that $v = x + x_k + y + y_k$, and

$$m_k(v - (x + y)) = m_k(x_k + y_k) \leq m_k(x_k) + m_k(y_k) < \frac{\epsilon_k}{2} + \frac{\epsilon_k}{2} = \epsilon_k,$$

so $v \in x + y + B_{m_k,\epsilon_k}$. This is true for each k, $1 \leq k \leq n$, so $v \in x + y + N$. Hence

$$(x + U) + (y + U) \subseteq x + y + N,$$

[2] Paul Garrett, *Seminorms and locally convex spaces*, `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/07b_seminorms.pdf`

showing that vector addition is continuous at $(x, y) \in X \times X$: for every basic open neighborhood $x+y+N$ of the image $x+y$, there is an open neighborhood $(x+U) \times (y+U)$ of (x, y) whose image under vector addition is contained in $x+y+N$.

Let $\alpha \in \mathbb{C}$, $x \in X$, and $N \in \mathscr{B}$, say $N = \bigcap_{k=1}^{n} B_{m_k,\epsilon_k}$. Let $\epsilon = \min\{\epsilon_k : 1 \leq k \leq n\}$, let $\delta > 0$ be small enough so that $\delta(\delta + |\alpha| + m_k(x)) < \epsilon$ for each $1 \leq k \leq n$, let $\Delta = \{\beta \in \mathbb{C} : |\beta - \alpha| < \delta\}$, and let $U = \bigcap_{k=1}^{n} B_{m_k,\delta}$. If $(\beta, v) \in \Delta \times (x+U)$ and $1 \leq k \leq n$, then

$$\begin{aligned} m_k(\beta v - \alpha x) &= m_k(\beta v - \beta x + \beta x - \alpha x) \\ &\leq m_k(\beta(v-x)) + m_k((\beta - \alpha)x) \\ &= |\beta| m_k(v-x) + |\beta - \alpha| m_k(x) \\ &< (\delta + |\alpha|)\delta + \delta m_k(x) \\ &= \delta(\delta + |\alpha| + m_k(x)) \\ &< \epsilon \\ &\leq \epsilon_k, \end{aligned}$$

showing that $\beta v \in \alpha x + B_{m_k,\epsilon_k}$. This is true for each k, so $\beta v \in N$, which shows that scalar multiplication is continuous at (α, x): for every basic open neighborhood $\alpha x + N$ of the image αx, there is an open neighborhood $\Delta \times (x+U)$ of (α, x) whose image under scalar multiplication is contained in $\alpha x + N$.

We have shown that X with the topology τ is a topological vector space. To show that X is a locally convex space it suffices to prove that each element of the local basis $\mathscr{B}$ is convex. An intersection of convex sets is a convex set, so to prove that each element of $\mathscr{B}$ is convex it suffices to prove that each $B_{m,\epsilon}$ is convex, $m \in \mathscr{F}$ and $\epsilon > 0$. If $0 \leq t \leq 1$ and $x, y \in B_{m,\epsilon}$, then

$$m(tx + (1-t)y) \leq m(tx) + m((1-t)y) = tm(x) + (1-t)m(y) < t\epsilon + (1-t)\epsilon = \epsilon,$$

showing that $tx + (1-t)y \in B_{m,\epsilon}$ and thus that $B_{m,\epsilon}$ is a convex set. Therefore, (X, τ) is a locally convex space. □

In the other direction, we will now explain how the topology of a locally convex space is induced by a separating family of seminorms. We say that a subset S of a vector space X is *absorbing* if $x \in X$ implies that there is some $t > 0$ such that $x \in tS$. The *Minkowski functional* $\mu_S : X \to [0, \infty)$ of an absorbing set S is defined by

$$\mu_S(x) = \inf\{t \geq 0 : x \in tS\}, \qquad x \in X.$$

If U is an open set containing 0 and $x \in X$, then $0 \cdot x = 0 \in U$, and because scalar multiplication is continuous there is some $t > 0$ such that $tx \in U$. Thus an open set containing 0 is absorbing. We say that a subset S of a vector space X is *balanced* if $|\alpha| \leq 1$ implies that $\alpha S \subseteq S$. One proves that in a topological vector space, every convex open neighborhood of 0 contains a balanced convex open neighhborhood of 0.[3] It follows that a locally convex space has a local basis at 0 whose elements are balanced convex open sets. The following lemma shows that the Minkowski functional of each member of this local basis is a seminorm.

Lemma 2. *If X is a topological vector space and U is a balanced convex open neighborhood of 0, then the Minkowski functional of U is a seminorm on X.*

Proof. Let $\alpha \in \mathbb{C}$ and $x \in X$. If $\alpha = 0$, then

$$\mu_U(\alpha x) = \mu_U(0) = 0 = |\alpha|\mu_U(x).$$

Otherwise, write $\alpha = ru$ with $r > 0$ and $|u| = 1$. Because U is balanced and $|u^{-1}| = 1$, we have

$$\begin{aligned}
\mu_U(\alpha x) &= \inf\{t \geq 0 : \alpha x \in tU\} \\
&= \inf\{t \geq 0 : rux \in tU\} \\
&= \inf\{t \geq 0 : x \in r^{-1}tu^{-1}U\} \\
&= \inf\{t \geq 0 : x \in r^{-1}tU\} \\
&= \inf\{rs \geq 0 : x \in sU\} \\
&= r \inf\{s \geq 0 : x \in sU\} \\
&= r\mu_U(x).
\end{aligned}$$

Therefore, if $\alpha \in \mathbb{C}$ and $x \in X$, then $\mu_U(\alpha x) = |\alpha|\mu_U(x)$.

Let $x, y \in X$. U is absorbing, so let $s = \mu_U(x)$ and $t = \mu_U(y)$. If $\epsilon > 0$ then $x \in (s+\epsilon)U$ and $y \in (t+\epsilon)U$. We have

$$x + y \in (s+\epsilon)U + (t+\epsilon)U = \{(s+\epsilon)u + (t+\epsilon)v : u, v \in U\},$$

and for $u, v \in U$, because U is convex we have

$$s'u + t'v = (s'+t')\left(\frac{s'}{s'+t'}u + \frac{t'}{s'+t'}v\right) \in (s'+t')U,$$

where $s' = s + \epsilon$ and $t' = t + \epsilon$, so

$$x + y \in (s + t + 2\epsilon)U.$$

This is true for every $\epsilon > 0$, which means that $\mu_U(x+y) \leq s + t$. Therefore

$$\mu_U(x+y) \leq s + t = \mu_U(x) + \mu_U(y),$$

showing that μ_U satisfies the triangle inequality and hence that μ_U is a seminorm on X. □

[3] Walter Rudin, *Functional Analysis*, second ed., p. 12, Theorem 1.14.

We proved above that the Minkowski functional of a balanced convex open neighborhood of 0 is a seminorm. The following lemma shows that the collection of Minkowski functionals corresponding to a balanced convex local basis at 0 are a separating family.[4]

Lemma 3. *If X is a topological vector space and U is a balanced convex open neighborhood of* 0*, then*

$$U = \{x \in X : \mu_U(x) < 1\}.$$

If $\mathscr{B}$ is a local basis at 0 *whose elements are balanced and convex, then*

$$\{\mu_U : U \in \mathscr{B}\}$$

is a separating family of seminorms on X.

Proof. Let $U \in \mathscr{B}$. If $x \in U$, then because $1 \cdot x \in U$ and scalar multiplication is continuous, there is some $\delta > 0$ and some open neighborhood N of x such that the image of $[1-\delta, 1+\delta] \times N$ under scalar multiplication is contained in U. In particular, if $(1+\delta)x \in U$ and so $x \in \frac{1}{1+\delta}U$. Thus we have

$$\mu_U(x) = \inf\{t \geq 0 : x \in tU\} \leq \frac{1}{1+\delta} < 1.$$

Therefore, if $x \in U$ then $\mu_U(x) < 1$. On the other hand, if $x \in X$ and $\mu_U(x) < 1$, then there is some $t < 1$ such that $x \in tU$. As U is balanced, we have $x \in U$. Therefore, if $\mu_U(x) < 1$ then $x \in U$. This establishes that if $U \in \mathscr{B}$ then

$$U = \{x \in X : \mu_U(x) < 1\}.$$

If $x \neq 0$, then because singletons are closed, the set $X \setminus \{x\}$ is open and contains 0, and thus there is some $U \in \mathscr{B}$ with $U \subseteq X \setminus \{x\}$. Hence $x \notin U$, which implies by the first claim that $\mu_U(x) \geq 1$. In particular, $\mu_U(x) \neq 0$, proving the second claim. □

If X is a locally convex space then there is a local basis at 0, call it $\mathscr{B}$, whose elements are balanced and convex, and we have established that $\mathscr{F} = \{\mu_U : U \in \mathscr{B}\}$ is a separating family of seminorms on X. Therefore by Theorem 1, X with the seminorm topology induced by $\mathscr{F}$ is a locally convex space. The following theorem states that the seminorm topology is equal to the original topology of the space.[5]

Theorem 4. *If (X, τ) is a locally convex space, then there is a separating family of seminorms on X such that τ is equal to the seminorm topology.*

[4] Walter Rudin, *Functional Analysis*, second ed., p. 27, Theorem 1.36.

[5] Paul Garrett, *Seminorms and locally convex spaces*, `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/07b_seminorms.pdf`

Proof. Let $\mathscr{B}$ be a local basis at 0 whose elements are balanced and convex and let $\mathscr{F} = \{\mu_U : U \in \mathscr{B}\}$. If $U \in \mathscr{B}$, then $U = \{x \in X : \mu_U(x) < 1\}$, which is an open neighborhood of 0 in the seminorm topology induced by $\mathscr{F}$, and this implies that the seminorm topology is at least as fine as τ.

If $U \in \mathscr{B}$ and $\epsilon > 0$, then

$$\{x \in X : \mu_U(x) < \epsilon\} = \left\{x \in X : \mu_U\left(\frac{x}{\epsilon}\right) < 1\right\} = \{\epsilon x \in X : \mu_U(x) < 1\} = \epsilon U.$$

$\epsilon U \in \tau$ and $0 \in \epsilon U$, and it follows that τ is at least as fine as the seminorm topology. Therefore τ is equal to the seminorm topology induced by $\mathscr{F}$. ☐

We have shown that if X is a vector space and $\mathscr{F}$ is a separating family of seminorms on X, then X with the seminorm topology induced by $\mathscr{F}$ is a locally convex space. Furthermore, we have shown that if X is a locally convex space then there is a separating family $\mathscr{F}$ of seminorms on X such that the topology of X is equal to the seminorm topology induced by $\mathscr{F}$. In other words, the topology of any locally convex space is the seminorm topology induced by some separating family of seminorms on the space.

A subset E of a topological vector space X is said to be *bounded* if for every open neighborhood N of 0 there is some $s > 0$ such that $t > s$ implies that $E \subseteq tN$.

Lemma 5. *If X is a locally convex space with the seminorm topology induced by a separating family $\mathscr{F}$ of seminorms on X, then a subset E of X is bounded if and only if each $m \in \mathscr{F}$ is a bounded function on E.*

Proof. Suppose that E is bounded and $m \in \mathscr{F}$. The set $U = \{x \in X : m(x) < 1\}$ is an open neighborhood of 0, so there is some $t > 0$ such that $E \subseteq tU$. Hence if $x \in E$ then $m(x) < t$, so m is a bounded function on E.

Suppose that for each $m \in \mathscr{F}$ there is some M_m such that $x \in E$ implies that $m(x) \leq M_m$. If U is an open neighborhood of 0, then there are $m_1, \ldots, m_n \in \mathscr{F}$ and $\epsilon_1, \ldots, \epsilon_n > 0$ such that

$$\bigcap_{k=1}^{n} \{x \in X : m_k(x) < \epsilon_k\} \subseteq U.$$

Let $M = \max\left\{\frac{M_{m_k}}{\epsilon_k} : 1 \leq k \leq n\right\}$. For $t > M$,

$$\bigcap_{k=1}^{n} \{tx \in X : m_k(x) < \epsilon_k\} \subseteq tU,$$

i.e.,

$$\bigcap_{k=1}^{n} \{x \in X : m_k(x) < \epsilon_k t\} \subseteq tU.$$

But if $x \in E$ and $1 \leq k \leq n$ then

$$m_k(x) \leq M_{m_k} \leq \epsilon_k M < \epsilon_k t,$$

hence x is in the above intersection and thus is in tU. Therefore $E \subseteq tU$, showing that E is bounded. □

We now prove that if the topology of a locally convex space is induced by a countable separating family of seminorms then the topology is metrizable.

Theorem 6. *If (X, τ) is a locally convex space with the seminorm topology induced by a countable separating family of seminorms $\{m_n : n \in \mathbb{N}\}$ and c_n is a summable nonincreasing sequence of positive numbers, then*

$$d(x, y) = \sum_{n=1}^{\infty} c_n \frac{m_n(x-y)}{1 + m_n(x-y)}, \qquad x, y \in X,$$

is a translation invariant metric on X, τ is equal to the metric topology for d, and with this metric the open balls centered at 0 are balanced.

Proof. For any $x, y \in X$ we have

$$d(x, y) < \sum_{n=1}^{\infty} c_n < \infty,$$

because the sequence c_n is summable. It is apparent that $d(x, y) = d(y, x)$.

If m is any seminorm on X, then

$$\frac{m(x) + m(y)}{1 + m(x) + m(y)} - \frac{m(x+y)}{1 + m(x+y)} = \frac{m(x) + m(y) - m(x+y)}{(1 + m(x) + m(y))(1 + m(x+y))} \geq 0,$$

so

$$\frac{m(x+y)}{1 + m(x+y)} \leq \frac{m(x) + m(y)}{1 + m(x) + m(y)}.$$

Also, it is straightforward to check that the function $f : [0, \infty) \to [0, \infty)$ defined by $f(a) = \frac{a}{1+a}$ satisfies $f(a + b) \leq f(a) + f(b)$. Define $d_0(x) = d(x, 0)$. If $x, y \in X$, then

$$\begin{aligned} d_0(x + y) &= \sum_{n=1}^{\infty} c_n \frac{m_n(x+y)}{1 + m_n(x+y)} \\ &\leq \sum_{n=1}^{\infty} c_n \frac{m_n(x) + m_n(y)}{1 + m_n(x) + m_n(y)} \\ &\leq \sum_{n=1}^{\infty} c_n \frac{m_n(x)}{1 + m_n(x)} + c_n \frac{m_n(y)}{1 + m_n(y)} \\ &= d_0(x) + d_0(y). \end{aligned}$$

Hence, for $x, y \in X$,

$$d(x, z) = d_0(x - y + y - z) \leq d_0(x - y) + d_0(y - z) = d(x, y) + d(y, z),$$

showing that d satisfies the triangle inequality.

If $d(x, y) = 0$, then

$$\sum_{n=1}^{\infty} c_n \frac{m_n(x-y)}{1 + m_n(x-y)} = 0.$$

As each term is nonnegative, each term must be equal to 0. As each c_n is positive, this implies that each $m_n(x - y)$ is equal to 0. But $\{m_n : n \in \mathbb{N}\}$ is a separating family so if $x - y \neq 0$ then there is some m_n with $m_n(x - y) \neq 0$, and this shows that $x - y = 0$, i.e. $x = y$. Therefore d is a metric on X.

If $x_0 \in X$, then $d(x + x_0, y + x_0) = d(x, y)$: the metric d is translation invariant.

If $|\alpha| \leq 1$ and $x \in X$, then

$$\begin{aligned} d_0(\alpha x) &= \sum_{n=1}^{\infty} c_n \frac{m_n(\alpha x)}{1 + m_n(\alpha x)} \\ &= \sum_{n=1}^{\infty} c_n \frac{|\alpha| m_n(x)}{1 + |\alpha| m_n(x)} \\ &= \sum_{n=1}^{\infty} c_n \frac{m_n(x)}{\frac{1}{|\alpha|} + m_n(x)} \\ &\leq \sum_{n=1}^{\infty} c_n \frac{m_n(x)}{1 + m_n(x)} \\ &= d_0(x). \end{aligned}$$

Thus, if $d(x, 0) < \epsilon$ and $|\alpha| \leq 1$ then $d(\alpha x, 0) < \epsilon$, so the open ball

$$\{x \in X : d(x, 0) < \epsilon\}$$

is balanced.

(X, τ) has a local basis at 0 whose elements are finite intersections of sets of the form $\{x \in X : m_n(x) < \epsilon\}$. Suppose that $\epsilon > 0$, let N be large enough so that $\sum_{n=N+1}^{\infty} c_n < \frac{\epsilon}{2}$, and let M be large enough so that $\frac{1}{M} \sum_{n=1}^{N} c_n < \frac{1}{2}$. If $x \in \bigcap_{n=1}^{N} \{y \in X : m_n(y) < \frac{\epsilon}{M}\}$, then

$$\begin{aligned} d(x, 0) &= \sum_{n=1}^{N} c_n \frac{m_n(x)}{1 + m_n(x)} + \sum_{n=N+1}^{\infty} c_n \frac{m_n(x)}{1 + m_n(x)} \\ &< \sum_{n=1}^{N} c_n m_n(x) + \sum_{n=N+1}^{\infty} c_n \\ &< \sum_{n=1}^{N} c_n \frac{\epsilon}{M} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

This shows that

$$\bigcap_{n=1}^{N} \left\{x \in X : m_n(x) < \frac{\epsilon}{M}\right\} \subseteq \{x \in X : d(x,0) < \epsilon\},$$

and this entails that τ is at least as fine as the metric topology induced by d.

Suppose that $0 < \epsilon < \frac{1}{2}$ and $N \in \mathbb{N}$. If $d(x,0) < c_N \epsilon$, then of course for each n we have

$$c_n \frac{m_n(x)}{1 + m_n(x)} < c_N \epsilon,$$

and hence if $1 \leq n \leq N$ then

$$\frac{m_n(x)}{1 + m_n(x)} < \frac{c_N}{c_n} \epsilon \leq \epsilon,$$

and hence if $1 \leq n \leq N$ then

$$m_n(x) < \frac{\epsilon}{1 - \epsilon} < 2\epsilon.$$

Therefore,

$$\{x \in X : d(x,0) < c_N \epsilon\} \subseteq \bigcap_{n=1}^{N} \{x \in X : m_n(x) < 2\epsilon\}.$$

It follows from this that the metric topology induced by d is at least as fine as τ. □

If a locally convex space is metrizable with a complete metric, then it is called a *Fréchet space*.

We now prove conditions under which a topological vector space is normable.

Theorem 7. *A topological vector space (X, τ) is normable if and only if there is a convex bounded open neighborhood of the origin.*

Proof. Suppose that V is a convex bounded open neighborhood of 0. V contains a balanced convex open neighborhood U of 0,[6] and because V is bounded so is U. We define $\|x\| = \mu_U(x)$, where μ_U is the Minkowski functional of U. If $x \neq 0$, then because $N = X \setminus \{x\}$ is an open neighborhood of 0 and U is bounded, there is some $t > 0$ such that $U \subseteq tN$. Hence $x \notin \frac{1}{t}U$, i.e., $tx \notin U$. As U is balanced, by Lemma 3 we get $\mu_U(tx) \geq 1$. μ_U is a seminorm, so $\mu_U(x) \geq \frac{1}{t} > 0$, showing that if $x \neq 0$ then $\mu_U(x) > 0$, and hence that $\|\cdot\|$ is a norm on X. Also, we check that

$$\{x \in X : \|x\| < r\} = rU.$$

Because U is bounded, for any open neighborhood N of 0 there is some $t > 0$ such that $U \subseteq tN$, hence

$$\left\{x \in X : \|x\| < \frac{1}{t}\right\} \subseteq N.$$

[6] Walter Rudin, *Functional Analysis*, second ed., p. 12, Theorem 1.14.

This implies that the norm topology for $\|\cdot\|$ is at least as fine as τ. And $\{x \in X : \|x\| < r\} = rU$ is an open set because scalar multiplication is continuous, so τ is at least as fine as the norm topology for $\|\cdot\|$. Therefore that (X, τ) is normable with the norm $\|\cdot\|$.

In the other direction, if τ is the norm topology for some norm $\|\cdot\|$ on X, then

$$U = \{x \in X : \|x\| < 1\}$$

is indeed a convex open neighborhood of the origin. Suppose that N is an open neighborhood of 0. There is some $r > 0$ such that

$$\{x \in X : \|x\| < r\} \subseteq N,$$

and thus such that $U \subseteq \frac{1}{r}N$, and hence U is bounded, showing that there exists a convex bounded open neighborhood of the origin. □

A topological vector space is called *locally bounded* if there is a bounded open neighborhood of the origin. A topological vector space is said to have the *Heine-Borel property* if every closed and bounded subset of it is compact.

Theorem 8. *If X is a topological vector space that is locally bounded and has the Heine-Borel property, then X has finite dimension.*

Proof. Let V be a bounded neighborhood of 0. It is a fact that the closure of a bounded set is itself bounded,[7] and therefore $\overline{V}$ is compact. For any $x \in X$, the set $x + \overline{V}$ is a compact neighborhood of x, hence X is locally compact. But a locally compact topological vector space is finite dimensional,[8] so X is finite dimensional. □

3 Continuous functions on the unit disc

Let $D = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc. Let $C(D)$ be the set of continuous functions $D \to \mathbb{C}$. $C(D)$ is a complex vector space. If K is a compact subset of D, define

$$\nu_K(f) = \sup\{|f(z)| : z \in K\}, \qquad f \in C(D).$$

It is straightforward to check that ν_K is a seminorm on $C(D)$. If $f \in C(D)$ is nonzero then there is some $z \in D$ with $f(z) \neq 0$, and hence $\nu_{\{z\}}(f) = |f(z)| > 0$, so the set of all ν_K is a separating family of seminorms on $C(D)$. Thus, $C(D)$ with the seminorm topology induced by the set of all ν_K is a locally convex space.

Define $K_n = \{z \in \mathbb{C} : |z| \leq 1 - \frac{1}{n}\}$, $n \geq 1$. If K is a compact subset of D, then there is some n with $K \subseteq K_n$, so $\nu_K(f) \leq \nu_{K_n}(f)$, and hence

$$\{f \in C(D) : \nu_{K_n}(f) < \epsilon\} \subseteq \{f \in C(D) : \nu_K(f) < \epsilon\}.$$

[7] Walter Rudin, *Functional Analysis*, second ed., p. 11, Theorem 1.13(f).
[8] Walter Rudin, *Functional Analysis*, second ed., p. 17, Theorem 1.22.

It follows that the seminorm topology induced by $\{\nu_{K_n} : n \in \mathbb{N}\}$ is at least as fine as the seminorm topology induced by $\{\nu_K : K \text{ is compact}\}$, thus the topologies are equal. Because the topology of $C(D)$ is induced by the countable family $\{\nu_{K_n} : n \in \mathbb{N}\}$, by Theorem 6 it is metrizable: for any summable nonincreasing sequence of positive real numbers c_n, the topology is induced by the metric

$$d(f,g) = \sum_{n=1}^{\infty} c_n \frac{\nu_{K_n}(f-g)}{1+\nu_{K_n}(f-g)}, \qquad f,g \in C(D). \tag{1}$$

Suppose that $f_i \in C(D)$ is a Cauchy sequence. For $n \in \mathbb{N}$, the fact that f_i is a Cauchy sequence in $C(D)$ implies that $\nu_{K_n}(f_i - f_j) \to 0$ as $i,j \to \infty$. $C(K_n)$ is a Banach space with the norm ν_{K_n}, and hence there is some $f_{K_n} \in C(K_n)$ satisfying $\nu_{K_n}(f_i - f_{K_n}) \to 0$ as $i \to \infty$. We define $f : D \to \mathbb{C}$ to be $f_{K_n}(z)$, for $z \in K_n$; this makes sense because the restriction of f_{K_n} to K_m is f_{K_m} if $n \geq m$. f is continuous at each point in D because for each point in D there is some K_n containing an open neighborhood of the point, and f_{K_n} is continuous. Hence $f \in C(D)$. Therefore $C(D)$ with the metric (1) is a complete metric space, which means that it is a Fréchet space.

Theorem 9. *The topology of $C(D)$ is not induced by a norm.*

Proof. Because the topology of $C(D)$ is the seminorm topology induced by the separating family of seminorms $\{\nu_{K_n} : n \in \mathbb{N}\}$, by Lemma 5 a subset E of $C(D)$ is bounded if and only if each ν_{K_n} is a bounded function on E, i.e., for each $n \in \mathbb{N}$ there is some M_n such that $f \in E$ implies $\nu_{K_n}(f) \leq M_n$.

Suppose by contradiction that there is a bounded convex open neighborhood V of the origin. Because $\nu_{K_n}(f) \leq \nu_{K_{n+1}}(f)$ for any $f \in C(D)$, there is some $N \in \mathbb{N}$ and some $\epsilon > 0$ such that

$$U = \{f \in C(D) : \nu_{K_N}(f) < \epsilon\} \subseteq V.$$

V being bounded implies that U is bounded. Let

$$\Delta_1 = \left\{ z \in \mathbb{C} : |z| < 1 - \frac{1}{N} + \frac{1}{N(N+1)} \right\}, \quad \Delta_2 = \left\{ z \in \mathbb{C} : 1 - \frac{1}{N} < |z| < 1 \right\},$$

and let ϕ_1, ϕ_2 be a partition of unity subordinate to this open cover of D. For any constant $M > 0$, the restriction of $M\phi_2$ to K_N is 0 and hence belongs to U. But $\nu_{K_{N+1}}(M\phi_2) = M$, so $\nu_{K_{N+1}}$ is not a bounded function on U, contradicting that U is bounded. Therefore, there is no bounded convex open neighborhood of 0. By Theorem 7, this tells us that $C(D)$ is not normable. $\square$

For each n, the set $C(K_n)$ is a Banach space with norm ν_{K_n}. If $n \geq m$ and $f \in C(K_n)$, let $r_{n,m}(f)$ be the restriction of f to K_m. For $n \geq m$, the function $r_{n,m}$ is a continuous linear map $C(K_n) \to C(K_m)$, and if $n \geq m \geq l$ then $r_{n,l} = r_{m,l} \circ r_{n,m}$. Thus the Banach spaces $C(K_n)$ and the maps $r_{n,m}$ are a *projective system* in the category of locally convex spaces, and it is a fact that any projective system in this category has a projective limit that is unique up to unique isomorphism.

Theorem 10. $C(D) = \varprojlim C(K_n)$.

Proof. Define $r_n : C(D) \to C(K_n)$ by taking $r_n(f)$ to be the restriction of f to K_n. Each r_n is continuous and linear. Certainly, if $n \geq m$ then $r_m = r_{n,m} \circ r_n$. Suppose the Y is a locally convex space, that $\phi_n : Y \to C(K_n)$ are continuous linear maps, and that if $n \geq m$ then

$$\phi_m = r_{n,m} \circ \phi_n. \tag{2}$$

If $z \in K_m$ and $n \geq m$, then by (2) we have $\phi_n(y)(z) = \phi_m(y)(z)$. For $z \in D$, eventually $z \in K_n$, and define $\phi(y)(z)$ to be $\phi_n(y)(z)$ for any n such that $z \in K_n$. For each $z \in D$ there is some n such that z is in the interior of K_n, and the restriction of $\phi(y)$ to K_n is equal to $\phi_n(y)$, hence $\phi(y)$ is continuous at z. Therefore $\phi(y) \in C(D)$, so $\phi : Y \to C(D)$.

Suppose that $y_1, y_2 \in Y$ and $\alpha \in \mathbb{C}$. If $z \in D$, then there is some n with $z \in K_n$, and because ϕ_n is linear,

$$\phi(\alpha y_1 + y_2)(z) = \phi_n(\alpha y_1 + y_2)(z) = \alpha\phi_n(y_1)(z) + \phi_n(y_2)(z) = \alpha\phi(y_1)(z) + \phi(y_2)(z).$$

Therefore ϕ is linear.

Suppose that $y_\alpha \in Y$ is a net with limit $y \in Y$. For $\phi(y_\alpha)$ to converge to $\phi(y)$ means that for each $n \in \mathbb{N}$ we have $\nu_{K_n}(\phi(y_\alpha) - \phi(y)) \to 0$. But

$$\nu_{K_n}(\phi(y_\alpha) - \phi(y)) = \nu_{K_n}(\phi_n(y_\alpha) - \phi_n(y)),$$

and $\phi_n(y_\alpha) \to \phi_n(y)$ because ϕ_n is continuous. Therefore, for each $n \in \mathbb{N}$ we have $\nu_{K_n}(\phi(y_\alpha) - \phi(y)) \to 0$, so ϕ is continuous. □

We proved in the above theorem that the Fréchet space $C(D)$ is the projective limit of the Banach spaces $C(K_n)$. It is a fact that the projective limit of any projective system of Banach spaces is a Fréchet space.[9]

A topological space is said to be *separable* if it has a countable subset that is dense.

Theorem 11. *$C(D)$ is separable.*

Proof. One proves using the Stone-Weierstrass theorem that the Banach space $C(K_n)$ is separable. The product of at most continuum many separable Hausdorff spaces each with at least two points is itself separable with the product topology.[10] Therefore, $\prod_{n=1}^{\infty} C(K_n)$ is separable. Because each $C(K_n)$ is a metric space, this countable product $\prod_{n=1}^{\infty} C(K_n)$ is metrizable, and any subset of a separable metric space is itself separable with the subspace topology. The projective limit of a projective system of topological vector spaces is a closed subspace of the product of the spaces; thus, using merely that the projective limit is a subset of the product $\prod_{n=1}^{\infty} C(K_n)$ and has the subspace topology inherited from the direct product, we get that $C(D)$ is separable. □

[9] J. L .Taylor, *Notes on locally convex topological vector spaces*, http://www.math.utah.edu/~taylor/LCS.pdf, p. 8, Proposition 2.6, and cf. Paul Garret, *Functions on circles: Fourier series, I*, http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/04_blevi_sobolev.pdf, p. 37, §13.

[10] Stephen Willard, *General Topology*, p. 109, Theorem 16.4.

4 Holomorphic functions on the unit disc

Let $H(D)$ be the set of holomorphic functions $D \to \mathbb{C}$. $H(D)$ is a linear subspace of $C(D)$. Let $H(D)$ have the subspace topology inherited from $C(D)$. One proves that this topology is equal to the seminorm topology induced by $\{\nu_{K_n} : n \in \mathbb{N}\}$. Any subset of a separable metric space with the subspace topology is separable. By Theorem 11 the Fréchet space $C(D)$ is separable, and thus $H(D)$ is separable too.

We now prove that $H(D)$ is a closed subspace of $C(D)$.[11] A closed linear subspace of a Fréchet space is itself a Fréchet space, hence this theorem shows that $H(D)$ is a Fréchet space.

Theorem 12. *$H(D)$ is a closed subset of $C(D)$.*

Proof. Suppose that $f_j \in H(D)$ is a net and that $f_j \to f \in C(D)$. We shall show that $f \in H(D)$. (In fact it suffices to prove this for a sequence of elements in $H(D)$ because we have shown that $C(D)$ is metrizable, but that will not simplify this argument.) To show this we have to prove that if $z \in D$ then $\frac{f(z+h)-f(z)}{h}$ has a limit as $h \to 0$, $h \in \mathbb{C}$. Let γ be a counterclockwise oriented circle contained in D with center z, say of radius $r = \frac{1-|z|}{2} > 0$. For each j the function f_j is holomorphic on D, and so Cauchy's integral formula gives

$$f_j(w) = \frac{1}{2\pi i}\int_\gamma \frac{f_j(\zeta)}{\zeta - w} d\zeta, \qquad w \in B_r(z).$$

Therefore

$$\begin{aligned} f(w) - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - w} d\zeta &= f(w) - f_j(w) + \frac{1}{2\pi i}\int_\gamma \frac{f_j(\zeta)}{\zeta - w} d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - w} d\zeta \\ &= f(w) - f_j(w) + \frac{1}{2\pi i}\int_\gamma \frac{f_j(w) - f(w)}{\zeta - w} d\zeta. \end{aligned}$$

As γ is a compact subset of D this gives us

$$\left| f(w) - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - w} d\zeta \right| \le |f(w) - f_j(w)| + \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{\nu_\gamma(f_j - f)}{r - |w - z|}.$$

The right-hand side tends to 0, while the left-hand side does not depend on j. Hence

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - w} d\zeta, \qquad w \in B_r(z). \tag{3}$$

Applying (3), we have for $0 \le |h| < r$,

$$f(z+h) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - (z+h)} d\zeta,$$

[11] Paul Garrett, *Holomorphic vector-valued functions*, http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/08b_vv_holo.pdf

hence

$$\begin{aligned}
f(z+h)-f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-(z+h)}d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left(\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z}\right)d\zeta \\
&= \frac{1}{2\pi i}\int_\gamma f(\zeta)\cdot\frac{h}{(\zeta-(z+h))(\zeta-z)}d\zeta,
\end{aligned}$$

thus

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-(z+h))(\zeta-z)}d\zeta.$$

For $\zeta\in\gamma$ we have $\left|\frac{f(\zeta)}{(\zeta-(z+h))(\zeta-z)}\right| \leq \frac{\nu_\gamma(f)}{(r-|h|)r}$, and so by the dominated convergence theorem we get

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^2}d\zeta.$$

Thus, for every $z \in D$, the function f is complex differentiable at z. Hence $f \in H(D)$, and therefore $H(D)$ is a closed subset of $C(D)$. □

We remind ourselves that a topological vector space is said to have the *Heine-Borel property* if every closed and bounded subset of it is compact. Lemma 5 tells us that a subset E of $H(D)$ is bounded if and only if each seminorm ν_{K_n} is a bounded function on E. The following theorem states that $H(D)$ has the Heine-Borel property.[12] An equivalent statement is called *Montel's theorem*.

Theorem 13 (Heine-Borel property)**.** *The Fréchet space $H(D)$ has the Heine-Borel property.*

That $H(D)$ has the Heine-Borel property is a useful tool, and lets us prove that the topology of $H(D)$ is not induced by a norm.

Theorem 14. *$H(D)$ is not normable.*

Proof. If $H(D)$ were normable then by Theorem 7 there would be a convex bounded open neighborhood of the origin. This would imply that $H(D)$ is locally bounded (has a bounded open neighborhood of the origin). But $H(D)$ has the Heine-Borel property, and a topological vector space that is locally bounded and has the Heine-Borel property is finite dimensional by Theorem 8. It is straightforward to check that $H(D)$ is not finite dimensional, and hence $H(D)$ is not normable. □

For $f \in H(D)$, let $(df)(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$. First, if $f\in H(D)$ then one proves that $df \in H(D)$. Then, the following theorem states that $d: H(D)\to H(D)$ is a morphism in the category of locally convex spaces.[13]

[12] Henri Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, pp. 162–167, chapter V, §4.

[13] Henri Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, p. 143, chapter V, §1.

Theorem 15. *Differentiation $H(D) \to H(D)$ is a continuous linear map.*

If K is a compact subset of D and $f \in H(D)$, let $r_K(f)$ be the restriction of f to K, and let $\overline{H}(K)$ be the closure in $C(K)$ of the set $\{r_K(f) : f \in H(D)\}$. Each element of $\overline{H}(K)$ is holomorphic on the interior of K. $C(K)$ is a Banach space with the norm ν_K, and hence $\overline{H}(K)$ is a Banach space with the same norm, because it is indeed a linear subspace. If $n \geq m$ and $f \in \overline{H}(K_n)$, let $r_{n,m}(f) = r_{K_m}(f) \in \overline{H}(K_m)$. The $r_{n,m}$ are continuous and linear, and if $n \geq m \geq l$ then $r_{n,l} = r_{m,l} \circ r_{n,m}$. Thus the Banach spaces $\overline{H}(K_n)$ and the continuous linear maps $r_{n,m}$ are a projective system in the category of locally convex spaces, and this projective system has a projective limit $\varprojlim \overline{H}(K_n)$. The following theorem states that this projective limit is equal to the Fréchet space $H(D)$.[14]

Theorem 16. $H(D) = \varprojlim \overline{H}(K_n)$.

5 Dual spaces

The *dual of a topological vector space* X is the set X^* of continuous linear maps $X \to \mathbb{C}$. If E is a bounded subset of X and $\lambda \in X^*$, then $\lambda(E)$ is a bounded subset of $\mathbb{C}$ (the image of a bounded set under a continuous linear map is a bounded set). Hence

$$p_E(\lambda) = \sup\{|\lambda x| : x \in E\} < \infty.$$

The function p_E is a seminorm on X^*, and if $\lambda \neq 0$ then there is some $x \in X$ with $\lambda x \neq 0$, hence $p_{\{x\}}(\lambda) > 0$. The *strong dual topology on* X^* is the seminorm topology induced by the separating family

$$\{p_E : E \text{ is a bounded subset of } X\}.$$

(To add to our vocabulary: the set of all bounded subsets of a topological vector space is called the *bornology of the space*. Similar to how one can define a topology as a collection of sets satisfying certain properties, one can also define a bornology on a set without first having the structure of a topological vector space.) We denote by X^*_β the dual space X^* with the strong dual topology. X^*_β is a locally convex space. If X is a normed space, one can prove[15] that X^*_β is normable with the operator norm

$$\|\lambda\| = \sup\{|\lambda x| : \|x\| \leq 1\}.$$

We say that a topological vector space X is *reflexive* if $(X^*_\beta)^*_\beta = X$; since the strong dual of a topological vector space is locally convex, for a topological vector space to be reflexive it is necessary that it be locally convex.

[14] J. L .Taylor, *Notes on locally convex topological vector spaces*, `http://www.math.utah.edu/~taylor/LCS.pdf`, p. 8

[15] K. Yosida, *Functional Analysis*, sixth ed., p. 111, Theorem 1.

Let X be a locally convex space. The Hahn-Banach separation theorem[16] yields that X^* separates X: if $x \neq 0$ then there is some $\lambda \in X^*$ with $\lambda x \neq 0$. If $\lambda \in X^*$, then $|\lambda|$ is a seminorm on X and $\{|\lambda| : \lambda \in X^*\}$ is therefore a separating family of seminorms on X. We call the seminorm topology induced by this separating family the *weak topology on* X, and X with the weak topology is a locally convex space. The original topology on X is at least as fine as the weak topology on X: any set that is open using the weak topology is open using the original topology.

The following lemma shows that a Fréchet space with the Heine-Borel property is reflexive, and therefore that $H(D)$ is reflexive.

Lemma 17. *If a Fréchet space has the Heine-Borel property, then it is reflexive.*

Proof. A subset of a locally convex space is called a *barrel* if it is closed, convex, balanced, and absorbing. A locally convex space is said to be *barreled* if each barrel is a neighborhood of 0. It is a fact that every Fréchet space is barreled.[17] A locally convex space is reflexive if and only if it is barreled and if every set that is closed, convex, balanced, and bounded is weakly compact.[18] Therefore, for a Fréchet space with the Heine-Borel property to be reflexive it is necessary and sufficient that every set that is compact, convex, and balanced be weakly compact. But if a subset of a locally convex space is compact then it is weakly compact, because the original topology is at least as fine as the weak topology and hence any cover of a set by elements of the weak topology is also a cover of the set by elements of the original topology. Therefore, any Fréchet space with the Heine-Borel property is reflexive. □

Morphisms in the category of locally convex spaces are continuous linear maps. If X and Y are locally convex spaces and $\phi : X \to Y$ is a morphism, the *dual of* ϕ is the morphism

$$\phi^* : Y^*_\beta \to X^*_\beta$$

defined by

$$\phi^*(\lambda) = \lambda \circ \phi, \qquad \lambda \in Y^*_\beta.$$

One verifies that ϕ^* is in fact a morphism. If the spaces X_j and the morphisms $\phi_{i,j} : X_i \to X_j$, $i \geq j$, are a projective system in the category of locally convex spaces, then the dual spaces $(X_j)^*_\beta$ and the morphisms $\phi^*_{i,j} : (X_j)^*_\beta \to (X_i)^*_\beta$, $i \geq j$, are a *direct system* in this category. It is a fact that the dual of a projective limit of Banach spaces is isomorphic to the direct limit of the duals of the Banach spaces.[19] Thus, as $H(D)$ is the projective limit of the Banach spaces $\overline{H}(K_n)$, its dual space $H^*(D) = (H(D))^*_\beta$ is isomorphic to the direct limit of the duals of these Banach spaces:

$$H^*(D) = \varinjlim (\overline{H}(K_n))^*_\beta.$$

[16]Walter Rudin, *Functional Analysis*, second ed., p. 59, Theorem 3.4.

[17]K. Yosida, *Functional Analysis*, sixth ed., p. 138, Corollary 1.

[18]K. Yosida, *Functional Analysis*, sixth ed., p. 140, Theorem 2.

[19]Paul Garrett, *Functions on circles: Fourier series, I*, `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/04_blevi_sobolev.pdf`, p. 15, Theorem 5.1.1.

Cooper[20] shows that $H^*(D)$ is isomorphic to the space of germs of functions on the complement of D in the extended complex plane that vanish at infinity. Let $\mathfrak{A}$ be those sequences $a \in \mathbb{C}^{\mathbb{N}}$ satisfying

$$\limsup |a_n|^{1/n} \leq 1.$$

By Hadamard's formula for the radius of convergence of a power series, these are precisely the sequences of coefficients of power series with radius of convergence ≥ 1, and $\mathfrak{A}$ is a complex vector space. The map

$$a \mapsto \sum_{n=0}^{\infty} a_n z^n$$

is linear and has the linear inverse

$$f \mapsto \left(\frac{f^{(n)}(0)}{n!} \right),$$

so $H(D)$ and $\mathfrak{A}$ are linearly isomorphic. For $0 < r < 1$, define

$$q_r(a) = \max\{|a_n| r^n : n \in \mathbb{N}\}.$$

Each q_r is a norm, yet we do not give $\mathfrak{A}$ the norm topology. Rather, we give $\mathfrak{A}$ the seminorm topology induced by the family $\{q_r : 0 < r < 1\}$, and with this topology $\mathfrak{A}$ is a locally convex space. One proves that the above two linear maps are continuous, and hence that $H(D)$ is isomorphic as a locally convex space to $\mathfrak{A}$. Then, one proves that the dual space of $\mathfrak{A}$ are those sequences $b \in \mathbb{C}^{\mathbb{N}}$ such that

$$\limsup |b_n|^{1/n} < 1,$$

and b corresponds to

$$\sum_{n=0}^{\infty} b_n \left(\frac{1}{z} \right)^{n+1}.$$

[20] J. B. Cooper, *Functional analysis– spaces of holomorphic functions and their duality*, `http://www.dynamics-approx.jku.at/lena/Cooper/holloc.pdf`, p. 11, §5.