

# Hermite functions

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## 1 Locally convex spaces

If  $V$  is a vector space and  $\{p_\alpha : \alpha \in A\}$  is a separating family of seminorms on  $V$ , then there is a unique topology with which  $V$  is a locally convex space and such that the collection of finite intersections of sets of the form

$$\{v \in V : p_\alpha(v) < \epsilon\}, \quad \alpha \in A, \quad \epsilon > 0$$

is a local base at 0.<sup>1</sup> We call this the **topology induced by the family of seminorms**. If  $\{p_n : n \geq 0\}$  is a separating family of seminorms, then

$$d(v, w) = \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(v-w)}{1+p_n(v-w)}, \quad v, w \in V,$$

is a metric on  $V$  that induces the same topology as the family of seminorms. If  $d$  is a complete metric, then  $V$  is called a **Fréchet space**.

## 2 Schwartz functions

For  $\phi \in C^\infty(\mathbb{R}, \mathbb{C})$  and  $n \geq 0$ , let

$$p_n(\phi) = \sup_{0 \leq k \leq n} \sup_{u \in \mathbb{R}} (1+u^2)^{n/2} |\phi^{(k)}(u)|.$$

We define  $\mathcal{S}$  to be the set of those  $\phi \in C^\infty(\mathbb{R}, \mathbb{C})$  such that  $p_n(\phi) < \infty$  for all  $n \geq 0$ .  $\mathcal{S}$  is a complex vector space and each  $p_n$  is a norm, and because each  $p_n$  is a norm, a fortiori  $\{p_n : n \geq 0\}$  is a separating family of seminorms. With the topology induced by this family of seminorms,  $\mathcal{S}$  is a Fréchet space.<sup>2</sup> As well,  $D : \mathcal{S} \rightarrow \mathcal{S}$  defined by

$$(D\phi)(x) = \phi'(x), \quad x \in \mathbb{R}$$

and  $M : \mathcal{S} \rightarrow \mathcal{S}$  defined by

$$(M\phi)(x) = x\phi(x), \quad x \in \mathbb{R}$$

are continuous linear maps.

<sup>1</sup><http://individual.utoronto.ca/jordanbell/notes/holomorphic.pdf>, Theorem 1 and Theorem 4.

<sup>2</sup>Walter Rudin, *Functional Analysis*, second ed., p. 184, Theorem 7.4.

### 3 Hermite functions

Let  $\lambda$  be Lebesgue measure on  $\mathbb{R}$  and let

$$(f, g)_{L^2} = \int_{\mathbb{R}} f \bar{g} d\lambda.$$

With this inner product,  $L^2(\lambda)$  is a separable Hilbert space. We write

$$\|f\|_{L^2}^2 = (f, f)_{L^2} = \int_{\mathbb{R}} |f|^2 d\lambda.$$

For  $n \geq 0$ , define  $H_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2},$$

which is a polynomial of degree  $n$ .  $H_n$  are called **Hermite polynomials**. It can be shown that

$$\exp(2zx - z^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^n, \quad z \in \mathbb{C}. \quad (1)$$

For  $m, n \geq 0$ ,

$$\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} d\lambda(x) = 2^n n! \sqrt{\pi} \delta_{m,n}.$$

For  $n \geq 0$ , define  $h_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} D^n e^{-x^2}.$$

$h_n$  are called **Hermite functions**. Then for  $m, n \geq 0$ ,

$$(h_m, h_n)_{L^2} = \int_{\mathbb{R}} h_m(x) h_n(x) d\lambda(x) = \delta_{m,n}.$$

One proves that  $\{h_n : n \geq 0\}$  is an orthonormal basis for  $(L^2(\lambda), (\cdot, \cdot)_{L^2})$ .<sup>3</sup>

We remind ourselves that for  $x \in \mathbb{R}$ ,<sup>4</sup>

$$e^{-x^2} = 2^{-1} \pi^{-1/2} \int_{\mathbb{R}} e^{-y^2/4} e^{-ixy} dy,$$

and by the dominated convergence theorem this yields

$$D^n e^{-x^2} = 2^{-1} \pi^{-1/2} \int_{\mathbb{R}} (-iy)^n e^{-y^2/4} e^{-ixy} dy,$$

and so

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \cdot 2^{-1} \pi^{-1/2} \int_{\mathbb{R}} (iy)^n e^{-y^2/4} e^{-ixy} dy. \quad (2)$$

<sup>3</sup><http://individual.utoronto.ca/jordanbell/notes/gaussian.pdf>, Theorem 8.

<sup>4</sup><http://individual.utoronto.ca/jordanbell/notes/completelymonotone.pdf>, Lemma 5.

## 4 Mehler's formula

We now prove **Mehler's formula** for the Hermite functions.<sup>5</sup>

**Theorem 1** (Mehler's formula). For  $z \in \mathbb{C}$  with  $|z| < 1$  and for  $x, y \in \mathbb{R}$ ,

$$\sum_{n=0}^{\infty} h_n(x)h_n(y)z^n = \pi^{-1/2}(1-z^2)^{-1/2} \exp\left(-\frac{1}{2} \cdot \frac{1+z^2}{1-z^2}(x^2+y^2) + \frac{2z}{1-z^2}xy\right).$$

*Proof.* Using (2),

$$\begin{aligned} & \sum_{n=0}^{\infty} h_n(x)h_n(y)z^n \\ &= \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{2^n n!} e^{(x^2+y^2)/2} z^n \left( \int_{\mathbb{R}} (2\pi i \xi)^n e^{-\pi^2 \xi^2} e^{-2\pi i x \xi} d\xi \right) \left( \int_{\mathbb{R}} (2\pi i \zeta)^n e^{-\pi^2 \zeta^2} e^{-2\pi i y \zeta} d\zeta \right) \\ &= \sqrt{\pi} e^{(x^2+y^2)/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi^2 \xi^2 - \pi^2 \zeta^2 - 2\pi i x \xi - 2\pi i y \zeta} \sum_{n=0}^{\infty} \frac{(-2\pi^2 \xi \zeta z)^n}{n!} d\xi d\zeta \\ &= \sqrt{\pi} e^{(x^2+y^2)/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi^2 \xi^2 - \pi^2 \zeta^2 - 2\pi i x \xi - 2\pi i y \zeta} e^{-2\pi^2 \xi \zeta z} d\xi d\zeta. \end{aligned}$$

Now, writing  $a = \frac{iy}{\pi} + \xi z$ , we calculate

$$\begin{aligned} \int_{\mathbb{R}} e^{-\pi^2 \zeta^2 - 2\pi i \zeta y - 2\pi^2 \xi \zeta z} d\zeta &= \int_{\mathbb{R}} e^{-\pi^2 (\zeta+a)^2 + \pi^2 a^2} d\zeta \\ &= \frac{1}{\sqrt{\pi}} e^{\pi^2 a^2} \\ &= \frac{1}{\sqrt{\pi}} \exp(-y^2 + 2\pi i y \xi z + \pi^2 \xi^2 z^2). \end{aligned}$$

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<sup>5</sup>Sundaram Thangavelu, *An Introduction to the Uncertainty Principle: Hardy's Theorem on Lie Groups*, p. 8, Proposition 1.2.1.

Then, for  $\alpha = (1 - z^2)\pi^2$ ,

$$\begin{aligned}
& \sum_{n=0}^{\infty} h_n(x)h_n(y)z^n \\
&= e^{(x^2+y^2)/2} \int_{\mathbb{R}} e^{-\pi^2\xi^2 - 2\pi ix\xi - y^2 + 2\pi iy\xi z + \pi^2\xi^2 z^2} d\xi \\
&= e^{(x^2-y^2)/2} \int_{\mathbb{R}} e^{-\alpha\xi^2 - 2\pi i(x-yz)\xi} d\xi \\
&= e^{(x^2-y^2)/2} \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2}{\alpha}(x-yz)^2\right) \\
&= \pi^{-1/2} e^{(x^2-y^2)/2} (1-z^2)^{-1/2} \exp\left(-\frac{(x-yz)^2}{1-z^2}\right) \\
&= \pi^{-1/2} (1-z^2)^{-1/2} \exp\left(-\frac{x^2}{1-z^2} + \frac{2xyz}{1-z^2} - \frac{y^2 z^2}{1-z^2} + \frac{x^2}{2} - \frac{y^2}{2}\right) \\
&= \pi^{-1/2} (1-z^2)^{-1/2} \exp\left(-\frac{1}{2} \frac{1+z^2}{1-z^2} (x^2+y^2) + \frac{2z}{1-z^2} xy\right).
\end{aligned}$$

□

## 5 The Hermite operator

We define  $A : \mathcal{S} \rightarrow \mathcal{S}$  by

$$(A\phi)(x) = -\phi''(x) + (x^2 + 1)\phi(x), \quad x \in \mathbb{R},$$

i.e.,

$$A = -D^2 + M^2 + 1,$$

which is a continuous linear map  $\mathcal{S} \rightarrow \mathcal{S}$ , which we call the **Hermite operator**.  $\mathcal{S}$  is a dense linear subspace of the Hilbert space  $L^2(\lambda)$ , and  $A : \mathcal{S} \rightarrow \mathcal{S}$  is a linear map, so  $A$  is a densely defined operator in  $L^2(\lambda)$ . For  $\phi, \psi \in \mathcal{S}$ , integrating by parts,

$$\begin{aligned}
(A\phi, \psi)_{L^2} &= \int_{\mathbb{R}} (-\phi''(x) + (x^2 + 1)\phi(x)) \overline{\psi(x)} d\lambda(x) \\
&= \int_{\mathbb{R}} -\phi''(x) \overline{\psi(x)} d\lambda(x) + \int_{\mathbb{R}} (x^2 + 1)\phi(x) \overline{\psi(x)} d\lambda(x) \\
&= \int_{\mathbb{R}} -\phi(x) \overline{\psi''(x)} d\lambda(x) + \int_{\mathbb{R}} (x^2 + 1)\phi(x) \overline{\psi(x)} d\lambda(x) \\
&= (\phi, A\psi)_{L^2},
\end{aligned}$$

showing that  $A : \mathcal{S} \rightarrow \mathcal{S}$  is symmetric. Furthermore, also integrating by parts,

$$(A\phi, \phi)_{L^2} = \int_{\mathbb{R}} (\phi'(x) \overline{\phi'(x)} + (x^2 + 1)\phi(x) \overline{\phi(x)}) d\lambda(x) \geq 0,$$

so  $A$  is a positive operator.

It is straightforward to check that each  $h_n$  belongs to  $\mathcal{S}$ . For  $n \geq 0$ , we calculate that

$$h_n''(x) + (2n + 1 - x^2)h_n(x) = 0,$$

and hence

$$(Ah_n)(x) = (2n + 1 - x^2)h_n(x) + x^2h_n(x) + h_n(x) = (2n + 2)h_n(x),$$

i.e.

$$Ah_n = (2n + 2)h_n.$$

Therefore, for each  $h_n$ ,  $A^{-1}h_n = \frac{1}{2n+2}h_n$ , and it follows that there is a unique bounded linear operator  $T : L^2(\lambda) \rightarrow L^2(\lambda)$  such that<sup>6</sup>

$$Th_n = A^{-1}h_n = (2n + 2)^{-1}h_n, \quad n \geq 0. \quad (3)$$

The operator norm of  $T$  is

$$\|T\| = \sup_{n \geq 0} \frac{1}{2n + 2} = \frac{1}{2}.$$

The Hermite functions are an orthonormal basis for  $L^2(\lambda)$ , so for  $f \in L^2(\lambda)$ ,

$$f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} h_n.$$

For  $f, g \in L^2(\lambda)$ ,

$$\begin{aligned} (Tf, g)_{L^2} &= \left( \sum_{n=0}^{\infty} (f, h_n)_{L^2} Th_n, \sum_{n=0}^{\infty} (g, h_n)_{L^2} h_n \right)_{L^2} \\ &= \left( \sum_{n=0}^{\infty} (f, h_n)_{L^2} (2n + 2)^{-1} h_n, \sum_{n=0}^{\infty} (g, h_n)_{L^2} h_n \right)_{L^2} \\ &= \sum_{n=0}^{\infty} (2n + 2)^{-1} (f, h_n)_{L^2} \overline{(g, h_n)_{L^2}}, \end{aligned}$$

from which it is immediate that  $T$  is self-adjoint.

For  $p \geq 0$ ,

$$|T^p h_n|_{L^2}^2 = |(2n + 2)^{-p} h_n|_{L^2}^2 = (2n + 2)^{-2p} |h_n|_{L^2}^2 = (2n + 2)^{-2p}.$$

Therefore for  $p \geq 1$ ,

$$\sum_{n=0}^{\infty} |T^p h_n|_{L^2}^2 = \sum_{n=0}^{\infty} (2n + 2)^{-2p} = 2^{-2p} \sum_{m=1}^{\infty} m^{-2p} = 2^{-2p} \zeta(2p).$$

This means that for  $p \geq 1$ ,  $T^p$  is a **Hilbert-Schmidt operator** with **Hilbert-Schmidt norm**<sup>7</sup>

$$\|T^p\|_{\text{HS}} = 2^{-p} \sqrt{\zeta(2p)}.$$

<sup>6</sup><http://individual.utoronto.ca/jordanbell/notes/traceclass.pdf>, Theorem 11.

<sup>7</sup><http://individual.utoronto.ca/jordanbell/notes/traceclass.pdf>, §7.

## 6 Creation and annihilation operators

Taking the derivative of (1) with respect to  $x$  gives

$$2 \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) z^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!} H'_n(x) z^n,$$

so  $H'_0 = 0$  and for  $n \geq 1$ ,  $\frac{1}{n!} H'_n(x) = \frac{1}{(n-1)!} 2H_{n-1}(x)$ , i.e.

$$H'_n = 2nH_{n-1},$$

and so

$$h'_n(x) = (2n)^{1/2} h_{n-1}(x) - xh_n(x),$$

i.e.

$$Dh_n = (2n)^{1/2} h_{n-1} - Mh_n.$$

Furthermore, from its definition we calculate

$$h'_n(x) = xh_n(x) - (2n+2)^{1/2} h_{n+1}(x),$$

i.e.

$$Dh_n = Mh_n - (2n+2)^{1/2} h_{n+1}.$$

We define  $B : \mathcal{S} \rightarrow \mathcal{S}$ , called the **annihilation operator**, by

$$(B\phi)(x) = \phi'(x) + x\phi(x), \quad x \in \mathbb{R},$$

i.e.

$$B = D + M,$$

which is a continuous linear map  $\mathcal{S} \rightarrow \mathcal{S}$ . For  $n \geq 1$ , we calculate

$$Bh_n = (2n)^{1/2} h_{n-1},$$

and  $h_0(x) = \pi^{-1/4} e^{-x^2/2}$ , so  $Bh_0 = 0$ .

We define  $C : \mathcal{S} \rightarrow \mathcal{S}$ , called the **creation operator**, by

$$(C\phi)(x) = -\phi'(x) + x\phi(x), \quad x \in \mathbb{R},$$

i.e.

$$C = -D + M,$$

which is a continuous linear map  $\mathcal{S} \rightarrow \mathcal{S}$ . For  $n \geq 0$ , we calculate

$$Ch_n = (2n+2)^{1/2} h_{n+1}.$$

Thus,

$$h_n = (2^n n!)^{-1/2} C^n h_0 = \pi^{-1/4} (2^n n!)^{-1/2} C^n (e^{-x^2/2}). \quad (4)$$

For  $\phi \in \mathcal{S}$ ,

$$B - C = 2D.$$

Furthermore,

$$BC = -D^2 + M^2 + 1 = A$$

and

$$CB = -D^2 + M^2 - 1 = A - 2.$$

## 7 The Fourier transform

Define  $\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S}$ , for  $\phi \in \mathcal{S}$ , by

$$(\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-i\xi x} \frac{dx}{(2\pi)^{1/2}}, \quad \xi \in \mathbb{R}.$$

For  $\xi \in \mathbb{R}$ , by the dominated convergence theorem we have

$$\lim_{h \rightarrow 0} \frac{\hat{\phi}(\xi + h) - \hat{\phi}(\xi)}{h} = \int_{\mathbb{R}} (-ix) \phi(x) e^{-i\xi x} \frac{dx}{(2\pi)^{1/2}},$$

i.e.

$$\widehat{x\phi(x)}(\xi) = -i^{-1} D\hat{\phi}(\xi) = iD\hat{\phi}(\xi),$$

in other words,

$$\mathcal{F}(M\phi) = iD(\mathcal{F}\phi). \quad (5)$$

Also, by the dominated convergence theorem we obtain

$$\widehat{D\phi}(\xi) = i\xi\hat{\phi}(\xi),$$

in other words,

$$\mathcal{F}(D\phi) = iM(\mathcal{F}\phi). \quad (6)$$

For  $\phi \in \mathcal{S}$ ,

$$\phi(x) = \int_{\mathbb{R}} \hat{\phi}(\xi) e^{ix\xi} \frac{d\xi}{(2\pi)^{1/2}}, \quad x \in \mathbb{R}. \quad (7)$$

$\phi \mapsto \hat{\phi}$  is an isomorphism of locally convex spaces  $\mathcal{S} \rightarrow \mathcal{S}$ .<sup>8</sup> Using (7) and the Cauchy-Schwarz inequality

$$\begin{aligned} \|\phi\|_{\infty} &\leq \int_{\mathbb{R}} (1 + \xi^2)^{1/2} (1 + \xi^2)^{-1/2} |\hat{\phi}(\xi)| \frac{d\xi}{(2\pi)^{1/2}} \\ &\leq (2\pi)^{-1/2} \left( \int_{\mathbb{R}} (1 + \xi^2)^{-1} d\xi \right)^{1/2} \left( \int_{\mathbb{R}} (1 + \xi^2) |\hat{\phi}(\xi)|^2 d\xi \right)^{1/2} \\ &= 2^{-1/2} \left( \int_{\mathbb{R}} (1 + \xi^2) |\hat{\phi}(\xi)|^2 d\xi \right)^{1/2}, \end{aligned}$$

and using (6) and the fact that  $|\hat{\phi}|_{L^2} = |\phi|_{L^2}$ ,

$$\begin{aligned} \|\phi\|_{\infty}^2 &\leq 2^{-1} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 d\xi + 2^{-1} \int_{\mathbb{R}} \xi^2 |\hat{\phi}(\xi)|^2 d\xi \\ &= 2^{-1} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 d\xi + 2^{-1} \int_{\mathbb{R}} |(\mathcal{F}\phi')(\xi)|^2 d\xi \\ &= 2^{-1} |\phi|_{L^2}^2 + 2^{-1} |\phi'|_{L^2}^2, \end{aligned}$$

<sup>8</sup>Walter Rudin, *Functional Analysis*, second ed., p. 186, Theorem 7.7.

and therefore

$$\|\phi\|_\infty \leq 2^{-1/2}(|\phi|_{L^2} + |\phi'|_{L^2}). \quad (8)$$

We remind ourselves that

$$A = -D^2 + M^2 + 1, \quad B = D + M, \quad C = -D + M.$$

Using

$$\mathcal{F}D = iM\mathcal{F}, \quad D\mathcal{F} = \frac{1}{i}\mathcal{F}M,$$

we get

$$\begin{aligned} \mathcal{F}A &= \mathcal{F}(-D^2 + M^2 + 1) \\ &= -(iM\mathcal{F})D + (iD\mathcal{F})M + \mathcal{F} \\ &= -iM(iM\mathcal{F}) + iD(iD\mathcal{F}) + \mathcal{F} \\ &= M^2\mathcal{F} - D^2\mathcal{F} + \mathcal{F} \\ &= A\mathcal{F}, \end{aligned}$$

and

$$\mathcal{F}B = \mathcal{F}(D + M) = iM\mathcal{F} + iD\mathcal{F} = iB\mathcal{F}$$

and

$$\mathcal{F}C = \mathcal{F}(-D + M) = -iM\mathcal{F} + iD\mathcal{F} = -iC\mathcal{F}.$$

We now determine the Fourier transform of the Hermite functions.

**Theorem 2.** For  $n \geq 0$ ,

$$\mathcal{F}h_n = (-i)^n h_n.$$

*Proof.* For  $n \geq 0$ , by induction, from  $\mathcal{F}C = -iC\mathcal{F}$  we get

$$\mathcal{F}C^n = (-iC)^n \mathcal{F}.$$

From (4),

$$h_n = \pi^{-1/4}(2^n n!)^{-1/2} C^n (e^{-x^2/2}).$$

Writing  $g(x) = e^{-x^2/2}$ , it is a fact that

$$\mathcal{F}g = g,$$

and using this with the above yields

$$\begin{aligned} \mathcal{F}h_n &= \pi^{-1/4}(2^n n!)^{-1/2} \mathcal{F}C^n g \\ &= \pi^{-1/4}(2^n n!)^{-1/2} (-iC)^n \mathcal{F}g \\ &= \pi^{-1/4}(2^n n!)^{-1/2} (-iC)^n g \\ &= \pi^{-1/4}(2^n n!)^{-1/2} (-i)^n \cdot \pi^{1/4}(2^n n!)^{1/2} h_n \\ &= (-i)^n h_n. \end{aligned}$$

□



There is a unique Hilbert space isomorphism  $\mathcal{F} : L^2(\lambda) \rightarrow L^2(\lambda)$  such that  $\mathcal{F}f = \hat{f}$  for all  $f \in \mathcal{S}$ .<sup>9</sup> For  $f \in L^2(\lambda)$ ,

$$f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} h_n,$$

and then

$$\mathcal{F}f = \sum_{n=0}^{\infty} (f, h_n)_{L^2} \mathcal{F}h_n = \sum_{n=0}^{\infty} (f, h_n)_{L^2} (-i)^n h_n.$$

## 8 Asymptotics

For  $x = 0$ , (1) reads

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(0) z^n = \exp(-z^2) = \sum_{n=0}^{\infty} \frac{(-z^2)^n}{n!},$$

thus

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0.$$

Similarly, taking the derivative of (1) with respect to  $x$  yields

$$H'_{2n}(0) = 0, \quad H'_{2n+1}(0) = 2(-1)^n \frac{(2n+1)!}{n!}.$$

For  $u(x) = e^{-x^2/2} H_n(x)$ ,<sup>10</sup>

$$u'(x) = -xu + e^{-x^2/2} H'_n(x), \quad u''(x) = -u - xu' - xe^{-x^2/2} H'_n(x) + e^{-x^2/2} H''_n(x).$$

Using

$$H'_n(x) = 2xH_n(x) - H_{n+1}(x), \quad H'_n(x) = 2nH_{n-1}(x)$$

we get

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0,$$

and thence

$$u'' = -u + x^2u - 2nu.$$

Thus, writing  $f(x) = x^2u(x)$ ,  $u$  satisfies the initial value problem

$$v'' + (2n+1)v = f, \quad v(0) = H_n(0), \quad v'(0) = H'_n(0). \quad (9)$$

Now, for  $\lambda > 0$ , two linearly independent solutions of  $v'' + \lambda v = 0$  are  $v_1(x) = \cos(\lambda^{1/2}x)$  and  $v_2(x) = \sin(\lambda^{1/2}x)$ . The Wronskian of  $(v_1, v_2)$  is  $W = \lambda^{1/2}$ , and

<sup>9</sup>Walter Rudin, *Functional Analysis*, second ed., p. 188, Theorem 7.9.

<sup>10</sup>N. N. Lebedev, *Special Functions and Their Applications*, p. 66, §4.14.

using variation of parameters, if  $v$  satisfies  $v'' + \lambda v = g$  then there are  $c_1, c_2$  such that

$$v(x) = c_1 v_1 + c_2 v_2 + A v_1 + B v_2,$$

where

$$A(x) = - \int_0^x \frac{1}{W} v_2(t) g(t) dt, \quad B(x) = \int_0^x \frac{1}{W} v_1(t) g(t) dt.$$

We calculate that the unique solution of the initial value problem  $v'' + \lambda v = g$ ,  $v(0) = a$ ,  $v'(0) = b$ , is

$$\begin{aligned} v(x) &= a v_1(x) + b \lambda^{-1/2} v_2(x) \\ &\quad - \lambda^{-1/2} v_1(x) \int_0^x v_2(t) g(t) dt + \lambda^{-1/2} v_2(x) \int_0^x v_1(t) g(t) dt \\ &= a \cos(\lambda^{1/2} x) + b \lambda^{-1/2} \sin(\lambda^{1/2} x) \\ &\quad + \lambda^{-1/2} \int_0^x (\cos(\lambda^{1/2} t) \sin(\lambda^{1/2} x) - \sin(\lambda^{1/2} t) \cos(\lambda^{1/2} x)) g(t) dt \\ &= a \cos(\lambda^{1/2} x) + b \lambda^{-1/2} \sin(\lambda^{1/2} x) + \lambda^{-1/2} \int_0^x \sin(\lambda^{1/2}(x-t)) g(t) dt. \end{aligned}$$

Therefore the unique solution of the initial value problem (9) is

$$\begin{aligned} v(x) &= H_n(0) \cos((2n+1)^{1/2} x) + H'_n(0) (2n+1)^{-1/2} \sin((2n+1)^{1/2} x) \\ &\quad + (2n+1)^{-1/2} \int_0^x \sin((2n+1)^{1/2}(x-t)) \cdot t^2 u(t) dt, \end{aligned}$$

where  $u(x) = e^{-x^2/2} H_n(x)$ . If  $n = 2k$  then

$$\begin{aligned} v(x) &= (-1)^k \frac{(2k)!}{k!} \cos((4k+1)^{1/2} x) \\ &\quad + (4k+1)^{-1/2} \int_0^x \sin((4k+1)^{1/2}(x-t)) \cdot t^2 u(t) dt \\ &= (-1)^k \frac{(2k)!}{k!} \cos((4k+1)^{1/2} x) + (4k+1)^{-1/2} r_{2k}(x). \end{aligned}$$

We calculate

$$\begin{aligned} |r_{2k}(x)|^2 &\leq \left( \int_0^{|x|} t^4 dt \right) \left( \int_0^{|x|} |u(t)|^2 dt \right) \\ &\leq \frac{|x|^5}{10} \cdot \int_{\mathbb{R}} e^{-t^2} |H_{2k}(t)|^2 dt \\ &= \frac{|x|^5}{10} \cdot 2^{2k} (2k)! \sqrt{\pi}, \end{aligned}$$

i.e.

$$|r_{2k}(x)| \leq \pi^{1/4} \frac{|x|^{5/2}}{\sqrt{10}} 2^k \sqrt{(2k)!}.$$

By Stirling's approximation,

$$\frac{2^k \sqrt{(2k)!}}{\frac{(2k)!}{k!}} = \frac{2^k k!}{\sqrt{(2k)!}} \sim \frac{2^k (2\pi k)^{1/2} k^k e^{-k}}{((4\pi k)^{1/2} (2k)^{2k} e^{-2k})^{1/2}} = \pi^{1/4} k^{1/4}.$$

Thus for  $\alpha_{2k} = \frac{(2k)!}{k!}$ ,

$$\frac{|r_{2k}(x)|}{\alpha_{2k}} = O(|x|^{5/2} \cdot k^{1/4} \cdot k^{-1/2}) = O(|x|^{5/2} k^{-1/4}).$$

Thangavelu states the following inequality and asymptotics without proof, and refers to Szegő and Muckenhoupt.<sup>11</sup>

**Lemma 3.** There are  $\gamma, C, \epsilon > 0$  such that for  $N = 2n + 1$ ,

$$\begin{aligned} |h_n(x)| &\leq C(N^{1/3} + |x^2 - N|)^{-1/4}, & x^2 \leq 2N \\ &\leq C e^{-\gamma x^2}, & x^2 > 2N, \end{aligned}$$

and

$$|h_n(x)| \leq N^{-1/8} (x - N^{1/2})^{-1/4} e^{-\epsilon N^{1/4} (x - N^{1/2})^{3/2}}$$

for  $N^{1/2} + N^{-1/6} \leq x \leq (2N)^{1/2}$ .

**Lemma 4.** For  $N = 2n + 1$ ,  $0 \leq x \leq N^{1/2} - N^{-1/6}$ , and  $\theta = \arccos(xN^{-1/2})$ ,

$$h_n(x) = \left(\frac{2}{\pi}\right)^{1/2} (N - x^2)^{-1/4} \cos\left(\frac{N(2\theta - \sin\theta) - \pi}{4}\right) + O(N^{1/2}(N - x^2)^{-7/4}).$$

**Theorem 5.** 1.  $\|h_n\|_p \asymp n^{\frac{1}{2p} - \frac{1}{4}}$  for  $1 \leq p < 4$ .

2.  $\|h_n\|_p \asymp n^{-\frac{1}{8}} \log n$  for  $p = 4$ .

3.  $\|h_n\|_p \asymp n^{-\frac{1}{6p} - \frac{1}{12}}$  for  $4 < p \leq \infty$ .

Rather than taking the  $p$ th power of  $h_n$ , one can instead take the  $p$ th power of  $H_n$  and integrate this with respect to Gaussian measure. Writing  $d\gamma(x) = (2\pi)^{-1/2} e^{-x^2/2} dx$  and taking  $H_n$  to be the Hermite polynomial that is monic, now write

$$\|H_n\|_p^p = \int_{\mathbb{R}} |H_n|^p d\gamma.$$

Larsson-Cohn<sup>12</sup> proves that for  $0 < p < 2$  there is an explicit  $c(p)$  such that

$$\|H_n\|_p = \frac{c(p)}{n^{1/4}} \sqrt{n!} (1 + O(n^{-1})),$$

<sup>11</sup>Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, pp. 26–27, Lemma 1.5.1 and Lemma 1.5.2; Gábor Szegő, *Orthogonal Polynomials*; Benjamin Muckenhoupt, *Mean convergence of Hermite and Laguerre series. II*, Trans. Amer. Math. Soc. **147** (1970), 433–470, Lemma 15.

<sup>12</sup>Lars Larsson-Cohn,  *$L^p$ -norms of Hermite polynomials and an extremal problem on Wiener chaos*, Ark. Mat. **40** (2002), 134–144.

and for  $2 < p < \infty$  there is an explicit  $c(p)$  such that

$$\|H_n\|_p = \frac{c(p)}{n^{1/4}} \sqrt{n!} (p-1)^{n/2} (1 + O(n^{-1})).$$

This uses the asymptotic expansion of Plancherel and Rotach.<sup>13</sup>

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<sup>13</sup>M. Plancherel and W. Rotach, *Sur les valeurs asymptotiques des polynomes d'Hermite*, *Commentarii mathematici Helvetici* **1** (1929), 227–254.