The Heisenberg group and Hermite functions

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1 The Heisenberg group

For $(z,t),(w,s) \in \mathbb{C}^n \times \mathbb{R}$, define the operation

$$(z,t)(w,s) = \left(z+w, t+s+\frac{1}{2}\mathrm{Im}\,(z\cdot\overline{w})\right),$$

which satisfies

$$(z,t)(0,0) = (z,t),$$

and because $\mathrm{Im}\,(z\cdot\overline{z}) = 0$,

$$(z,t)^{-1} = (-z,-t).$$

We denote $\mathbb{C}^n \times \mathbb{R}$ with this operation by H^n. This is a Lie group of dimension $2n+1$, called the **Heisenberg group**.

Writing $z = x+iy$ define

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \qquad 1 \le j \le n,$$

and

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}, \qquad 1 \le j \le n,$$

and

$$T = \frac{\partial}{\partial t}.$$

We calculate the Lie brackets of these vector fields. For X_j and X_k,

$$\begin{aligned} X_jX_k &= \left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x_k} - \frac{1}{2}y_k\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial x_j\partial x_k} - \frac{1}{2}y_k\frac{\partial}{\partial x_j}\frac{\partial}{\partial t} - \frac{1}{2}y_j\frac{\partial}{\partial t}\frac{\partial}{\partial x_k} + \frac{1}{4}y_jy_k\frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[X_j, X_k] = X_jX_k - X_kX_j = 0.$$

For Y_j and Y_k,

$$\begin{aligned} Y_jY_k &= \left(\frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y_k} + \frac{1}{2}x_k\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial y_j \partial y_k} + \frac{1}{2}x_k\frac{\partial}{\partial y_j}\frac{\partial}{\partial t} + \frac{1}{2}x_j\frac{\partial}{\partial t}\frac{\partial}{\partial y_k} + \frac{1}{4}x_jx_k\frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[Y_j, Y_k] = Y_jY_k - Y_kY_j = 0.$$

For X_j and Y_j,

$$\begin{aligned} X_jY_j &= \left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial x_j \partial y_j} + \frac{1}{2}\frac{\partial}{\partial t} + \frac{1}{2}x_j\frac{\partial}{\partial x_j}\frac{\partial}{\partial t} - \frac{1}{2}y_j\frac{\partial}{\partial t}\frac{\partial}{\partial y_j} - \frac{1}{4}y_jx_j\frac{\partial^2}{\partial t^2}, \end{aligned}$$

and

$$\begin{aligned} Y_jX_j &= \left(\frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial y_j \partial x_j} - \frac{1}{2}\frac{\partial}{\partial t} - \frac{1}{2}y_j\frac{\partial}{\partial y_j}\frac{\partial}{\partial t} + \frac{1}{2}x_j\frac{\partial}{\partial t}\frac{\partial}{\partial x_j} - \frac{1}{4}x_jy_j\frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[X_j, Y_j] = X_jY_j - Y_jX_j = \frac{\partial}{\partial t} = T.$$

For X_j and Y_k with $j \neq k$,

$$\begin{aligned} X_jY_k &= \left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y_k} + \frac{1}{2}x_k\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial x_j \partial y_k} + \frac{1}{2}x_k\frac{\partial}{\partial x_j}\frac{\partial}{\partial t} - \frac{1}{2}y_j\frac{\partial}{\partial t}\frac{\partial}{\partial y_k} - \frac{1}{4}y_jx_k\frac{\partial^2}{\partial t^2} \end{aligned}$$

and

$$\begin{aligned} Y_kX_j &= \left(\frac{\partial}{\partial y_k} + \frac{1}{2}x_k\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right) \\ &= \frac{\partial^2}{\partial y_k \partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial y_k}\frac{\partial}{\partial t} + \frac{1}{2}x_k\frac{\partial}{\partial t}\frac{\partial}{\partial x_j} - \frac{1}{4}x_ky_j\frac{\partial^2}{\partial t^2}, \end{aligned}$$

yielding

$$[X_j, Y_k] = 0.$$

For X_j and T,

$$X_jT = \left(\frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial t} - \frac{1}{2}y_j\frac{\partial^2}{\partial t^2} = TX_j,$$

yielding

$$[X_j, T] = 0.$$

For Y_j and T,

$$Y_j T = \left(\frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t} \right) \frac{\partial}{\partial t} = \frac{\partial}{\partial y_j} \frac{\partial}{\partial t} + \frac{1}{2} x_j \frac{\partial^2}{\partial t^2} = T Y_j,$$

yielding

$$[Y_j, T] = 0.$$

We summarize the above calculations in the following theorem.

Theorem 1. The Lie brackets of the vector fields X_j, Y_j, $1 \leq j \leq n$, and T are:

- $[X_j, X_k] = 0$
- $[Y_j, Y_k] = 0$
- $[X_j, Y_j] = T$
- $[X_j, Y_k] = 0$ for $j \neq k$
- $[X_j, T] = 0$
- $[Y_j, T] = 0$

The Lie algebra of the H^n is called the **Heisenberg Lie algebra** and is denoted $\mathfrak{h}^n$. The above vector fields are left-invariant and are a basis for $\mathfrak{h}^n$.[1]

2 Representation theory

For a Hilbert space H, we denote by $\mathscr{B}(H)$ the set of bounded linear operators $H \to H$, which is a Banach algebra with the operator norm. We denote by $\mathscr{B}_0(H)$ the set of compact operators $H \to H$, which is a closed ideal of the Banach algebra $\mathscr{B}(H)$. We denote by $\mathscr{B}_{\text{HS}}(H)$ the collection of Hilbert-Schmidt operators $H \to H$: if $\{e_i : i \in I\}$ is an orthonormal basis of H, a linear map $A : H \to H$ is called a **Hilbert-Schmidt operator** if

$$\|A\|_{\text{HS}}^2 = \sum_{i \in I} \|Ae_i\|^2 < \infty.$$

This satisfies $\|A\| \leq \|A\|_{\text{HS}}$. A Hilbert-Schmidt operator is a compact operator. A linear map $U : H \to H$ is called a **unitary operator** if it is a bijection and satisfies

$$\langle Ux, Uy \rangle = xy, \qquad x, y \in H.$$

We denote the set of unitary operators $H \to H$ by $\mathscr{U}(H)$.

[1] Sundaram Thangavelu, *An Introduction to the Uncertainty Principle: Hardy's Theorem on Lie Groups*, p. 47, §2.1.

For $\lambda \in \mathbb{R}, \lambda \neq 0$, for $(x+iy,t) \in H^n$, and for $f \in L^2(\mathbb{R}^n)$, define

$$\pi_\lambda(x+iy,t)f(\xi) = e^{i\lambda t}e^{i\lambda\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}f(\xi+y), \qquad \xi \in \mathbb{R}^n.$$

It is apparent that $\pi_\lambda(z,t)$ is a linear map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

For $(x+iy,t),(u+iv,s) \in H^n$ we calculate

$$\begin{aligned}\pi_\lambda(x+iy,t)\pi_\lambda(u+iv,s)f(\xi) &= \pi_\lambda(x+iy,t)e^{i\lambda s}e^{i\lambda\left(u\cdot\xi+\frac{1}{2}u\cdot v\right)}f(\xi+v)\\ &= e^{i\lambda t}e^{i\lambda\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}e^{i\lambda s}e^{i\lambda\left(u\cdot(\xi+y)+\frac{1}{2}u\cdot v\right)}f(\xi+y+v)\\ &= e^{i\lambda(t+s)}e^{i\lambda\left((x+u)\cdot\xi+\frac{1}{2}x\cdot y+u\cdot y+\frac{1}{2}u\cdot v\right)}f(\xi+y+v).\end{aligned}$$

On the other hand, with $z = x+iy$ and $w = u+iv$,

$$\begin{aligned}(z,t)(w,s) &= \left(z+w,t+s+\frac{1}{2}\mathrm{Im}\,(z\cdot\overline{w})\right)\\ &= \left(x+iy+u+iv,t+s+\frac{1}{2}\mathrm{Im}\,((x+iy)\cdot(u-iv))\right)\\ &= \left(x+u+i(y+v),t+s+\frac{1}{2}\mathrm{Im}\,(x\cdot u-ix\cdot v+iy\cdot u+y\cdot v)\right)\\ &= \left(x+u+i(y+v),t+s-\frac{1}{2}x\cdot v+\frac{1}{2}y\cdot u\right),\end{aligned}$$

for which

$$\begin{aligned}\pi_\lambda((z,t)(w,s))f(\xi) &= e^{i\lambda\left(t+s-\frac{1}{2}x\cdot v+\frac{1}{2}y\cdot u\right)}e^{i\lambda\left((x+u)\cdot\xi+\frac{1}{2}(x+u)\cdot(y+v)\right)}f(\xi+y+v)\\ &= e^{i\lambda(t+s)}e^{i\lambda\left((x+u)\cdot\xi+\frac{1}{2}x\cdot y+y\cdot u+\frac{1}{2}u\cdot v\right)}f(\xi+y+v),\end{aligned}$$

and therefore

$$\pi_\lambda(x+iy,t)\pi_\lambda(u+iv,s) = \pi_\lambda((z,t)(w,s)).$$

We calculate

$$\pi_\lambda(0,0)f(\xi) = f(\xi)$$

and

$$\pi_\lambda(x+iy,t)\pi_\lambda((x+iy,t)^{-1})f = \pi_\lambda(0,0)f = f.$$

For $f,g \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}&\langle\pi_\lambda(x+iy,t)f,\pi_\lambda(x+iy,t)g\rangle\\ &\int_{\mathbb{R}^n}\pi_\lambda(x+iy,t)f(\xi)\overline{\pi_\lambda(x+iy,t)g(\xi)}d\xi\\ &=\int_{\mathbb{R}^n}e^{i\lambda t}e^{i\lambda\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}f(\xi+y)e^{-i\lambda t}e^{-i\lambda\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}\overline{g(\xi+y)}d\xi\\ &=\int_{\mathbb{R}^n}f(\xi+y)\overline{g(\xi+y)}d\xi\\ &=\langle f,g\rangle.\end{aligned}$$

Therefore $\pi_\lambda(z,t)$ is a unitary operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and

$$\pi_\lambda : H^n \to \mathscr{U}(L^2(\mathbb{R}^n))$$

is a group homomorphism, namely, π_λ is a unitary representation of H^n on $L^2(\mathbb{R}^n)$.[2] Furthermore, using that $y \mapsto f(\cdot + y)$ is continuous $\mathbb{R}^n \to L^2(\mathbb{R}^n)$,

$$\|\pi_\lambda(x+iy,t)f - f\|^2 = \int_{\mathbb{R}^n} |e^{i\lambda t} e^{i\lambda\left(x\cdot\xi + \frac{1}{2}x\cdot y\right)} f(\xi+y) - f(\xi)|^2 d\xi \to 0$$

as $(z,t) \to 0$, showing that $\pi_\lambda : H^n \to \mathscr{U}(L^2(\mathbb{R}^n))$ is **strongly continuous**. (That is, it is continuous when $\mathscr{U}(L^2(\mathbb{R}^n))$ is assigned the strong operator topology.)

Theorem 2. For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the map π_λ defined by

$$\pi_\lambda(x+iy,t)f(\xi) = e^{i\lambda t} e^{i\lambda\left(x\cdot\xi + \frac{1}{2}x\cdot y\right)} f(\xi+y),$$

for $(x+iy,t) \in H^n$, $f \in L^2(\mathbb{R}^n)$, and $\xi \in \mathbb{R}^n$, is a strongly continuous unitary representation of H^n on $L^2(\mathbb{R}^n)$.

We call π_1 the **Schrödinger representation**. Its kernel is

$$\Gamma = \{(0, 2\pi k) : k \in \mathbb{Z}\}.$$

For $f \in L^1(H^n/\Gamma)$ we define

$$\pi_1(f) = \int_{H^n/\Gamma} f(z,t)\pi_1(z,t)dzdt.$$

For $f, g \in L^1(H^n/\Gamma)$,

$$(f * g)(z,t) = \int_{H^n/\Gamma} f((z,t)\cdot(w,s)^{-1})g(w,s)dwds, \qquad (z,t) \in H^n/\Gamma.$$

It is a fact that Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ is a bi-invariant Haar measure on H^n, and using this we calculate

$$\begin{aligned}
&\pi_1(f*g) \\
=&\int_{H^n/\Gamma} \left(\int_{H^n/\Gamma} f((z,t)\cdot(w,s)^{-1})g(w,s)dwds \right) \pi_1(z,t)dzdt \\
=&\int_{H^n/\Gamma} g(w,s) \left(\int_{H^n/\Gamma} f((z,t)\cdot(w,s)^{-1})\pi_1((z,t)\cdot(w,s)^{-1})dzdt \right) \pi_1(w,s)dwds \\
=&\int_{H^n/\Gamma} g(w,s)\pi_1(f)dwds \\
=&\pi_1(f)\pi_1(g).
\end{aligned}$$

[2] cf. `https://www.math.ubc.ca/~cass/research/pdf/Unitary.pdf`

Lemma 3. For $f, g \in L^1(H^n/\Gamma)$,

$$\pi_1(f * g) = \pi_1(f)\pi_1(g).$$

We define

$$W(z) = \pi_1(z, 0),$$

with which

$$\pi_1(z, t) = e^{it} W(z).$$

Define

$$f_1(z) = (2\pi)^{-1/2} \int_0^{2\pi} f(z,t) e^{it} dt.$$

Then

$$\begin{aligned}
\pi_1(f) &= \int_{H^n/\Gamma} f(z,t) e^{it} W(z) dz dt \\
&= \int_{\mathbb{C}^n} W(z) \left(\int_0^{2\pi} f(z,t) e^{it} dt \right) dz \\
&= (2\pi)^{1/2} \int_{\mathbb{C}^n} f_1(z) W(z) dz.
\end{aligned}$$

For $f \in L^1(\mathbb{C}^n)$, define

$$f^\#(z,t) = (2\pi)^{-1} e^{-it} f(z).$$

$f^\# \in L^1(H^n/\Gamma)$, and

$$f_1^\#(z) = (2\pi)^{-1/2} \int_0^{2\pi} f^\#(z,t) e^{it} dt = (2\pi)^{-1/2} f(z),$$

thus

$$\pi_1(f^\#) = (2\pi)^{1/2} \int_{\mathbb{C}^n} f^\#(z) W(z) dz = \int_{\mathbb{C}^n} f(z) W(z) dz.$$

We define $W : L^1(\mathbb{C}^n) \to \mathscr{U}(L^2(\mathbb{R}^n))$ by

$$W(f) = \pi_1(f^\#),$$

called the **Weyl transform**.

For $f, g \in L^1(\mathbb{C}^n)$ and for $(z,t) \in H^n/\Gamma$,

$$\begin{aligned}
&(f^{\#} * g^{\#})(z,t) \\
=&\int_{H^n/\Gamma} f^{\#}((z,t)\cdot(w,s)^{-1})g^{\#}(w,s)dwds \\
=&\int_{H^n/\Gamma} f^{\#}((z,t)\cdot(-w,-s))g^{\#}(w,s)dwds \\
=&\int_{H^n/\Gamma} f^{\#}\left(z-w, t-s-\frac{1}{2}\mathrm{Im}\,(z\cdot\overline{w})\right)g^{\#}(w,s)dwds \\
=&\int_{H^n/\Gamma} (2\pi)^{-2}e^{-i\left(t-s-\frac{1}{2}\mathrm{Im}\,(z\cdot\overline{w})\right)}f(z-w)e^{-is}g(w)dwds \\
=&(2\pi)^{-1}e^{-it}\int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\mathrm{Im}\,(z\cdot\overline{w})}dw \\
=&(f\times g)^{\#}(z,t),
\end{aligned}$$

for

$$(f\times g)(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\mathrm{Im}\,(z\cdot\overline{w})}dw,$$

called the **twisted convolution**. Using what we have established so far gives the following.

Lemma 4. For $f, g \in L^1(\mathbb{C}^n)$,

$$W(f\times g) = \pi_1((f\times g)^{\#}) = \pi_1(f^{\#} * g^{\#}) = \pi_1(f^{\#})\pi_1(g^{\#}) = W(f)W(g)$$

For $\phi \in L^1(\mathbb{C}^n)$, we define

$$K_\phi(\xi,\eta) = \int_{\mathbb{R}^n} \phi(x+i(\eta-\xi))e^{\frac{i}{2}(\xi+\eta)\cdot x}dx, \qquad (\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n,$$

which satisfies, for $f \in L^2(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
W(\phi)f(\xi) &= \int_{\mathbb{C}^n}\phi(z)W(z)f(\xi)dz \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(x+iy)e^{i\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}f(\xi+y)dydx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(x+i(y-\xi))e^{\frac{i}{2}(x\cdot\xi+x\cdot y)}f(y)dydx \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\phi(x+i(y-\xi))e^{\frac{i}{2}(\xi+y)\cdot x)}dx\right)f(y)dy \\
&= \int_{\mathbb{R}^n}K_\phi(\xi,y)f(y)dy.
\end{aligned}$$

Thus K_ϕ is an integral kernel for the operator $W(\phi)$.

We show in the following theorem that the Weyl transform sends elements of $L^1(\mathbb{C}^n)$ to compact operators on $L^2(\mathbb{R}^n)$, and that it sends square integrable functions to Hilbert-Schmidt operators.[3]

Theorem 5. $W : L^1(\mathbb{C}^n) \to \mathscr{B}_0(L^2(\mathbb{R}^n))$, and for $\phi \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$ we have $W(\phi) \in \mathscr{B}_{\mathrm{HS}}(L^2(\mathbb{R}^n))$ and

$$\|\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \|W(\phi)\|_{\mathrm{HS}}.$$

Proof. First take $\phi \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n)$. It follows from this that $K_\phi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, and because K_ϕ is the integral kernel of $W(\phi)$ this implies[4] that $W(\phi) \in \mathscr{B}_{\mathrm{HS}}(L^2(\mathbb{R}^n))$ and

$$\|W(\phi)\|_{\mathrm{HS}}^2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} |K(\xi, \eta)|^2 d\xi d\eta.$$

□

3 Hermite functions

For $\phi \in \mathscr{S}(\mathbb{R}^n)$, define

$$\widehat{\phi}(\xi) = (\mathscr{F}\phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(x) e^{-ix\cdot\xi} dx, \qquad \xi \in \mathbb{R}^n.$$

$\mathscr{S}(\mathbb{R}^n)$ is a dense linear subspace of $L^2(\mathbb{R}^n)$, and the Fourier transform extends to a unique Hilbert space isomorphism $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. For $f, g \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx.$$

For $\phi \in \mathscr{S}(\mathbb{R})$, let

$$(D\phi)(x) = \phi'(x), \qquad (M\phi)(x) = x\phi(x), \qquad x \in \mathbb{R},$$

and let

$$A = -D + M, \qquad B = D + M.$$

Let

$$H = \sum_{j=1}^n (-D_j^2 + M_j^2) = \frac{1}{2} \sum_{j=1}^n (A_j B_j + B_j A_j),$$

which satisfies

$$(H\phi)(x) = -(\Delta\phi)(x) + |x|^2 \phi(x),$$

[3] Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, p. 13, Theorem 1.2.1.

[4] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics, volume I: Functional Analysis*, revised and enlarged edition, p. 210, Theorem VI.23.

called the **Hermite operator**.

For $k \geq 0$, define

$$H_k(x) = (-1)^k e^{x^2} D^k e^{-x^2}$$

and

$$h_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_k(x).$$

The Hermite functions are an orthonormal basis for $L^2(\mathbb{R})$. Let $\mathbb{N}$ be the nonnegative integers, and for $\alpha \in \mathbb{N}^n$ let

$$\Phi_\alpha = h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_n},$$

which are an orthonormal basis for $L^2(\mathbb{R}^n)$. It is a fact that

$$A_j \Phi_\alpha = (2\alpha_j + 2)^{1/2} \Phi_{\alpha+e_j}, \qquad B_j \Phi_\alpha = (2\alpha_j)^{1/2} \Phi_{\alpha-e_j}$$

and

$$H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha.$$

It is a fact that

$$\widehat{h}_k = (-i)^k h_k,$$

whence

$$\widehat{\Phi}_\alpha = (-i)^{|\alpha|} \Phi_\alpha.$$

Because $\{\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, for $f \in L^2(\mathbb{R}^n)$,

$$f = \sum_\alpha \langle f, \Phi_\alpha \rangle \Phi_\alpha.$$

and then

$$\widehat{f} = \sum_\alpha \langle f, \Phi_\alpha \rangle (-i)^{|\alpha|} \Phi_\alpha.$$

Let E_k be the linear span of $\{\Phi_\alpha : |\alpha| = k\}$, which has dimension $\binom{k+n-1}{k}$. For $f \in E_k$, $Hf = (2k+n)f$. Let $P_k : L^2(\mathbb{R}^n) \to E_k$ be the projection:

$$P_k f = \sum_{|\alpha|=k} \langle f, \Phi_\alpha \rangle \Phi_\alpha, \qquad f \in L^2(\mathbb{R}^n).$$

Let

$$\Phi_k(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y), \qquad x, y \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$ we calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi_k(x,y) f(y) dy &= \sum_{|\alpha|=k} \Phi_\alpha(y) \int_{\mathbb{R}^n} f(y)\Phi_\alpha(y) dy \\
&= \sum_{|\alpha|=k} \Phi_\alpha(y) \langle f, \Phi_\alpha \rangle \\
&= (P_k f)(y),
\end{aligned}$$

thus Φ_k is a kernel for the projection operator P_k.

Using the 1-dimensional Mehler's formula we obtain the n-dimensional Mehler's formula:

$$\sum_\alpha r^{|\alpha|}\Phi_\alpha(x)\Phi_\alpha(y) = \pi^{-\frac{n}{2}}(1-r^2)^{-\frac{n}{2}}\exp\left(-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2r}{1-r^2}x\cdot y\right).$$

4 Special Hermite functions

We first define the **Fourier-Wigner transform**. For $f, g \in L^2(\mathbb{R}^n)$ and $z = x + iy \in \mathbb{C}^n$,

$$V(f,g)(z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f\left(\xi+\frac{1}{2}y\right)\overline{g\left(\xi-\frac{1}{2}y\right)}d\xi.$$

The following theorem relates the inner product on $L^2(\mathbb{R}^n)$ and the inner product on $L^2(\mathbb{C}^n)$.[5]

Theorem 6. For $f, g, \phi, \psi \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{C}^n} V(f,g)(z)\overline{V(\phi,\psi)(z)}dz = \langle f,\phi\rangle\langle\psi,g\rangle.$$

We now define the **special Hermite functions** on $\mathbb{C}^n$. For $\alpha, \beta \in \mathbb{N}^n$, let

$$\Phi_{\alpha\beta}(z) = V(\Phi_\alpha, \Phi_\beta)(z).$$

We calculate

$$\begin{aligned}\langle W(z)\Phi_\alpha, \Phi_\beta\rangle &= \int_{\mathbb{R}^n} W(z)\Phi_\alpha(\xi)\Phi_\beta(\xi)d\xi \\ &= \int_{\mathbb{R}^n} e^{i\left(x\cdot\xi+\frac{1}{2}x\cdot y\right)}\Phi_\alpha(\xi+y)\Phi_\beta(\xi)d\xi \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}\Phi_\alpha\left(\xi+\frac{1}{2}y\right)\Phi_\beta\left(\xi-\frac{1}{2}y\right)d\xi \\ &= (2\pi)^{n/2}V(\Phi_\alpha,\Phi_\beta).\end{aligned}$$

Lemma 7. For $\alpha, \beta \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$,

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-n/2}\langle W(z)\Phi_\alpha, \Phi_\beta\rangle.$$

Using that the Hermite functions Φ_α are an orthonormal basis for $L^2(\mathbb{R}^n)$, it is proved that the special Hermite functions $\Phi_{\alpha\beta}$ are an orthonormal basis for $L^2(\mathbb{C}^n)$.[6]

[5] Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, p. 14, Proposition 1.3.1.

[6] Sundaram Thangavelu, *Lectures on Hermite and Laguerre Expansions*, p. 16, Theorem 1.3.2.