# The heat kernel on the torus

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## 1 Heat kernel on $\mathbb{T}$

For $t > 0$, define $k_t : \mathbb{R} \to (0, \infty)$ by[1]

$$k_t(x) = (4\pi t)^{-1/2} \exp\left(-\frac{x^2}{4t}\right), \qquad x \in \mathbb{R}.$$

For $t > 0$, define $g_t : \mathbb{R} \to (0, \infty)$ by

$$g_t(x) = 2\pi \sum_{k \in \mathbb{Z}} k_t(x + 2\pi k), \qquad x \in \mathbb{R},$$

which one checks indeed converges for all $x \in \mathbb{R}$. Of course, $g_t(x + 2\pi k) = g_t(x)$ for any $k \in \mathbb{Z}$, so we can interpret $g_t$ as a function on $\mathbb{T}$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Let $m$ be Haar measure on $\mathbb{T}$: $dm(x) = (2\pi)^{-1}dx$, and so $m(\mathbb{T}) = 1$. With $\|f\|_1 = \int_{\mathbb{T}} |f| dm$ for $f : \mathbb{T} \to \mathbb{C}$, we have, because $g_t > 0$,

$$\|g_t\|_1 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} k_t(x + 2\pi k) dx = \int_{\mathbb{R}} k_t(x) dx = 1.$$

Hence $g_t \in L^1(\mathbb{T})$. For $\xi \in \mathbb{Z}$, we compute

$$\begin{aligned}
\hat{g}_t(\xi) &= \int_{\mathbb{T}} g_t(x) e^{-i\xi x} dm(x) \\
&= \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} k_t(x + 2\pi k) e^{-i\xi x} dx \\
&= \sum_{k \in \mathbb{Z}} \int_{\mathbb{T}} k_t(x + 2\pi k) e^{-i\xi(x + 2\pi k)} dx \\
&= \int_{\mathbb{R}} k_t(x) e^{-i\xi x} dx \\
&= \hat{k}_t\left(\frac{\xi}{2\pi}\right) \\
&= e^{-\xi^2 t}.
\end{aligned}$$

[1] Most of this note is my working through of notes by Patrick Maheux. `http://www.univ-orleans.fr/mapmo/membres/maheux/InfiniteTorusV2.pdf`

**Lemma 1.** *For $t > 0$ and $x \in \mathbb{R}$,*

$$g_t(x) = \sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{4t}\right) \left(1 + 2\sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right) \cosh\left(\frac{\pi k x}{t}\right)\right).$$

*Proof.* Using the definition of $g_t$,

$$\begin{aligned} g_t(x) &= 2\pi \sum_{k\in\mathbb{Z}} k_t(x + 2\pi k) \\ &= 2\pi \sum_{k\in\mathbb{Z}} (4\pi t)^{-1/2} \exp\left(-\frac{(x+2\pi k)^2}{4t}\right) \\ &= \sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{4t}\right) \sum_{k\in\mathbb{Z}} \exp\left(-\frac{\pi k x}{t}\right) \exp\left(-\frac{\pi^2 k^2}{t}\right) \\ &= \sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{4t}\right) \\ &\quad \left(1 + \sum_{k\geq 1} \left(\exp\left(\frac{\pi k x}{t}\right) + \exp\left(-\frac{\pi k x}{t}\right)\right) \exp\left(-\frac{\pi^2 k^2}{t}\right)\right), \end{aligned}$$

which gives the claim, using $\cosh y = \frac{e^y + e^{-y}}{2}$. □

**Definition 2.** *For $x \in \mathbb{R}$, let $\|x\| = \inf\{|x - 2\pi k| : k \in \mathbb{Z}\}$.*

For $k \in \mathbb{Z}$, $\|x + 2\pi k\| = \|x\|$, so it makes sense to talk about $\|x\|$ for $x \in \mathbb{T}$.

**Theorem 3.** *For $t > 0$ and $x \in \mathbb{R}$,*

$$\exp\left(-\frac{\|x\|^2}{4t}\right) g_t(0) \leq g_t(x) \leq \exp\left(-\frac{\|x\|^2}{4t}\right) \left(\sqrt{\frac{\pi}{t}} + g_t(0)\right).$$

*Proof.* Let $x = 2\pi m + \theta$ with $|\theta| \leq \pi$, so that $\|x\| = \|\theta\| = |\theta|$, and $g_t(x) = g_t(\theta)$. Using Lemma 1 and the fact that $\cosh y \geq 1$, we get

$$g_t(\theta) \geq \exp\left(-\frac{\theta^2}{4t}\right) \sqrt{\frac{\pi}{t}} \left(1 + 2\sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right)\right) = \exp\left(-\frac{\theta^2}{4t}\right) g_t(0),$$

hence

$$g_t(x) \geq \exp\left(-\frac{\|x\|^2}{4t}\right) g_t(0),$$

the lower bound we wanted to prove.

Write

$$S = 1 + 2\sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right) \cosh\left(\frac{\pi k \theta}{t}\right).$$

For any $k \geq 1$, using $|\theta| \leq \pi$,

$$2\cosh\left(\frac{\pi k\theta}{t}\right) \leq 2\cosh\left(\frac{\pi^2 k}{t}\right) = \exp\left(\frac{\pi^2 k}{t}\right)+\exp\left(-\frac{\pi^2 k}{t}\right) \leq 1+\exp\left(\frac{\pi^2 k}{t}\right).$$

Hence

$$\begin{aligned} S &\leq 1 + \sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right)\left(1+\exp\left(\frac{\pi^2 k}{t}\right)\right) \\ &= 1 + \sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right) + \exp\left(-\frac{\pi^2 k(k-1)}{t}\right) \\ &\leq 1 + \sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right) + \exp\left(-\frac{\pi^2 (k-1)^2}{t}\right) \\ &= 2 + 2\sum_{k\geq 1} \exp\left(-\frac{\pi^2 k^2}{t}\right) \\ &= 1 + \sqrt{\frac{t}{\pi}} g_t(0). \end{aligned}$$

But $g_t(\theta) = \sqrt{\frac{\pi}{t}} \exp\left(-\frac{\theta^2}{4t}\right) S$, so

$$g_t(\theta) \leq \exp\left(-\frac{\theta^2}{4t}\right)\left(\sqrt{\frac{\pi}{t}} + g_t(0)\right) = \exp\left(-\frac{\|x\|^2}{4t}\right)\left(\sqrt{\frac{\pi}{t}} + g_t(0)\right),$$

the upper bound we wanted to prove. □

Applying Lemma 1 with $x = 0$ gives $g_t(0) \geq \sqrt{\frac{\pi}{t}}$, and using this with the above theorem we obtain

$$g_t(x) \leq 2\exp\left(-\frac{\|x\|^2}{4t}\right) g_t(0). \tag{1}$$

**Theorem 4.** *For $t > 0$,*

$$\sqrt{\frac{\pi}{t}} \leq g_t(0) \leq 1 + \sqrt{\frac{\pi}{t}}$$

*and*

$$2e^{-t} \leq g_t(0) - 1 \leq \frac{2e^{-t}}{1-e^{-t}}.$$

*Proof.* Using Lemma 1 we have

$$g_t(0) \geq \sqrt{\frac{\pi}{t}}.$$

For each $x \in \mathbb{R}$ we have

$$g_t(x) = \sum_{k\in\mathbb{Z}} \hat{g}_t(k)e^{ikx} = \sum_{k\in\mathbb{Z}} e^{-k^2t}e^{ikx} = 1 + 2\sum_{k\geq 1} e^{-k^2t}\cos(kx).$$

Writing $\phi(t) = \sum_{k\geq 1} e^{-k^2t}$, we then have

$$g_t(0) = 1 + 2\phi(t).$$

But as $e^{-x^2t}$ is positive and decreasing, bounding a sum by an integral we get

$$\phi(t) \leq \int_0^\infty e^{-x^2t}dx = \frac{1}{\sqrt{t}}\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{t}},$$

hence

$$g_t(0) = 1 + 2\phi(t) \leq 1 + \sqrt{\frac{\pi}{t}}.$$

Moreover, because $\phi(t) \geq e^{-t}$ (lower bounding the sum by the first term), we have

$$g_t(0) = 1 + 2\phi(t) \geq 1 + 2e^{-t}.$$

Finally, because $e^{-tk^2} \leq e^{-tk}$ for $k \geq 1$,

$$\phi(t) \leq \sum_{k\geq 1} e^{-tk} = e^{-t}\frac{1}{1-e^{-t}},$$

thus

$$g_t(0) \leq 1 + \frac{2e^{-t}}{1-e^{-t}}.$$

□

Taking $t \to 0$ and $t \to \infty$ in the above theorem gives the following asymptotics.

**Corollary 5.**

$$g_t(0) \sim \sqrt{\frac{\pi}{t}}, \qquad t \to 0$$

*and*

$$g_t(0) - 1 \sim 2e^{-t}, \qquad t \to \infty.$$

# 2 Heat kernel on $\mathbb{T}^n$

Fix $n \geq 1$, and let $\mathscr{A} = (a_1, \ldots, a_n)$, $a_i$ positive real numbers. We define $g_t^{\mathscr{A}} : \mathbb{R}^n \to (0, \infty)$ by

$$g_t^{\mathscr{A}}(x) = \prod_{k=1}^n g_{a_kt}(x_k), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$ and $\xi \in \mathbb{Z}^n$ we have

$$g_t^{\mathscr{A}}(x+2\pi\xi) = \prod_{k=1}^n g_{a_k t}(x_k + 2\pi\xi_k) = \prod_{k=1}^n g_{a_k t}(x_k) = g_t^{\mathscr{A}}(x),$$

so $g_t^{\mathscr{A}}$ can be interpreted as a function on $\mathbb{T}^n$.

Let $m_n$ be Haar measure on $\mathbb{T}^n$:

$$dm_n(x) = \prod_{k=1}^n dm(x_k) = \prod_{k=1}^n (2\pi)^{-1} dx_k = (2\pi)^{-n} dx,$$

which satisfies $m_n(\mathbb{T}^n) = 1$. Define $\mu_t^{\mathscr{A}}$ to be the measure on $\mathbb{T}^n$ whose density with respect to $m_n$ is $g_t^{\mathscr{A}}$:

$$d\mu_t^{\mathscr{A}} = g_t^{\mathscr{A}} dm_n.$$

We now calculate the Fourier coefficients of $g_t^{\mathscr{A}}$. For $\xi \in \mathbb{Z}^n$,

$$\begin{aligned}
\mathscr{F}(g_t^{\mathscr{A}})(\xi) &= \int_{\mathbb{T}^n} g_t^{\mathscr{A}}(x) e^{-i\xi\cdot x} dm_n(x) \\
&= \int_{\mathbb{T}^n} \prod_{k=1}^n g_{a_k t}(x_k) e^{-i\xi_1 x_1 - \cdots - i\xi_n x_n} dm_n(x) \\
&= \prod_{k=1}^n \int_{\mathbb{T}} g_{a_k t}(x_k) e^{-i\xi_k x_k} dm(x_k) \\
&= \prod_{k=1}^n \hat{g}_{a_k t}(\xi_k) \\
&= \prod_{k=1}^n e^{-\xi_k^2 a_k t} \\
&= e^{-tq(\xi)},
\end{aligned}$$

where

$$q(\xi) = \sum_{k=1}^n a_k \xi_k^2, \qquad \xi \in \mathbb{Z}^n.$$

**Definition 6.** *For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define*

$$\|x\|_{\mathscr{A}}^2 = \frac{1}{a_1}\|x_1\|^2 + \cdots + \frac{1}{a_n}\|x_n\|^2,$$

*with $\mathscr{A} = (a_1, \ldots, a_n)$.*

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Z}^n$, because $\|x_k + 2\pi\xi_k\| = \|x_k\|$, we have $\|x + 2\pi\xi\|_{\mathscr{A}} = \|x\|_{\mathscr{A}}$, so it makes sense to talk about $\|\cdot\|_{\mathscr{A}}$ on $\mathbb{T}^n$.

Using Theorem 3 and (1) we get the following.

**Theorem 7.** *For $t > 0$ and $x \in \mathbb{R}^n$,*

$$\exp\left(-\frac{\|x\|_{\mathscr{A}}^2}{4t}\right) g_t^{\mathscr{A}}(0) \leq g_t^{\mathscr{A}}(x) \leq 2^n \exp\left(-\frac{\|x\|_{\mathscr{A}}^2}{4t}\right) g_t^{\mathscr{A}}(0).$$

Combining this with Theorem 4 we obtain the following. The first inequality is appropriate for $t \to 0^+$ and the second inequality for $t \to \infty$.

**Theorem 8.** *For $t > 0$ and $x \in \mathbb{R}^n$,*

$$\exp\left(-\frac{\|x\|_{\mathscr{A}}^2}{4t}\right) \prod_{k=1}^n \sqrt{\frac{\pi}{a_k t}} \leq g_t^{\mathscr{A}}(x) \leq 2^n \exp\left(-\frac{\|x\|_{\mathscr{A}}^2}{4t}\right) \prod_{k=1}^n \left(1 + \sqrt{\frac{\pi}{a_k t}}\right)$$

*and*

$$\exp\left(-\frac{\|x\|_{\mathscr{A}}^2}{4t}\right) \prod_{k=1}^n \left(1 + 2e^{-a_k t}\right) \leq g_t^{\mathscr{A}}(x) \leq 2^n \exp\left(-\frac{\|x\|_{\mathscr{A}}^2}{4t}\right) \prod_{k=1}^n \left(1 + \frac{2e^{-a_k t}}{1 - e^{-a_k t}}\right).$$

# 3 The infinite-dimensional torus

$\mathbb{T}^\infty$ with the product topology is a compact abelian group. Let $m_\infty$ be Haar measure on $\mathbb{T}^\infty$:

$$dm_\infty(x) = \prod_{k=1}^\infty dm(x_k), \qquad x = (x_1, x_2, \ldots) \in \mathbb{T}^\infty,$$

where $m$ is Haar measure on $\mathbb{T}$.

For $t > 0$, let $\mu_t$ be the measure on $\mathbb{T}$ whose density with respect to Haar measure $m$ is $g_t$:

$$d\mu_t = g_t dm.$$

This is a probability measure on $\mathbb{T}$.

Let $\mathscr{A} = (a_1, a_2, \ldots) \in \mathbb{N}^\infty$. For $t > 0$ we define

$$\mu_t^{\mathscr{A}} = \prod_{k=1}^\infty \mu_{a_k t}.$$

This is a probability measure on $\mathbb{T}^\infty$.[2]

The Pontryagin dual of $\mathbb{T}^\infty$ is the direct sum $\bigoplus_{k=1}^\infty \mathbb{Z}$, which we denote by $\mathbb{Z}^{(\infty)}$, which is a discrete abelian group. For $\xi \in \mathbb{Z}^{(\infty)}$ and $x \in \mathbb{T}^\infty$, we write

$$e_\xi(x) = \exp\left(i \sum_{k=1}^\infty \xi_k x_k\right).$$

[2]Christian Berg determines conditions on $\mathscr{A}$ and $t$ so that $\mu_t^{\mathscr{A}}$ is absolutely continuous with respect to Haar measure $m_\infty$ on $\mathbb{T}^\infty$: *Potential theory on the infinite dimensional torus*, Invent. Math. **32** (1976), no. 1, 49–100.

The Fourier transform of $\mu_t^{\mathscr{A}}$ is $\mathscr{F}(\mu_t^{\mathscr{A}}) : \mathbb{Z}^{(\infty)} \to \mathbb{C}$ defined by

$$\mathscr{F}(\mu_t^{\mathscr{A}})(\xi) = \int_{\mathbb{T}^\infty} e_{-\xi}(x) dm_\infty(x), \qquad \xi \in \mathbb{Z}^{(\infty)},$$

which is

$$\begin{aligned}
\int_{\mathbb{T}^\infty} e_{-\xi}(x) dm_\infty(x) &= \int_{\mathbb{T}^\infty} \exp\left(-i \sum_{k=1}^{\infty} \xi_k x_k\right) d\mu_t^{\mathscr{A}}(x) \\
&= \int_{\mathbb{T}^\infty} \prod_{k=1}^{\infty} \exp(-i\xi_k x_k) d\mu_t^{\mathscr{A}}(x) \\
&= \prod_{k=1}^{\infty} \int_{\mathbb{T}} \exp(-i\xi_k x_k) g_{a_k t}(x_k) dm(x_k) \\
&= \prod_{k=1}^{\infty} \hat{g}_{a_k t}(\xi_k) \\
&= \prod_{k=1}^{\infty} \exp(-\xi_k^2 a_k t) \\
&= \exp\left(-t \sum_{k=1}^{\infty} a_k \xi_k^2\right).
\end{aligned}$$

# 4 Convergence of infinite products

If $c_k \geq 0$, then for any $n$,

$$1 + \sum_{k=1}^{n} c_k \leq \prod_{k=1}^{n} (1 + c_k) \leq \exp\left(\sum_{k=1}^{n} c_k\right).$$

Thus, the limit of $\prod_{k=1}^{n}(1 + c_k)$ as $n \to \infty$ exists if and only if

$$\sum_{k=1}^{\infty} c_k < \infty.$$

For the second inequality in Theorem 8, the limit of $\prod_{k=1}^{n}\left(1 + 2e^{-a_k t}\right)$ as $n \to \infty$ exists if and only if

$$\sum_{k=1}^{\infty} 2e^{-a_k t} < \infty.$$