Harmonic polynomials and the spherical Laplacian

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1 Topological groups

Let G be a topological group: $(x, y) \mapsto xy$ is continuous $G \times G \to G$ and $x \mapsto x^{-1}$ is continuous $G \to G$. For $g \in G$, the maps $L_g(x) = gx$ and $R_g(x) = xg$ are homeomorphisms. If U is an open subset of G and X is a subset of G, for each $x \in X$ the set $Ux = \{ux : u \in U\}$ is open because U is open and $u \mapsto ux$ is a homeomorphism. Therefore

$$UX = \{ux : u \in U, x \in X\} = \bigcup_{x \in X} Ux$$

is open, being a union of open sets.

For a subgroup H of G, not necessarily a normal subgroup, define $q : G \to G/H$ by

$$q(g) = gH, \qquad g \in G,$$

and assign G/H the final topology for q, the finest topology on G/H such that $q : G \to G/H$ is continuous (namely, the quotient topology). If U is an open subset of G, then UH is open, and we check that $q^{-1}(q(U)) = UH$. Because G/H has the final topology for q, this means that $q(U)$ is an open set in G/H. Therefore, $q : G \to G/H$ is an **open map**.

Theorem 1. If G is a topological group and H is a closed subgroup, then G/H is a Hausdorff space.

Proof. For a topological space X, define $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$. It is a fact that X is Hausdorff if and only if $\Delta(X)$ is a closed subset of $X \times X$. Thus the quotient space G/H is Hausdorff if and only if the image of $\Delta : G/H \to G/H \times G/H$ is closed. The complement of $\Delta(G/H)$ is

$$(G/H \times G/H) - \Delta(G/H) = \{(xH, yH) : xH \neq yH\} = \{(q(x), q(y)) : x^{-1}y \notin H\}.$$

Call this set U and let $p = q \times q$, which is a product of open maps and thus is itself open $G \times G \to G/H \times G/H$ and likewise is surjective. We check that

$$p^{-1}(U) = \{(x, y) \in G \times G : x^{-1}y \notin H\}.$$

The map $f : G \times G \to G$ defined by $f(x, y) = x^{-1}y$ is continuous and $G - H$ is open in G, so $f^{-1}(G - H)$ is open in $G \times G$. But

$$f^{-1}(G - H) = \{(x, y) \in G \times G : x^{-1}y \notin H\} = p^{-1}(U),$$

thus $p^{-1}(U)$ is open. As p is surjective, $p(p^{-1}(U)) = U$, and because p is an open map and $p^{-1}(U)$ is an open set, U is an open set. Because U is the complement of $\Delta(G \times H)$, that set is closed and it follows that G/H is Hausdorff. □

Let G be a compact group, let K be a compact Hausdorff space. A **left action** of G on K is a continuous map $\alpha : G \times K \to K$, denoted

$$\alpha(g, k) = g \cdot k,$$

satisfying $e \cdot k = k$ and $(g_1g_2) \cdot k = g_1 \cdot (g_2 \cdot k)$. The action is called **transitive** if for $k_1, k_2 \in K$ there is some $g \in G$ such that $g \cdot k_1 = k_2$.

Let H be a closed subgroup of G and let $q : G \to G/H$ be the quotient map. We have established that q is open and that G/H is Hausdorff. Because G is compact and q is surjective and continuous, $q(G) = G/H$ is a compact space. We define $\beta : G \times G/H \to G/H$ by

$$\beta(g, xH) = g \cdot (xH) = (gx)H, \qquad g \in G, \quad xH \in G/H.$$

If $xH = yH$, then $(gx)H = (gy)H$, so indeed this makes sense.[1]

Lemma 2. $\beta : G \times G/H \to G/H$ is a transitive left action.

Proof. Write $\mu(x, y) = xy$. For an open subset V in G/H, we check that

$$(L_e \times q)^{-1}(\beta^{-1}(V)) = \mu^{-1}(q^{-1}(V)),$$

hence $(L_e \times q)^{-1}(\beta^{-1}(V))$ is open in G. Because $L_e : G \to G$ and $q : G \to G/H$ are surjective open maps, the product $L_e \times q : G \times G \to G \times G/H$ is a surjective open map, so

$$(L_e \times q)((L_e \times q)^{-1}(\beta^{-1}(V))) = \beta^{-1}(V)$$

is open in $G \times G/H$, showing that β is continuous.

For $xH \in G/H$, $e \cdot (xH) = (ex)H = xH$, and for $g_1, g_2 \in G$,

$$(g_1g_2) \cdot (xH) = (g_1g_2xH) = g_1 \cdot (g_2xH) = g_1 \cdot (g_2 \cdot (xH)).$$

Therefore β is a left action.

For $xH, yH \in G/H$,

$$(yx^{-1}) \cdot xH = (yx^{-1}xH) = yH,$$

showing that β is transitive. □

[1] cf. Mamoru Mimura and Hiroshi Toda, *Topology of Lie Groups, I and II*, Chapter I.

Let G be a compact group and let α be a transitive action of G on a compact Hausdorff space K. For any $k_0 \in K$, let $H = \{g \in G : \alpha(g, k_0) = k_0\}$, the **isotropy group of** k_0, which is a closed subgroup of G. A theorem of Weil[2] states that $\phi : G/H \to K$ defined by

$$\phi(xH) = \alpha(x, k_0), \qquad xH \in G/H$$

is a homeomorphism that satisfies

$$\phi(\beta(g, xH)) = \alpha(g, \phi(xH)), \qquad g \in G, \quad xH \in G/H,$$

called an **isomorphism of G-spaces**.

A Borel measure m on G is called **left-invariant** if $m(gE) = m(E)$ for all Borel sets E and **right-invariant** if $m(Eg) = m(E)$ for all Borel sets E. It is proved that there is a unique regular Borel probability measure m on G that is left-invariant.[3] This measure is right-invariant, and satisfies

$$\int_G f(x) dm(x) = \int_G f(x^{-1}) dm(x), \qquad f \in C(G).$$

We call m the **Haar probability measure** on the compact group G.

Let H be the above isotropy group, and define $m_{G/H}$ on the Borel σ-algebra of G/H by

$$m_{G/H} = m \circ q^{-1}.$$

This is a regular Borel probability measure on G/H, and satisfies

$$m_{G/H}(g \cdot E) = m_{G/H}(E)$$

for Borel sets E in G/H and for $g \in G$; we say that $m_{G/H}$ is **G-invariant**. A theorem attributed to Weil states that this is the unique G-invariant regular Borel probability measure on G/H.[4] Then define m_K on the Borel σ-algebra of K by

$$m_K = m_{G/H} \circ \phi^{-1} = m \circ q^{-1} \circ \phi^{-1}.$$

This is the unique G-invariant regular Borel probability measure on K.

2 Spherical surface measure

$SO(n)$ is a compact Lie group. S^{n-1} is a topological group, and it is a fact that $\alpha : SO(n) \times S^{n-1} \to S^{n-1}$ defined by

$$\alpha(g, k) = gk, \qquad g \in SO(n), \quad k \in S^{n-1},$$

[2] Joe Diestel and Angela Spalsbury, *The Joys of Haar Measure*, p. 148, Theorem 6.1.
[3] Walter Rudin, *Functional Analysis*, second ed., p. 130, Theorem 5.14.
[4] Joe Diestel and Angela Spalsbury, *The Joys of Haar Measure*, p. 149, Theorem 6.2.

is a transitive left-action. We check that the isotropy group of e_n is $SO(n-1)$. Let $q : SO(n) \to SO(n)/SO(n-1)$ be the projection map and define $\phi : SO(n)/SO(n-1) \to S^{n-1}$ by

$$\phi(xSO(n-1)) = \alpha(x, e_n) = xe_n, \qquad xSO(n-1) \in SO(n)/SO(n-1).$$

Then for m the Borel probability measure on $SO(n)$,[5] the unique $SO(n)$-invariant regular Borel probability measure on S^{n-1} is

$$m_{S^{n-1}} = m \circ q^{-1} \circ \phi^{-1}. \tag{1}$$

It is a fact that the volume of the unit ball in $\mathbb{R}^n$ is

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)},$$

and that the surface area of S^{n-1} in $\mathbb{R}^n$ is

$$A_{n-1} = n\omega_n = n\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

For E a Borel set in S^{n-1}, define

$$\sigma(E) = A_{n-1} m_{S^{n-1}}(E).$$

Then σ is a $SO(n)$-invariant regular Borel measure on S^{n-1}, with total measure

$$\sigma(S^{n-1}) = A_{n-1} m_{S^{n-1}}(S^{n-1}) = A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

We call σ the **spherical surface measure**.[6]

For $\gamma \in SO(n)$ and $f \in C(S^{n-1})$, define

$$(\gamma \cdot f)(x) = f(\gamma^{-1}x) = (f \circ \gamma^{-1})(x), \qquad x \in S^{n-1}.$$

Let $\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, which satisfies

$$\int_{\mathbb{R}^n} \gamma_n(x) dx = 1,$$

[5] $SO(n)$ is a compact Lie group, and more than merely a compact group, it has a natural volume, rather than merely volume 1. It is

$$\mathrm{Vol}(SO(n)) = \frac{2^{n-1}\pi^{\frac{(n-1)(n+2)}{4}}}{\prod_{d=2}^{n}\Gamma(d/2)}.$$

See Luis J. Boya, E. C. G. Sudarshan, and Todd Tilma, *Volumes of compact manifolds*, `http://repository.ias.ac.in/51021/`.

[6] cf. Jacques Faraut, *Analysis on Lie Groups: An Introduction*, p. 186, §9.1 and Claus Müller, *Analysis of Spherical Symmetries in Euclidean Spaces*, Chapter 1.

and define $I : C(S^{n-1}) \to \mathbb{C}$ by

$$I(f) = \int_{\mathbb{R}^n} f(x/|x|)\gamma_n(x)dx, \qquad f \in C(S^{n-1}),$$

which is a positive linear functional. S^{n-1} is a compact Hausdorff space, so by the Riesz representation theorem there is a unique regular Borel measure μ on S^{n-1} such that

$$I(f) = \int_{S^{n-1}} f d\mu, \qquad f \in C(S^{n-1}).$$

Because $I(f) = \int_{\mathbb{R}^n} \gamma_n(x)dx = 1$, μ is a probability measure. For $\gamma \in SO(n)$, write $g = \gamma \cdot f$, for which $g(x/|x|) = f(\gamma^{-1}(x/|x|))$, and because $|\gamma^{-1}x| = |x|$ for $x \in \mathbb{R}^n$ and because Lebesgue measure on $\mathbb{R}^n$ is invariant under $SO(n)$, by the change of variables theorem we have

$$I(\gamma \cdot f) = I(g) = \int_{\mathbb{R}^n} f\left(\frac{1}{|x|}\gamma^{-1}x\right)(2\pi)^{-n/2}e^{-|x|^2/2}dx = I(f).$$

Now define $\nu(E) = \mu(\gamma(E)) = ((\gamma)_*^{-1}\mu)(E)$, the pushforward of μ by γ^{-1}. This is a regular Borel probability measure on S^{n-1}, and by the change of variables theorem,

$$\int_{S^{n-1}} f d\nu = \int_{S^{n-1}} f \circ \gamma^{-1} d\mu = \int_{S^{n-1}} \gamma \cdot f d\mu = I(\gamma \cdot f) = I(f).$$

Because $I(f) = \int_{S^{n-1}} f d\nu$ for all $f \in C(S^{n-1})$, it follows that $\nu = \mu$. Because $\gamma \in SO(n)$ is arbitrary, this measn that μ is $SO(n)$-invariant. But $m_{S^{n-1}}$ in (1) is the unique $SO(n)$-invariant regular Borel probability measure on S^{n-1}, so $\mu = m_{S^{n-1}}$, so

$$\int_{S^{n-1}} f d\sigma = A_{n-1}\int_{S^{n-1}} f d\mu = A_{n-1}\int_{\mathbb{R}^n} f(x/|x|)(2\pi)^{-n/2}e^{-|x|^2/2}dx,$$

where $A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

3 $L^2(S^{n-1})$ and the spherical Laplacian

For $f, g \in C(S^{n-1})$, let

$$\langle f, g\rangle = \int_{S^{n-1}} f\overline{g} d\sigma,$$

and let $L^2(S^1)$ be the completion of $C(S^{n-1})$ with respect to this inner product.

For $\gamma \in SO(n)$ and $f \in C(S^{n-1})$ we have defined

$$(\gamma \cdot f)(x) = f(\gamma^{-1}x) = (f \circ \gamma^{-1})(x), \qquad x \in S^{n-1}.$$

Because σ is $SO(n)$-invariant,

$$\begin{aligned}\langle \gamma \cdot f, \gamma \cdot g\rangle &= \int_{S^{n-1}} f(\gamma^{-1}x)\overline{g}(\gamma^{-1}x)d\sigma(x)\\ &= \int_{S^{n-1}} f(x)\overline{g}(x)d((\gamma^{-1})_*\sigma)(x)\\ &= \int_{S^{n-1}} f(x)\overline{g}(x)d\sigma(x)\\ &= \langle f, g\rangle .\end{aligned}$$

For $f : S^{n-1} \to \mathbb{C}$, define $F : \mathbb{R}^n - \{0\} \to \mathbb{C}$ by

$$F(x) = f(x/|x|).$$

We take f to belong to $C^k(S^{n-1})$ when $F \in C^k(\mathbb{R}^n - \{0\})$, $0 \le k \le \infty$, and we define $\Delta_{S^{n-1}} f$ be the restriction of ΔF to S^{n-1}. We call $\Delta_{S^{n-1}}$ the **spherical Laplacian**.[7]

Theorem 3. Let $F : \mathbb{R}^n \to \mathbb{C}$ be positive-homogeneous of degree s and harmonic and let f be the restriction of F to S^{n-1}. Then

$$\Delta_{S^{n-1}} f = -s(n+s-2)f.$$

Proof. Let $H(x) = F(x/|x|) = |x|^{-s}F(x)$ and let $r(x) = |x| = (x_1^2+\cdots+x_n^2)^{1/2}$. We calculate

$$\begin{aligned}\Delta H &= \sum_{i=1}^n \partial_i^2((r^2)^{-\frac{s}{2}}F)\\ &= \sum_{i=1}^n \partial_i\left(-sx_i(r^2)^{-\frac{s}{2}-1}F + (r^2)^{-\frac{s}{2}}\partial_i F\right)\\ &= \sum_{i=1}^n -s(r^2)^{-\frac{s}{2}-1}F - sx_i(2x_i)\left(-\frac{s}{2}-1\right)(r^2)^{-\frac{s}{2}-2}F - sx_i(r^2)^{\frac{s}{2}-1}\partial_i F\\ &- sx_i(r^2)^{-\frac{s}{2}-1}\partial_i F + (r^2)^{-\frac{s}{2}}\partial_i^2 F\\ &= -ns(r^2)^{-\frac{s}{2}-1}F + (r^2)^{-\frac{s}{2}-2}\sum_{i=1}^n\left(-s(-s-2)x_i^2F - sx_ir^2\partial_iF - sx_ir^2\partial_iF\right)\\ &+ (r^2)^{-\frac{s}{2}}\Delta F\\ &= -ns(r^2)^{-\frac{s}{2}-1}F + (r^2)^{-\frac{s}{2}-2}\sum_{i=1}^n(s^2x_i^2F + 2sx_i^2F - 2sx_ir^2\partial_iF).\end{aligned}$$

[7] cf. N. J. Vilenkin, *Special Functions and the Theory of Group Representations*, Chapter IX, §1.

Euler's identity for positive-homogeneous functions[8] states that if $G : \mathbb{R}^n - \{0\} \to \mathbb{C}$ is positive-homogeneous of degree s then $x \cdot (\nabla G)(x) = sG(x)$ for all x. Therefore

$$\begin{aligned}\Delta H &= -ns(r^2)^{-\frac{s}{2}-1}F + (r^2)^{-\frac{s}{2}-2}(s^2+2s)|x|^2 F - (r^2)^{-\frac{s}{2}-2} \cdot 2sr^2 \cdot sF \\ &= -ns(r^2)^{-\frac{s}{2}-1}F + (r^2)^{-\frac{s}{2}-1}(s^2+2s)F - (r^2)^{-\frac{s}{2}-1} \cdot 2s^2 F \\ &= -sr^{-s-2}(n+s-2)F.\end{aligned}$$

For $x \in \mathbb{R}^n - \{0\}$,

$$f(x/|x|) = F(x/|x|) = H(x).$$

Then $\Delta_{S^{n-1}} f$ is equal to the restriction of ΔH to S, thus for $x \in S$, for which $|r| = 1$,

$$(\Delta_{S^{n-1}} f)(x) = -sr^{-s-2}(n+s-2)F(x) = -s(n+s-2)f(x).$$

□

Theorem 4. If $f \in C^2(S^{n-1})$ satisfies $\Delta_{S^{n-1}} f = \lambda f$, then $\lambda \leq 0$.
If $g \in C^2(S^{n-1})$ satisfies $\Delta_{S^{n-1}} g = \mu g$ with $\lambda \neq \mu$, then $\langle f, g \rangle = 0$.

Proof. Say $\lambda \neq 0$. Then

$$\begin{aligned}\langle f, f \rangle &= \frac{1}{\lambda} \langle \Delta_{S^{n-1}} f, f \rangle \\ &= \frac{1}{\lambda} \int_{S^{n-1}} (\Delta_{S^{n-1}} f) \overline{f} d\sigma \\ &= \frac{1}{\lambda} \int_{S^{n-1}} f \Delta_{S^{n-1}} \overline{f} d\sigma \\ &= \frac{1}{\lambda} \int_{S^{n-1}} f \overline{\Delta_{S^{n-1}} f} d\sigma \\ &= \frac{1}{\lambda} \int_{S^{n-1}} f \overline{\lambda f} d\sigma \\ &= \frac{\overline{\lambda}}{\lambda} \langle f, f \rangle .\end{aligned}$$

Because $\lambda \neq 0$, it is not the case that $f = 0$, hence $\langle f, f \rangle > 0$. Hence $\frac{\overline{\lambda}}{\lambda} = 1$, which means that $\lambda \in \mathbb{R}$. Furthermore,

$$\lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle \Delta_{S^{n-1}} f, f \rangle = \int_{S^{n-1}} (\Delta_{S^{n-1}} f) \overline{f} d\sigma < 0,$$

which implies that $\lambda < 0$. □

We now prove that $\Delta_{S^{n-1}}$ is invariant under the action of $SO(n)$.

[8] cf. John L. Greenberg, *Alexis Fontaine's 'Fluxio-differential Method' and the Origins of the Calculus of Several Variables*, Annals of Science **38** (1981), 251–290.

Theorem 5. If $f \in C^2(S^{n-1})$ and $\gamma \in SO(n)$ then

$$\Delta_{S^{n-1}}(\gamma \cdot f) = \gamma \cdot (\Delta_{S^{n-1}} f).$$

Proof. Let $F(x) = f(x/|x|)$, let $g = \gamma \cdot f$, and let $G(x) = g(x/|x|) = f(\gamma^{-1}x/|\gamma^{-1}x|)$. For $x \in \mathbb{R}^n - \{0\}$,

$$(\gamma \cdot F)(x) = F(\gamma^{-1}x) = f(\gamma^{-1}x/|\gamma^{-1}x|) = G(x),$$

so $\gamma \cdot F = G$. It is a fact that $\Delta(\gamma \cdot F) = \gamma \cdot (\Delta F)$.[9] Thus for $x \in S^{n-1}$,

$$(\Delta_{S^{n-1}} g)(x) = (\Delta G)(x) = (\gamma \cdot (\Delta F))(x) = (\Delta F)(\gamma^{-1}x) = (\Delta_{S^{n-1}} f)(\gamma^{-1}x),$$

namely $\Delta_{S^{n-1}}(\gamma \cdot f) = \gamma \cdot (\Delta_{S^{n-1}} f)$. □

We now prove that $\Delta_{S^{n-1}}$ is symmetric and negative-definite.[10]

Theorem 6. For $f, g \in C^2(S^{n-1})$,

$$\int_{S^{n-1}} (\Delta_{S^{n-1}} f) \cdot g d\sigma = \int_{S^{n-1}} f \cdot \Delta_{S^{n-1}} g d\sigma.$$

$\Delta_{S^{n-1}}$ is negative-definite:

$$\int_{S^{n-1}} (\Delta_{S^{n-1}} f) \cdot \overline{f} \leq 0,$$

and this is equal to 0 only when f is constant.

Proof. It is a fact that if F is positive-homogeneous of degree s then ΔF is positive-homogeneous of degree $s-2$. Let $F(x) = f(x/|x|)$ and $G(x) = g(x/|x|)$, with which

$$(\Delta_{S^{n-1}} f)(x) = (\Delta F)(x), \qquad (\Delta_{S^{n-1}} g)(x) = (\Delta G)(x), \qquad x \in S^{n-1}$$

and, because F and G are positive-homogeneous of degree 0,

$$\begin{aligned}
\int_{S^{n-1}} (\Delta_{S^{n-1}} f)(x) \cdot g(x) d\sigma(x) &= \int_{S^{n-1}} (\Delta F)(x) \cdot G(x) d\sigma(x) \\
&= A_{n-1} \int_{\mathbb{R}^n} (\Delta F)(x/|x|) \cdot G(x/|x|) \gamma_n(x) dx \\
&= A_{n-1} \int_{\mathbb{R}^n} |x|^2 (\Delta F)(x) \cdot G(x) \gamma_n(x) dx.
\end{aligned}$$

Because

$$\partial_i(|x|^2 G \gamma_n) = 2x_i G \gamma_n + |x|^2 \gamma_n \partial_i G + |x|^2 G(-x_i \gamma_n),$$

[9] Gerald B. Folland, *Introduction to Partial Differential Equations*, second ed., p. 67, Theorem 2.1.

[10] http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/09_spheres.pdf, p. 9, Proposition 4.0.1.

integrating by parts and using Euler's identity for positive-homogeneous functions gives us

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\Delta F)(x) \cdot |x|^2 G(x) \gamma_n(x) dx \\
&= -\int_{\mathbb{R}^n} \sum_{i=1}^n (\partial_i F)(x) \partial_i (|x|^2 G(x) \gamma_n(x)) dx \\
&= -\int_{\mathbb{R}^n} \sum_{i=1}^n ((2G\gamma_n - |x|^2 G\gamma_n) \cdot x_i \partial_i F + |x|^2 \gamma_n \partial_i F \partial_i G) dx \\
&= -\int_{\mathbb{R}^n} \sum_{i=1}^n |x|^2 \gamma_n \partial_i F \cdot \partial_i G dx.
\end{aligned}$$

Because the above expression is the same when F and G are switched, this establishes

$$\int_{S^{n-1}} (\Delta_{S^{n-1}} f) \cdot g d\sigma = \int_{S^{n-1}} f \cdot \Delta_{S^{n-1}} g d\sigma.$$

For $g = \overline{f}$ we have $G = \overline{F}$ and

$$\int_{S^{n-1}} (\Delta_{S^{n-1}} f) \cdot \overline{f} d\sigma = -A_{n-1} \int_{\mathbb{R}^n} \sum_{i=1}^n |x|^2 \gamma_n |\partial_i F|^2 dx,$$

which is ≤ 0. If it is equal to 0 then $(\partial_i F)(x) = 0$ for all $x \in \mathbb{R}^n$, which means that F is constant and hence that f is constant. □

4 Homogeneous polynomials

For $P(x_1, \dots, x_n) = \sum a_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ write

$$P(\partial) = \sum a_\alpha \partial^\alpha, \qquad \overline{P}(x_1, \dots, x_n) = \sum \overline{a_\alpha} x^\alpha, \qquad \overline{P}(\partial) = \sum \overline{a_\alpha} \partial^\alpha.$$

For $P, Q \in \mathbb{C}[x_1, \dots, x_n]$, define[11]

$$(P, Q) = (\overline{Q}(\partial P)\Big|_{x=0}.$$

For $P = \sum a_\alpha x^\alpha$ and $Q = \sum b_\beta x^\beta$,

$$(P, Q) = \left(\sum_\beta \overline{b_\beta} \partial^\beta \sum_\alpha a_\alpha x^\alpha \right) \Big|_{x=0} = \sum_\beta \overline{b_\beta} a_\beta \cdot \beta!. \tag{2}$$

Lemma 7. $(\cdot, \cdot)$ is a positive-definite Hermitian form on $\mathbb{C}[x_1, \dots, x_n]$.

[11] cf. John E. Gilbert and Margaret A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, p. 164, Chapter 3, §3.

Proof. It is apparent that $(\cdot,\cdot)$ is $\mathbb{C}$-linear in its first argument and conjugate linear in its second argument. From (2), it satisfies $(P,Q)=\overline{(Q,P)}$, namely, $(\cdot,\cdot)$ is a Hermitian form. For $P\in\mathbb{C}[x_1,\ldots,x_n]$,

$$(P,P)=\sum_\alpha a_\alpha\overline{a_\alpha}\cdot\alpha!=\sum_\alpha|a_\alpha|^2\cdot\alpha!\geq 0,$$

and if $(P,P)=0$ then each a_α is equal to 0, showing that $(\cdot,\cdot)$ is postive-definite. □

For $P=\sum_\alpha a_\alpha x^\alpha$ and $Q=\sum_\beta b_\beta x^\beta$,

$$(\Delta P)(x)=\sum_\alpha a_\alpha\sum_{i=1}^n\partial_i^2x^\alpha=\sum_\alpha a_\alpha\sum_{i=1}^n\frac{\alpha!}{(\alpha-2e_i)!}x^{\alpha-2e_i},$$

and we calculate

$$(\Delta P,Q)=\sum_\beta\overline{b_\beta}\sum_{i=1}^n a_{\beta+2e_i}(\beta+2e_i)!.$$

On the other hand,

$$r^2Q(x_1,\ldots,x_n)=\sum_\beta b_\beta x^\beta\sum_{i=1}^n x_i^2=\sum_\beta b_\beta\sum_{i=1}^n x^{\beta+2e_i},$$

and we calculate

$$(P,r^2Q)=\sum_\beta\overline{b_\beta}\sum_{i=1}^n a_{\beta+2e_i}.$$

Lemma 8. For $P,Q\in\mathbb{C}[x_1,\ldots,x_n]$,

$$(\Delta P,Q)=(P,r^2Q).$$

Let $\mathscr{P}_d$ be the set of homogeneous polynomials of degree d in $\mathbb{C}[x_1,\ldots,x_n]$, i.e. those $P(x_1,\ldots,x_n)\in\mathbb{C}[x_1,\ldots,x_n]$ of the form

$$P(x_1,\ldots,x_n)=\sum_{|\alpha|=d}a_\alpha x^\alpha.$$

We include the polynomial $P=0$, and $\mathscr{P}_d$ is a complex vector space. We calculate[12]

$$\dim_{\mathbb{C}}\mathscr{P}_d=\{\alpha:|\alpha|=d\}=\binom{n+d-1}{d}. \tag{3}$$

Let $\mathscr{A}_d$ be the set of those $P\in\mathscr{P}_d$ satisfying $\Delta P=0$, i.e. the homogeneous harmonic polynomials of degree d.

We prove that $\Delta:\mathscr{P}_d\to\mathscr{P}_{d-2}$ is surjective.[13]

[12] cf. Arthur T. Benjamin and Jennifer J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, p. 71, Identity 143 and p. 74, Identity 149.

[13] `http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/09_spheres.pdf`, p. 8, Claim 3.0.3.

Theorem 9. The map $\Delta : \mathscr{P}_d \to \mathscr{P}_{d-2}$ is surjective. Its kernel is $\mathscr{A}_d$, and

$$\mathscr{A}_d^{\perp} = r^2 \mathscr{P}_{d-2}.$$

Proof. By Lemma 8,

$$0 = (\Delta P, Q) = (P, r^2 Q).$$

In particular, $(r^2 Q, r^2 Q) = 0$, and because $(\cdot, \cdot)$ is nondegenerate this means that $r^2 Q = 0$, and therefore $Q = 0$. Because $\mathscr{P}_{d-2}$ is a finite-dimensional Hilbert space and the orthogonal complement of the image $\Delta \mathscr{P}_d$ is equal to $\{0\}$, it follows that $\Delta \mathscr{P}_d = \mathscr{P}_{d-2}$.

If $P \in (r^2 \mathscr{P}_{d-2})^{\perp}$ then $(P, r^2 Q) = 0$ for all $Q \in \mathscr{P}_{d-2}$, hence $(\Delta P, Q) = 0$. In particular $(\Delta P, \Delta P) = 0$ and so $\Delta P = 0$, which means that $P \in \mathscr{A}_d$. On the other hand if $P \in \mathscr{A}_d$ then $(P, r^2 Q) = (\Delta P, Q) = 0$, so we get that $(r^2 \mathscr{P}_{d-2})^{\perp} = \mathscr{A}_d$. Because $\mathscr{P}_d$ is a finite-dimensional Hilbert space, this implies that $\mathscr{A}_d^{\perp} = (r^2 \mathscr{P}_{d-2})^{\perp\perp} = r^2 \mathscr{P}_{d-2}$. □

The above theorem tells us that

$$\mathscr{P}_d = \mathscr{A}_d \oplus \mathscr{A}_d^{\perp} = \mathscr{A}_d \oplus r^2 \mathscr{P}_{d-2}.$$

Then,

$$\mathscr{P}_{d-2} = \mathscr{A}_{d-2} \oplus r^2 \mathscr{P}_{d-4},$$

and by induction,

$$\mathscr{P}_d = \mathscr{A}_d \oplus r^2 \mathscr{A}_{d-2} \oplus r^2 \mathscr{A}_{d-4} \oplus \cdots .$$

For $P \in \mathscr{P}_d$, there are unique $F_0 \in \mathscr{A}_d$, $F_2 \in \mathscr{A}_{d-2}$, $F_4 \in \mathscr{A}_{d-4}$, etc., such that

$$P = F_0 + r^2 F_2 + r^4 F_4 + \cdots .$$

Let p be the restriction of P to S^{n-1} and let f_i be the restriction of F_i to S^{n-1}. Since $r^2 = 1$ for $x \in S^{n-1}$,

$$p = f_0 + f_2 + f_4 + \cdots .$$

We have established the following.

Theorem 10. The restriction of a homogeneous polynomial to S^{n-1} is equal to a sum of the restrictions of homogeneous harmonic polynomials to S^{n-1}.

Using $\mathscr{P}_d = \mathscr{A}_d \oplus r^2 \mathscr{P}_{d-2}$, we have $\dim_{\mathbb{C}} \mathscr{P}_d = \dim_{\mathbb{C}} \mathscr{A}_d + \dim_{\mathbb{C}} \mathscr{P}_{d-2}$, and then using the (3) for $\dim_{\mathbb{C}} \mathscr{P}_d$ we get the following.

Theorem 11.

$$\dim_{\mathbb{C}} \mathscr{A}_d = \binom{n+d-1}{d} - \binom{n+d-3}{d-2} = \binom{n+d-2}{n-2} + \binom{n+d-3}{n-2}.$$

With n fixed, using the asymptotic formula

$$\binom{z+k}{k} = \frac{k^z}{\Gamma(z+1)}\left(1 + \frac{z(z+1)}{2k} + O(k^{-2})\right), \qquad k \to \infty,$$

we get from the above lemma

$$\dim_{\mathbb{C}} \mathscr{A}_d \sim \frac{2}{(n-2)!} d^{n-2}.$$

Let $\mathscr{H}_d$ be the restrictions of $P \in \mathscr{A}_d$ to S^{n-1}. We get the following from Theorem 3.

Lemma 12. For $Y \in \mathscr{H}_d$,

$$\Delta_{S^{n-1}} Y = \lambda_d Y$$

where

$$\lambda_d = -d(d+n-2) = -\left(d + \frac{n-2}{2}\right)^2 + \left(\frac{n-2}{2}\right)^2.$$

$\lambda_d = 0$ if and only if $d = 0$; if $d_1 < d_2$ then $\lambda_{d_2} < \lambda_{d_1} \leq 0$; and $\lambda_d \to -\infty$ as $d \to \infty$.

5 The Hilbert space $L^2(S^{n-1})$

We prove that when $d_1 \neq d_2$, the subspaces $\mathscr{H}_{d_1}$ and $\mathscr{H}_{d_2}$ of $L^2(S^{n-1})$ are mutually orthogonal.

Theorem 13. For $d_1 \neq d_2$, for $Y_1 \in \mathscr{H}_{d_1}$ and for $Y_2 \in \mathscr{H}_{d_2}$,

$$\langle Y_1, Y_2 \rangle = 0.$$

Proof. From Lemma 12,

$$\Delta_{S^{n-1}} Y_1 = \lambda_{d_1} Y_1, \qquad \Delta_{S^{n-1}} Y_2 = \lambda_{d_2} Y_2.$$

where $\lambda_d = -d(d+n-2)$. Because $d_1 \neq d_2$ it follows that $\lambda_{d_1} \neq \lambda_{d_2}$ and then by Theorem 4, $\langle Y_1, Y_2 \rangle = 0$. $\square$

For $\phi \in C(S^{n-1})$, write

$$\|\phi\|_{C^0} = \sup_{x \in S^{n-1}} |\phi(x)|.$$

Let A be the set of restrictions of all $P \in \mathbb{C}[x_1, \ldots, x_n]$ to S^{n-1}. A is a **self-adjoint algebra**: it is a linear subspace of $C(S^{n-1})$; for $p, q \in A$, with $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$ such that p is the restriction of P to S^{n-1} and q is the restriction of Q to S^{n-1}, the product PQ belongs to $\mathbb{C}[x_1, \ldots, x_n]$ and pq is equal to the restriction of PQ to S^{n-1}, showing that A is an algebra; and $\overline{p}$

is the restriction of $\overline{P} \in \mathbb{C}[x_1, \dots, x_n]$ to S^{n-1}, showing that A is self-adjoint. For distinct $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n)$ in S^{n-1}, say with $u_k \neq v_k$, let $P(x_1, \dots, x_n) = x_k$ and let p be the restriction of P to S^{n-1}. Then $p(u) = u_k$ and $p(v) = v_k$, showing that A **separates points**. For $u \in S^{n-1}$, let $P(x_1, \dots, x_n) = 1$ and let p be the restriction of P to S^{n-1}. Then $p(u) = 1$, showing that A is **nowhere vanishing**. Because S^{n-1} is a compact Hausdorff space, we obtain from the **Stone-Weierstrass theorem**[14] that A is dense in the Banach space $C(S^{n-1})$: for any $\phi \in C(S^{n-1})$ and for $\epsilon > 0$, there is some $p \in A$ such that $\|p - \phi\|_{C^0} \leq \epsilon$.

$L^2(S^{n-1})$ is the completion of $C(S^{n-1})$ with respect to the inner product

$$\langle f, g \rangle = \int_{S^{n-1}} f \cdot \overline{g} d\sigma.$$

For $f \in L^2(S^{n-1})$ and for $\epsilon > 0$, there is some $\phi \in C(S^{n-1})$ with $\|\phi - f\|_{L^2} \leq \epsilon$, and there is some $p \in A$ with $\|p - \phi\|_{C^0} \leq \epsilon$. But for $\psi \in C(S^{n-1})$,

$$\|\psi\|_{L^2} = \left(\int_{S^{n-1}} |\psi|^2 d\sigma \right)^{1/2} \leq \|\psi\|_{C^0} \cdot \sqrt{\sigma(S^{n-1})}.$$

Then

$$\begin{aligned} \|p - f\|_{L^2} &\leq \|p - \phi\|_{L^2} + \|\phi - f\|_{L^2} \\ &\leq \|p - \phi\|_{C^0} \cdot \sqrt{\sigma(S^{n-1})} + \epsilon \\ &\leq \epsilon \cdot \sqrt{\sigma(S^{n-1})} + \epsilon. \end{aligned}$$

This shows that A is dense in $L^2(S^{n-1})$ with respect to the norm $\|\cdot\|_{L^2}$.

An element of $\mathbb{C}[x_1, \dots, x_n]$ can be written as a finite linear combination of homogeneous polynomials. By Theorem 10, the restriction to S^{n-1} of each of these homogeneous polynomials is itself equal to a finite linear combination of homogeneous harmonic polynomials. Thus for $p \in A$ there are $Y_1 \in \mathscr{H}_{d_1}, \dots, Y_m \in \mathscr{H}_{d_m}$ with $p = Y_1 + \cdots + Y_m$. Therefore, the collection of all finite linear combinations of restrictions to S^{n-1} of homogeneous harmonic polynomials is dense in $L^2(S^{n-1})$. Now, Theorem 13 says that for $d_1 \neq d_2$, the subspaces $\mathscr{H}_{d_1}$ and $\mathscr{H}_{d_2}$ are mutually orthogonal. Putting the above together gives the following.

Theorem 14. $L^2(S^{n-1}) = \bigoplus_{d \geq 0} \mathscr{H}_d$.

For $\phi \in C(S^{n-1})$,

$$\|\phi\|_{L^2} \leq \sqrt{\sigma(S^{n-1})} \cdot \|\phi\|_{C^0}.$$

Similar to Nikolsky's inequality for the Fourier transform, for $Y \in \mathscr{H}_d$, the norm $\|Y\|_{C^0}$ is upper bounded by a multiple of the norm $\|Y\|_{L^2}$ that depends on d.[15]

[14] Walter Rudin, *Functional Analysis*, second ed., p. 122, Theorem 5.7.

[15] `http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/09_spheres.pdf`, p. 12, Proposition 6.0.1.

Theorem 15. For $Y \in \mathscr{H}_d$,

$$\|Y\|_{C^0} \leq \sqrt{\frac{\dim_{\mathbb{C}} \mathscr{H}_d}{\sigma(S^{n-1})}} \cdot \|Y\|_{L^2}.$$

6 Sobolev embedding

Let $P_d : L^2(S^{n-1}) \to \mathscr{H}_d$ the projection operator. Thus

$$f = \sum_{d \geq 0} P_d f$$

in $L^2(S^{n-1})$.

We prove the **Sobolev embedding** for S^{n-1}.[16]

Theorem 16 (Sobolev embedding)**.** For $f \in L^2(S^{n-1})$, if $s > n-1$ and

$$\sum_{d \geq 0} (1+d)^s \cdot \|P_d f\|_{L^2}^2 < \infty$$

then there is some $\phi \in C(S^{n-1})$ such that $\phi = \sum_{d \geq 0} P_j f$ in $C(S^{n-1})$, and $f = \phi$ almost everywhere.

Proof. By Theorem 11 there is some C_n such that

$$\dim_{\mathbb{C}} \mathscr{H}_d \leq C_n (1+d)^{n-2}.$$

Then by Theorem 15 and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\sum_{d \geq 0} \|P_d f\|_{C^0} &\leq \sum_{d \geq 0} \sqrt{\frac{\dim_{\mathbb{C}} \mathscr{H}_d}{\sigma(S^{n-1})}} \cdot \|P_d f\|_{L^2} \\
&\leq \sqrt{\frac{C_n}{\sigma(S^{n-1})}} \sum_{d \geq 0} (1+d)^{\frac{n-2}{2}} \cdot \|P_d f\|_{L^2} \\
&= \sqrt{\frac{C_n}{\sigma(S^{n-1})}} \sum_{d \geq 0} (1+d)^{\frac{s}{2}} \|P_d f\|_{L^2} \cdot (1+d)^{-\frac{s-n+2}{2}} \\
&\leq \sqrt{\frac{C_n}{\sigma(S^{n-1})}} \left(\sum_{d \geq 0} (1+d)^s \|P_d f\|_{L^2}^2 \right) \left(\sum_{d \geq 0} (1+d)^{-(s-n+2)} \right) \\
&= \sqrt{\frac{C_n}{\sigma(S^{n-1})}} \cdot \zeta(s-n+2) \cdot \sum_{d \geq 0} (1+d)^s \|P_d\|_{L^2}^2 \\
&< \infty.
\end{aligned}$$

[16] `http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/09_spheres.pdf`, p. 14, Corollary 7.0.1; cf. Kendall Atkinson and Weimin Han, *Spherical Harmonics and Approximations on the Unit Sphere: An Introduction*, p. 119, §3.8

Therefore $\sum_{d=0}^m P_d f$ is a Cauchy sequence in the Banach space $C(S^{n-1})$, and hence converges to some $\phi \in C(S^{n-1})$. Because

$$\left\| \sum_{d=0}^m P_d f - \phi \right\|_{L^2} \le \sqrt{\sigma(S^{n-1})} \cdot \left\| \sum_{d=0}^m P_d f - \phi \right\|_{C^0},$$

the partial sums converge to ϕ in $L^2(S^{n-1})$, and hence $\phi = f$ in $L^2(S^{n-1})$, which implies that $\phi = f$ almost everywhere. □

7 Hecke's identity

Hecke's identity tells us the Fourier transform of a product of an element of $\mathscr{A}_d$ and a Gaussian.[17]

Theorem 17 (Hecke's identity)**.** For $f(u) = e^{-\pi|u|^2} P(u)$ with $P \in \mathscr{A}_d$,

$$\widehat{f}(v) = (-i)^d f(v), \qquad v \in \mathbb{R}^n.$$

Proof. Let $v \in \mathbb{R}^n$. The map $z \mapsto e^{-\pi z \cdot z} P(z - iv)$ is a holomorphic separately in $z_1, \ldots, z_n$, and applying Cauchy's integral theorem separately for $z_1, \ldots, z_n$,

$$\int_{\mathbb{R}^n} e^{-\pi(u+iv)\cdot(u+iv)} P(u) du = \int_{\mathbb{R}^n} e^{-\pi u \cdot u} P(u - iv) du.$$

Define $Q : \mathbb{C}^n \to \mathbb{C}$ by

$$Q(z) = \int_{\mathbb{R}^n} e^{-\pi|u|^2} P(z + u) du, \qquad z \in \mathbb{C}^n,$$

and thus

$$\begin{aligned} Q(-iv) &= \int_{\mathbb{R}^n} e^{-\pi(u+iv)\cdot(u+iv)} P(u) du \\ &= \int_{\mathbb{R}^n} e^{-\pi|u|^2 + \pi|v|^2 - 2\pi i u \cdot v} P(u) du \\ &= e^{\pi|v|^2} \widehat{f}(v). \end{aligned}$$

On the other hand, for $t \in \mathbb{R}^n$, using spherical coordinates, using the mean value property for the harmonic function P, and then using spherical coordinates

[17] Elias M. Stein and Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, p. 155, Theorem 3.4; `http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/09_spheres.pdf`, p. 17, Theorem 9.0.1.

again,

$$\begin{aligned}
Q(t) &= \int_0^\infty e^{-\pi r^2} \left(\int_{S^{n-1}} P(t+w) d\sigma(w) \right) r^{n-1} dr \\
&= \int_0^\infty e^{-\pi r^2} \sigma(S^{n-1}) P(t) \cdot r^{n-1} dr \\
&= P(t) \int_0^\infty e^{-\pi r^2} \left(\int_{S^{n-1}} d\sigma \right) r^{n-1} dr \\
&= P(t) \int_{\mathbb{R}^n} e^{-\pi |x|^2} dx \\
&= P(t).
\end{aligned}$$

Because $P \in \mathbb{C}[x_1, \dots, x_n]$, P has an analytic continuation to $\mathbb{C}^n$, and then $P(z) = Q(z)$ for all $z \in \mathbb{C}^n$. Therefore

$$P(-iv) = Q(-iv) = e^{\pi |v|^2} \widehat{f}(v).$$

But because P is a homogeneous polynomial of degree d, $P(-iv) = (-i)^d P(v)$, so

$$(-i)^d P(v) = e^{\pi |v|^2} \widehat{f}(v),$$

i.e.

$$\widehat{f}(v) = (-i)^d e^{-\pi |v|^2} P(v) = (-i)^d f(v),$$

proving the claim. □

8 Representation theory

Let a complex Hilbert space H with $\langle \cdot, \cdot \rangle$, let $\mathscr{U}(H)$ be the group of unitary operators $H \to H$. For a Lie group G, a **unitary representation of** G **on** H is a group homomorphism $\pi : G \to \mathscr{U}(H)$ such that for each $f \in H$ the map $\gamma \mapsto \pi(\gamma)(f)$ is continuous $G \to H$.

We have defined σ as a unique $SO(n)$-invariant regular Borel measure on S^{n-1}. It does not follow a priori that σ is $O(n)$-invariant. But in fact, using that $|\gamma x| = |x|$ for $x \in \mathbb{R}^n$ and that Lebesgue measure on $\mathbb{R}^n$ is $O(n)$-invariant, we check that σ is $O(n)$-invariant: for $\gamma \in O(n)$ and a Borel set E in S^{n-1}, $\sigma(\gamma E) = \sigma(E)$, i.e. $\gamma_*^{-1} \sigma = \sigma$.

For $\gamma \in O(n)$ and $f \in L^2(S^{n-1})$, define

$$\pi(\gamma)(f) = f \circ \gamma^{-1}.$$

$\pi(\gamma)$ is linear. For $f, g \in L^2(\gamma)$,

$$\begin{aligned}\langle \pi(\gamma)(f), \pi(\gamma)(g)\rangle &= \int_{S^{n-1}} f \circ \gamma^{-1} \cdot \overline{g \circ \gamma^{-1}} d\sigma \\ &= \int_{S^{n-1}} f \cdot \overline{g} d(\gamma^{-1})_* \sigma \\ &= \int_{S^{n-1}} f \cdot \overline{g} d\sigma \\ &= \langle f, g\rangle .\end{aligned}$$

For $f \in L^2(S^{n-1})$, let $g = f \circ \gamma$, for which

$$\pi(\gamma)(g) = g \circ \gamma^{-1} = f \circ \gamma \circ \gamma^{-1} = f,$$

showing that $\pi(\gamma)$ is surjective. Hence $\pi(\gamma) \in \mathscr{U}(L^2(S^{n-1}))$.

For $\gamma_1, \gamma \in O(n)$ and $f \in L^2(S^{n-1})$,

$$\begin{aligned}\pi(\gamma_1\gamma_2)(f) &= f \circ (\gamma_1\gamma_2)^{-1} \\ &= f \circ (\gamma_2^{-1}\gamma_1^{-1}) \\ &= (f \circ \gamma_2^{-1}) \circ \gamma_1^{-1} \\ &= \pi(\gamma_1)(\pi(\gamma_2^{-1}(f))),\end{aligned}$$

which means that $\pi(\gamma_1\gamma_2) = \pi(\gamma_1)\pi(\gamma_2)$, namely $\pi : O(n) \to \mathscr{U}(L^2(S^{n-1}))$ is a group homomorphism.

For $\phi \in C(S^{n-1})$ and for $\gamma_0, \gamma \in O(n)$,

$$\|\pi(\gamma)(\phi) - \pi(\gamma_0)(\phi)\|_{L^2}^2 = \left\|\pi(\gamma_0^{-1}\gamma)(\phi) - \phi\right\|_{L^2}^2 \leq \sigma(S^{n-1}) \cdot \left\|\pi(\gamma_0^{-1}\gamma)(\phi) - \phi\right\|_{C^0}^2 .$$

We take as given that $\left\|\pi(\gamma_0^{-1}\gamma)(\phi) - \phi\right\|_{C^0} \to 0$ as $\gamma \to \gamma_0$ in $O(n)$. Using that $C(S^{n-1})$ is dense in $L^2(S^{n-1})$, one then proves that for each $f \in L^2(S^{n-1})$, the map $\gamma \mapsto \pi(\gamma)(f)$ is continuous $O(n) \to L^2(S^{n-1})$.

Lemma 18. *π is a unitary representation of the compact Lie group $O(n)$ on the complex Hilbert space $L^2(S^{n-1})$.*

It is a fact that if $\gamma \in O(n)$ and $P \in \mathscr{P}_d$ then $\gamma \cdot P \in \mathscr{P}_d$. Furthermore, for $\phi \in C^2(S^{n-1})$, $\Delta(\gamma \cdot \phi) = \gamma \cdot (\Delta\phi)$, hence if $P \in \mathscr{A}_d$ then $\gamma \cdot P \in \mathscr{A}_d$. Then for $Y \in \mathscr{H}_d$, $\pi(\gamma)(Y) \in \mathscr{H}_d$. This means that each $\mathscr{H}_d$ is a π-invariant subspace.[18]

[18] cf. Feng Dai and Yuan Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Chapter 1.