Fréchet derivatives and Gâteaux derivatives

Jordan Bell

April 3, 2014

1 Introduction

In this note all vector spaces are real. If X and Y are normed spaces, we denote by $\mathscr{B}(X,Y)$ the set of bounded linear maps $X \to Y$, and write $\mathscr{B}(X) = \mathscr{B}(X,X)$. $\mathscr{B}(X,Y)$ is a normed space with the operator norm.

2 Remainders

If X and Y are normed spaces, let $o(X,Y)$ be the set of all maps $r : X \to Y$ for which there is some map $\alpha : X \to Y$ satisfying:

- $r(x) = \|x\| \alpha(x)$ for all $x \in X$,
- $\alpha(0) = 0$,
- α is continuous at 0.

Following Penot,[1] we call elements of $o(X,Y)$ *remainders*. It is immediate that $o(X,Y)$ is a vector space.

If X and Y are normed spaces, if $f : X \to Y$ is a function, and if $x_0 \in X$, we say that f is *stable at* x_0 if there is some $\epsilon > 0$ and some $c > 0$ such that $\|x - x_0\| \leq \epsilon$ implies that $\|f(x - x_0)\| \leq c \|x - x_0\|$. If $T : X \to Y$ is a bounded linear map, then $\|Tx\| \leq \|T\| \|x\|$ for all $x \in X$, and thus a bounded linear map is stable at 0. The following lemma shows that the composition of a remainder with a function that is stable at 0 is a remainder.[2]

Lemma 1. Let X, Y be normed spaces and let $r \in o(X,Y)$. If W is a normed space and $f : W \to X$ is stable at 0, then $r \circ f \in o(W,Y)$. If Z is a normed space and $g : Y \to Z$ is stable at 0, then $g \circ r \in o(X,Z)$.

Proof. $r \in o(X,Y)$ means that there is some $\alpha : X \to Y$ satisfying $r(x) = \|x\| \alpha(x)$ for all $x \in X$, that takes the value 0 at 0, and that is continuous at

[1] Jean-Paul Penot, *Calculus Without Derivatives*, p. 133, §2.4.

[2] Jean-Paul Penot, *Calculus Without Derivatives*, p. 134, Lemma 2.41.

0. As f is stable at 0, there is some $\epsilon > 0$ and some $c > 0$ for which $\|w\| \le \epsilon$ implies that $\|f(w)\| \le c\,\|w\|$. Define $\beta : W \to Y$ by

$$\beta(w) = \begin{cases} \frac{\|f(w)\|}{\|w\|}\alpha(f(w)) & w \neq 0 \\ 0 & w = 0, \end{cases}$$

for which we have

$$(r \circ f)(w) = \|w\|\,\beta(w), \qquad w \in W.$$

If $\|w\| \le \epsilon$, then $\|\beta(w)\| \le c\,\|\alpha(f(w))\|$. But $f(w) \to 0$ as $w \to 0$, and because α is continuous at 0 we get that $\alpha(f(w)) \to \alpha(0) = 0$ as $w \to 0$. So the above inequality gives us $\beta(w) \to 0$ as $w \to 0$. As $\beta(0) = 0$, the function $\beta : W \to Y$ is continuous at 0, and therefore $r \circ f$ is remainder.

As g is stable at 0, there is some $\epsilon > 0$ and some $c > 0$ for which $\|y\| \le \epsilon$ implies that $\|g(y)\| \le c\,\|y\|$. Define $\gamma : X \to Z$ by

$$\gamma(x) = \begin{cases} \frac{g(\|x\|\alpha(x))}{\|x\|} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

For all $x \in X$,

$$(g \circ r)(x) = g(\|x\|\,\alpha(x)) = \|x\|\,\gamma(x).$$

Since $\alpha(0) = 0$ and α is continuous at 0, there is some $\delta > 0$ such that $\|x\| \le \delta$ implies that $\|\alpha(x)\| \le \epsilon$. Therefore, if $\|x\| \le \delta \wedge 1$ then

$$\|g(\|x\|\,\alpha(x))\| \le c\,\|x\|\,\|\alpha(x)\| \le c\,\|x\|\,\epsilon,$$

and hence if $\|x\| \le \delta \wedge 1$ then $\|\gamma(x)\| \le c\epsilon$. This shows that $\gamma(x) \to 0$ as $x \to 0$, and since $\gamma(0) = 0$ the function $\gamma : X \to Z$ is continuous at 0, showing that $g \circ r$ is a remainder. $\square$

If $Y_1, \ldots, Y_n$ are normed spaces where Y_k has norm $\|\cdot\|_k$, then $\|(y_1, \ldots, y_n)\| = \max_{1 \le k \le n} \|y_k\|_k$ is a norm on $\prod_{k=1}^n Y_k$, and one can prove that the topology induced by this norm is the product topology.

Lemma 2. If X and $Y_1, \ldots, Y_n$ are normed spaces, then a function $r : X \to \prod_{k=1}^n Y_k$ is a remainder if and only if each of $r_k : X \to Y_k$ are remainders, $1 \le k \le n$, where $r(x) = (r_1(x), \ldots, r_n(x))$ for all $x \in X$.

Proof. Suppose that there is some function $\alpha : X \to \prod_{k=1}^n Y_k$ such that $r(x) = \|x\|\,\alpha(x)$ for all $x \in X$. With $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x))$, we have

$$r_k(x) = \|x\|\,\alpha_k(x), \qquad x \in X.$$

Because $\alpha(x) \to 0$ as $x \to 0$, for each k we have $\alpha_k(x) \to 0$ as $x \to 0$, which shows that r_k is a remainder.

Suppose that each r_k is a remainder. Thus, for each k there is a function $\alpha_k : X \to Y_k$ satisfying $r_k(x) = \|x\|\,\alpha_k(x)$ for all $x \in X$ and $\alpha_k(x) \to 0$ as $x \to 0$. Then the function $\alpha : X \to \prod_{k=1}^n Y_k$ defined by $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x))$ satisfies $r(x) = \|x\|\,\alpha(x)$. Because $\alpha_k(x) \to 0$ as $x \to 0$ for each of the finitely many k, $1 \le k \le n$, we have $\alpha(x) \to 0$ as $x \to 0$. $\square$

3 Definition and uniqueness of Fréchet derivative

Suppose that X and Y are normed spaces, that U is an open subset of X, and that $x_0 \in U$. A function $f : U \to Y$ is said to be *Fréchet differentiable at x_0* if there is some $L \in \mathscr{B}(X, Y)$ and some $r \in o(X, Y)$ such that

$$f(x) = f(x_0) + L(x - x_0) + r(x - x_0), \qquad x \in U. \tag{1}$$

Suppose there are bounded linear maps L_1, L_2 and remainders r_1, r_2 that satisfy the above. Writing $r_1(x) = \|x\| \, \alpha_1(x)$ and $r_2(x) = \|x\| \, \alpha_2(x)$ for all $x \in X$, we have

$$L_1(x-x_0)+\|x - x_0\| \, \alpha_1(x-x_0) = L_2(x-x_0)+\|x - x_0\| \, \alpha_2(x-x_0), \qquad x \in U,$$

i.e.,

$$L_1(x - x_0) - L_2(x - x_0) = \|x - x_0\| \, (\alpha_2(x - x_0) - \alpha_1(x - x_0)), \qquad x \in U.$$

For $x \in X$, there is some $h > 0$ such that for all $|t| \le h$ we have $x_0 + tx \in U$, and then

$$L_1(tx) - L_2(tx) = \|tx\| \, (\alpha_2(tx) - \alpha_1(tx)),$$

hence, for $0 < |t| \le h$,

$$L_1(x) - L_2(x) = \|x\| \, (\alpha_2(tx) - \alpha_1(tx)).$$

But $\alpha_2(tx) - \alpha_1(tx) \to 0$ as $t \to 0$, which implies that $L_1(x) - L_2(x) = 0$. As this is true for all $x \in X$, we have $L_1 = L_2$ and then $r_1 = r_2$. If f is Fréchet differentiable at x_0, the bounded linear map L in (1) is called *the Fréchet derivative of f at x_0*, and we define $Df(x_0) = L$. Thus,

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + r(x - x_0), \qquad x \in U.$$

If U_0 is the set of those points in U at which f is Fréchet differentiable, then $Df : U_0 \to \mathscr{B}(X, Y)$.

Suppose that X and Y are normed spaces and that U is an open subset of X. We denote by $C^1(U, Y)$ the set of functions $f : U \to Y$ that are Fréchet differentiable at each point in U and for which the function $Df : U \to \mathscr{B}(X, Y)$ is continuous. We say that an element of $C^1(U, Y)$ is *continuously differentiable.* We denote by $C^2(U, Y)$ those elements f of $C^1(U, Y)$ such that

$$Df \in C^1(U, \mathscr{B}(X, Y));$$

that is, $C^2(U, Y)$ are those $f \in C^1(U, Y)$ such that the function $Df : U \to \mathscr{B}(X, Y)$ is Fréchet differentiable at each point in U and such that the function

$$D(Df) : U \to \mathscr{B}(X, \mathscr{B}(X, Y))$$

is continuous.[3]

The following theorem characterizes continuously differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$.[4]

Theorem 3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable at each point in $\mathbb{R}^n$, and write

$$f = (f_1, \dots, f_m).$$

$f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ if and only if for each $1 \le i \le m$ and $1 \le j \le n$ the function

$$\frac{\partial f_i}{\partial x_j} : \mathbb{R}^n \to \mathbb{R}$$

is continuous.

4 Properties of the Fréchet derivative

If $f : X \to Y$ is Fréchet differentiable at x_0, then because a bounded linear map is continuous and in particular continuous at 0, and because a remainder is continuous at 0, we get that f is continuous at x_0.

We now prove that Fréchet differentiation at a point is linear.

Lemma 4 (Linearity)**.** Let X and Y be normed spaces, let U be an open subset of X and let $x_0 \in U$. If $f_1, f_2 : U \to Y$ are both Fréchet differentiable at x_0 and if $\alpha \in \mathbb{R}$, then $\alpha f_1 + f_2$ is Fréchet differentiable at x_0 and

$$D(\alpha f_1 + f_2)(x_0) = \alpha D f_1(x_0) + D f_2(x_0).$$

Proof. There are remainders $r_1, r_2 \in o(X, Y)$ such that

$$f_1(x) = f_1(x_0) + Df_1(x_0)(x - x_0) + r_1(x - x_0), \qquad x \in U,$$

and

$$f_2(x) = f_2(x_0) + Df_2(x_0)(x - x_0) + r_2(x - x_0), \qquad x \in U.$$

Then for all $x \in U$,

$$\begin{aligned} (\alpha f_1 + f_2)(x) - (\alpha f_1 + f_2)(x_0) &= \alpha f_1(x) - \alpha f_1(x_0) + f_2(x) - f_2(x_0) \\ &= \alpha D f_1(x_0)(x - x_0) + \alpha r_1(x - x_0) \\ &\quad + D f_2(x_0)(x - x_0) + r_2(x - x_0) \\ &= (\alpha D f_1(x_0) + D f_2(x_0))(x - x_0) \\ &\quad + (\alpha r_1 + r_2)(x - x_0), \end{aligned}$$

and $\alpha r_1 + r_2 \in o(X, Y)$. $\square$

[3] See Henri Cartan, *Differential Calculus*, p. 58, §5.1, and Jean Dieudonné, *Foundations of Modern Analysis*, enlarged and corrected printing, p. 179, Chapter VIII, §12.

[4] Henri Cartan, *Differential Calculus*, p. 36, §2.7.

The following lemma gives an alternate characterization of a function being Fréchet differentiable at a point.[5]

Lemma 5. Suppose that X and Y are normed space, that U is an open subset of X, and that $x_0 \in U$. A function $f : U \to Y$ is Fréchet differentiable at x_0 if and only if there is some function $F : U \to \mathscr{B}(X, Y)$ that is continuous at x_0 and for which

$$f(x) - f(x_0) = F(x)(x - x_0), \qquad x \in U.$$

Proof. Suppose that there is a function $F : U \to \mathscr{B}(X, Y)$ that is continuous at x_0 and that satisfies $f(x) - f(x_0) = F(x)(x - x_0)$ for all $x \in U$. Then, for $x \in U$,

$$\begin{aligned} f(x) - f(x_0) &= F(x)(x - x_0) - F(x_0)(x - x_0) + F(x_0)(x - x_0) \\ &= F(x_0)(x - x_0) + r(x - x_0), \end{aligned}$$

where $r : X \to Y$ is defined by

$$r(x) = \begin{cases} (F(x + x_0) - F(x_0))(x) & x + x_0 \in U \\ 0 & x + x_0 \notin U. \end{cases}$$

We further define

$$\alpha(x) = \begin{cases} \frac{(F(x+x_0) - F(x_0))(x)}{\|x\|} & x + x_0 \in U, x \neq 0 \\ 0 & x + x_0 \notin U \\ 0 & x = 0, \end{cases}$$

with which $r(x) = \|x\| \, \alpha(x)$ for all $x \in X$. To prove that r is a remainder it suffices to prove that $\alpha(x) \to 0$ as $x \to 0$. Let $\epsilon > 0$. That $F : U \to \mathscr{B}(X, Y)$ is continuous at x_0 tells us that there is some $\delta > 0$ for which $\|x\| < \delta$ implies that $\|F(x + x_0) - F(x_0)\| < \epsilon$ and hence

$$\|(F(x + x_0) - F(x_0))(x)\| \leq \|F(x + x_0) - F(x_0)\| \, \|x\| < \epsilon \, \|x\| .$$

Therefore, if $\|x\| < \delta$ then $\|\alpha(x)\| < \epsilon$, which establishes that r is a remainder and therefore that f is Fréchet differentiable at x_0, with Fréchet derivative $Df(x_0) = F(x_0)$.

Suppose that f is Fréchet differentiable at x_0: there is some $r \in o(X, Y)$ such that

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + r(x - x_0), \qquad x \in U,$$

where $Df(x_0) \in \mathscr{B}(X, Y)$. As r is a remainder, there is some $\alpha : X \to Y$ satisfying $r(x) = \|x\| \, \alpha(x)$ for all $x \in X$, and such that $\alpha(0) = 0$ and $\alpha(x) \to 0$ as $x \to 0$. For each $x \in X$, by the Hahn-Banach extension theorem[6] there is some $\lambda_x \in X^*$ such that $\lambda_x x = \|x\|$ and $|\lambda_x v| \leq \|v\|$ for all $v \in X$. Thus,

$$r(x) = (\lambda_x x)\alpha(x), \qquad x \in X.$$

[5] Jean-Paul Penot, *Calculus Without Derivatives*, p. 136, Lemma 2.46.

[6] Walter Rudin, *Functional Analysis*, second ed., p. 59, Corollary to Theorem 3.3.

Define $F : U \to \mathscr{B}(X, Y)$ by

$$F(x) = Df(x_0) + (\lambda_{x-x_0})\alpha(x - x_0),$$

i.e. for $x \in U$ and $v \in X$,

$$F(x)(v) = Df(x_0)(v) + (\lambda_{x-x_0} v)\alpha(x - x_0) \in Y.$$

Then for $x \in U$,

$$r(x - x_0) = (\lambda_{x-x_0}(x - x_0))\alpha(x - x_0) = F(x)(x - x_0) - Df(x_0)(x - x_0),$$

and hence

$$f(x) = f(x_0) + F(x)(x - x_0), \qquad x \in U.$$

To complete the proof it suffices to prove that F is continuous at x_0. But both $\lambda_0 = 0$ and $\alpha(0) = 0$ so $F(x_0) = Df(x_0)$, and for $x \in U$ and $v \in X$,

$$\begin{aligned} \|(F(x) - F(x_0))(v)\| &= \|(\lambda_{x-x_0} v)\alpha(x - x_0)\| \\ &= |\lambda_{x-x_0} v| \, \|\alpha(x - x_0)\| \\ &\leq \|v\| \, \|\alpha(x - x_0)\|, \end{aligned}$$

so $\|F(x) - F(x_0)\| \leq \|\alpha(x - x_0)\|$. From this and the fact that $\alpha(0) = 0$ and $\alpha(x) \to 0$ as $x \to 0$ we get that F is continuous at x_0, completing the proof. $\square$

We now prove the chain rule for Fréchet derivatives.[7]

Theorem 6 (Chain rule)**.** Suppose that X, Y, Z are normed spaces and that U and V are open subsets of X and Y respectively. If $f : U \to Y$ satisfies $f(U) \subseteq V$ and is Fréchet differentiable at x_0 and if $g : V \to Z$ is Fréchet differentiable at $f(x_0)$, then $g \circ f : U \to Z$ is Fréchet differentiable at x_0, and its Fréchet derivative at x_0 is

$$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0).$$

Proof. Write $y_0 = f(x_0)$, $L_1 = Df(x_0)$, and $L_2 = Dg(y_0)$. Because f is Fréchet differentiable at x_0, there is some $r_1 \in o(X, Y)$ such that

$$f(x) = f(x_0) + L_1(x - x_0) + r_1(x - x_0), \qquad x \in U,$$

and because g is Fréchet differentiable at y_0 there is some $r_2 \in o(Y, Z)$ such that

$$g(y) = g(y_0) + L_2(y - y_0) + r_2(y - y_0), \qquad y \in V.$$

For all $x \in U$ we have $f(x) \in V$, and using the above formulas,

$$\begin{aligned} g(f(x)) &= g(y_0) + L_2(f(x) - y_0) + r_2(f(x) - y_0) \\ &= g(y_0) + L_2\Big(L_1(x - x_0) + r_1(x - x_0)\Big) + r_2\Big(L_1(x - x_0) + r_1(x - x_0)\Big) \\ &= g(y_0) + L_2(L_1(x - x_0)) + L_2(r_1(x - x_0)) + r_2\Big(L_1(x - x_0) + r_1(x - x_0)\Big). \end{aligned}$$

[7] Jean-Paul Penot, *Calculus Without Derivatives*, p. 136, Theorem 2.47.

Define $r_3 : X \to Z$ by $r_3(x) = r_2(L_1 x + r_1(x))$, and fix any $c > \|L_1\|$. Writing $r_1(x) = \|x\| \alpha_1(x)$, the fact that $\alpha(0) = 0$ and that α is continuous at 0 gives us that there is some $\delta > 0$ such that if $\|x\| < \delta$ then $\|\alpha(x)\| < c - \|L_1\|$, and hence if $\|x\| < \delta$ then $\|r_1(x)\| \leq (c - \|L_1\|) \|x\|$. Then, $\|x\| < \delta$ implies that

$$\|L_1 x + r_1(x)\| \leq \|L_1 x\| + \|r_1(x)\| \leq \|L_1\| \|x\| + (c - \|L_1\|) \|x\| = c \|x\| .$$

This shows that $x \mapsto L_1 x + r_1(x)$ is stable at 0 and so by Lemma 1 that $r_3 \in o(X, Z)$. Then, $r : X \to Z$ defined by $r = L_1 \circ r_1 + r_3$ is a sum of two remainders and so is itself a remainder, and we have

$$g \circ f(x) = g \circ f(x_0) + L_2 \circ L_1(x - x_0) + r(x - x_0), \qquad x \in U.$$

But $L_1 \in \mathscr{B}(X, Y)$ and $L_2 \in \mathscr{B}(Y, Z)$, so $L_2 \circ L_1 \in \mathscr{B}(X, Z)$. This shows that $g \circ f$ is Fréchet differentiable at x_0 and that its Fréchet derivative at x_0 is

$$L_2 \circ L_1 = Dg(y_0) \circ Df(x_0) = Dg(f(x_0)) \circ Df(x_0).$$

□

The following is the product rule for Fréchet derivatives. By $f_1 \cdot f_2$ we mean the function $x \mapsto f_1(x) f_2(x)$.

Theorem 7 (Product rule)**.** Suppose that X is a normed space, that U is an open subset of X, that $f_1, f_2 : U \to \mathbb{R}$ are functions, and that $x_0 \in U$. If f_1 and f_2 are both Fréchet differentiable at x_0, then $f_1 \cdot f_2$ is Fréchet differentiable at x_0, and its Fréchet derivative at x_0 is

$$D(f_1 \cdot f_2)(x_0) = f_2(x_0) Df_1(x_0) + f_1(x_0) Df_2(x_0).$$

Proof. There are $r_1, r_2 \in o(X, \mathbb{R})$ with which

$$f_1(x) = f_1(x_0) + Df_1(x_0)(x - x_0) + r_1(x - x_0), \qquad x \in U$$

and

$$f_2(x) = f_2(x_0) + Df_2(x_0)(x - x_0) + r_2(x - x_0), \qquad x \in U.$$

Multiplying the above two formulas,

$$\begin{aligned} f_1(x) f_2(x) \quad &= \quad f_1(x_0) f_2(x_0) + f_2(x_0) Df_1(x_0)(x - x_0) + f_1(x_0) Df_2(x_0)(x - x_0) \\ &\quad + Df_1(x_0)(x - x_0) Df_2(x_0)(x - x_0) + r_1(x - x_0) r_2(x - x_0) \\ &\quad + f_1(x_0) r_2(x - x_0) + r_2(x - x_0) Df_1(x_0)(x - x_0) \\ &\quad + f_2(x_0) r_1(x - x_0) + r_1(x - x_0) Df_2(x_0)(x - x_0). \end{aligned}$$

Define $r : X \to \mathbb{R}$ by

$$\begin{aligned} r(x) \quad &= \quad Df_1(x_0) x Df_2(x_0) x + r_1(x) r_2(x) + f_1(x_0) r_2(x) + r_2(x) Df_1(x_0) x \\ &\quad + f_2(x_0) r_1(x) + r_1(x) Df_2(x_0) x, \end{aligned}$$

for which we have, for $x \in U$,

$$f_1(x)f_2(x) = f_1(x_0)f_2(x_0)+f_2(x_0)Df_1(x_0)(x-x_0)+f_1(x_0)Df_2(x_0)(x-x_0)+r(x-x_0).$$

Therefore, to prove the claim it suffices to prove that $r \in o(X, \mathbb{R})$. Define $\alpha : X \to \mathbb{R}$ by $\alpha(0) = 0$ and $\alpha(x) = \frac{Df_1(x_0)xDf_2(x_0)x}{\|x\|}$ for $x \neq 0$. For $x \neq 0$,

$$\begin{aligned}
|\alpha(x)| &= \frac{|Df_1(x_0)x||Df_2(x_0)x|}{\|x\|} \\
&\leq \frac{\|Df_1(x_0)\| \, \|x\| \, \|Df_2(x_0)\| \, \|x\|}{\|x\|} \\
&= \|Df_1(x_0)\| \, \|Df_2(x_0)\| \, \|x\| \, .
\end{aligned}$$

Thus $\alpha(x) \to 0$ as $x \to 0$, showing that the first term in the expression for r belongs to $o(X, \mathbb{R})$. Likewise, each of the other five terms in the expression for r belongs to $o(X, \mathbb{R})$, and hence $r \in o(X, \mathbb{R})$, completing the proof. $\square$

5 Dual spaces

If X is a normed space, we denote by X^* the set of bounded linear maps $X \to \mathbb{R}$, i.e. $X^* = \mathscr{B}(X, \mathbb{R})$. X^* is itself a normed space with the operator norm. If X is a normed space, the *dual pairing* $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ is

$$\langle x, \psi \rangle = \psi(x), \qquad x \in X, \psi \in X^*.$$

If U is an open subset of X and if a function $f : U \to \mathbb{R}$ is Fréchet differentiable at $x_0 \in U$, then $Df(x_0)$ is a bounded linear map $X \to \mathbb{R}$, and so belongs to X^*. If U_0 are those points in U at which $f : U \to \mathbb{R}$ is Fréchet differentiable, then

$$Df : U_0 \to X^*.$$

In the case that X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, the Riesz representation theorem shows that $R : X \to X^*$ defined by $R(x)(y) = \langle y, x \rangle$ is an isometric isomorphism. If $f : U \to \mathbb{R}$ is Fréchet differentiable at $x_0 \in U$, then we define

$$\nabla f(x_0) = R^{-1}(Df(x_0)),$$

and call $\nabla f(x_0) \in X$ the *gradient of f at x_0*. With U_0 denoting the set of those points in U at which f is Fréchet differentiable,

$$\nabla f : U_0 \to X.$$

(To define the gradient we merely used that R is a bijection, but to prove properties of the gradient one uses that R is an isometric isomorphism.)

Example. Let X be a Hilbert space, $A \in \mathscr{B}(X)$, $v \in X$, and define

$$f(x) = \langle Ax, x \rangle - \langle x, v \rangle \, , \qquad x \in X.$$

For all $x_0, x \in X$ we have, because the inner product of a real Hilbert space is symmetric,

$$\begin{aligned}
f(x) - f(x_0) &= \langle Ax, x\rangle - \langle x, v\rangle - \langle Ax_0, x_0\rangle + \langle x_0, v\rangle \\
&= \langle Ax, x\rangle - \langle Ax_0, x\rangle + \langle Ax_0, x\rangle - \langle Ax_0, x_0\rangle - \langle x - x_0, v\rangle \\
&= \langle A(x - x_0), x\rangle + \langle Ax_0, x - x_0\rangle - \langle x - x_0, v\rangle \\
&= \langle x - x_0, A^*x\rangle + \langle x - x_0, Ax_0\rangle - \langle x - x_0, v\rangle \\
&= \langle x - x_0, A^*x + Ax_0 - v\rangle \\
&= \langle x - x_0, A^*x - A^*x_0 + A^*x_0 + Ax_0 - v\rangle \\
&= \langle x - x_0, (A^* + A)x_0 - v\rangle + \langle x - x_0, A^*(x - x_0)\rangle .
\end{aligned}$$

With $Df(x_0)(x-x_0) = \langle x - x_0, (A^* + A)x_0 - v\rangle$, or $Df(x_0)(x) = \langle x, (A^* + A)x_0 - v\rangle$, we have that f is Fréchet differentiable at each $x_0 \in X$. Furthermore, its gradient at x_0 is

$$\nabla f(x_0) = (A^* + A)x_0 - v.$$

For each $x_0 \in X$, the function $f : X \to \mathbb{R}$ is Fréchet differentiable at x_0, and thus

$$Df : X \to X^*,$$

and we can ask at what points Df has a Fréchet derivative. For $x_0, x, y \in X$,

$$\begin{aligned}
(Df(x) - Df(x_0))(y) &= \langle y, (A^* + A)x - v\rangle - \langle y, (A^* + A)x_0 - v\rangle \\
&= \langle y, (A^* + A)(x - x_0)\rangle .
\end{aligned}$$

For $D(Df)(x_0)(x - x_0)(y) = \langle y, (A^* + A)(x - x_0)\rangle$, in other words with

$$D^2 f(x_0)(x)(y) = D(Df)(x_0)(x)(y) = \langle y, (A^* + A)x\rangle ,$$

we have that Df is Fréchet differentiable at each $x_0 \in X$. Thus

$$D^2 f : X \to \mathscr{B}(X, X^*).$$

Because $D^2 f(x_0)$ does not depend on x_0, it is Fréchet differentiable at each point in X, with $D^3 f(x_0) = 0$ for all $x_0 \in X$. Here $D^3 f : X \to \mathscr{B}(X, \mathscr{B}(X, X^*))$.

6 Gâteaux derivatives

Let X and Y be normed spaces, let U be an open subset of X, let $f : U \to Y$ be a function, and let $x_0 \in U$. If there is some $T \in \mathscr{B}(X, Y)$ such that for all $v \in X$ we have

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = Tv, \tag{2}$$

then we say that f is *Gâteaux differentiable at* x_0 and call T the *Gâteaux derivative of* f *at* x_0.[8] It is apparent that there is at most one $T \in \mathscr{B}(X, Y)$ that

[8] Our definition of the Gâteaux derivative follows Jean-Paul Penot, *Calculus Without Derivatives*, p. 127, Definition 2.23.

satisfies (2) for all $v \in X$. We write $f'(x_0) = T$. Thus, f' is a map from the set of points in U at which f is Gâteaux differentiable to $\mathscr{B}(X,Y)$. If $V \subseteq U$ and f is Gâteaux differentiable at each element of V, we say that f is *Gâteaux differentiable on* V.

Example. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = \frac{x_1^4 x_2}{x_1^6 + x_2^3}$ for $(x_1, x_2) \neq (0,0)$ and $f(0,0) = 0$. For $v = (v_1, v_2) \in \mathbb{R}^2$ and $t \neq 0$,

$$\frac{f(0+tv) - f(0)}{t} = \frac{f(tv_1, tv_2)}{t} = \begin{cases} \frac{1}{t} \cdot \frac{t^5 v_1^4 v_2}{t^6 v_1^6 + t^3 v_2^3} & v \neq (0,0) \\ 0 & v = (0,0) \end{cases} = \begin{cases} \frac{t v_1^4 v_2}{t^3 v_1^6 + v_2^3} & v \neq (0,0) \\ 0 & v = (0,0). \end{cases}$$

Hence, for any $v \in \mathbb{R}^2$, we have $\frac{f(0+tv)-f(0)}{t} \to 0$ as $t \to 0$. Therefore, f is Gâteaux differentiable at $(0,0)$ and $f'(0,0)v = 0 \in \mathbb{R}$ for all $v \in \mathbb{R}^2$, i.e. $f'(0,0) = 0$. However, for $(x_1, x_2) \neq (0,0)$,

$$f(x_1, x_1^2) = \frac{x_1^6}{x_1^6 + x_1^6} = \frac{1}{2},$$

from which it follows that f is not continuous at $(0,0)$. We stated in §4 that if a function is Fréchet differentiable at a point then it is continuous at that point, and so f is not Fréchet differentiable at $(0,0)$. Thus, a function that is Gâteaux differentiable at a point need not be Fréchet differentiable at that point.

We prove that being Fréchet differentiable at a point implies being Gâteaux differentiable at the point, and that in this case the Gâteaux derivative is equal to the Fréchet derivative.

Theorem 8. Suppose that X and Y are normed spaces, that U is an open subset of X, that $f \in Y^U$, and that $x_0 \in U$. If f is Fréchet differentiable at x_0, then f is Gâteaux differentiable at x_0 and $f'(x_0) = Df(x_0)$.

Proof. Because f is Fréchet differentiable at x_0, there is some $r \in o(X,Y)$ for which

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + r(x - x_0), \qquad x \in U.$$

For $v \in X$ and nonzero t small enough that $x_0 + tv \in U$,

$$\frac{f(x_0 + tv) - f(x_0)}{t} = \frac{Df(x_0)(x_0 + tv - x_0) + r(x_0 + tv - x_0)}{t} = \frac{tDf(x_0)v + r(tv)}{t}.$$

Writing $r(x) = \|x\| \, \alpha(x)$,

$$\frac{f(x_0 + tv) - f(x_0)}{t} = \frac{tDf(x_0) + \|tv\| \, \alpha(tv)}{t} = Df(x_0)v + \|v\| \, \alpha(tv).$$

Hence,

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = Df(x_0)v.$$

This holds for all $v \in X$, and as $Df(x_0) \in \mathscr{B}(X,Y)$ we get that f is Gâteaux differentiable at x_0 and that $f'(x_0) = Df(x_0)$. $\square$

If X is a vector space and $u, v \in X$, let

$$[u,v] = \{(1-t)u + tv : 0 \leq t \leq 1\},$$

namely, the line segment joining u and v. The following is a mean value theorem for Gâteaux derivatives.[9]

Theorem 9 (Mean value theorem)**.** Let X and Y be normed spaces, let U be an open subset of X, and let $f : U \to Y$ be Gâteaux differentiable on U. If $u, v \in U$ and $[u,v] \subset U$, then

$$\|f(u) - f(v)\| \leq \sup_{w \in [u,v]} \|f'(w)\| \cdot \|u - v\| .$$

Proof. If $f(u) = f(v)$ then immediately the claim is true. Otherwise, $f(v) - f(u) \neq 0$, and so by the Hahn-Banach extension theorem[10] there is some $\psi \in Y^*$ satisfying $\psi(f(v) - f(u)) = \|f(v) - f(u)\|$ and $\|\psi\| = 1$. Define $h : [0,1] \to \mathbb{R}$ by

$$h(t) = \langle f((1-t)u + tv), \psi \rangle .$$

For $0 < t < 1$ and $\tau \neq 0$ satisfying $t + \tau \in [0,1]$, we have

$$\begin{aligned}
\frac{h(t+\tau) - h(t)}{\tau} &= \frac{1}{\tau} \langle f((1-t-\tau)u + (t+\tau)v), \psi \rangle - \frac{1}{\tau} \langle f((1-t)u + tv), \psi \rangle \\
&= \left\langle \frac{f((1-t)u + tv + (v-u)\tau) - f((1-t)u + tv)}{\tau}, \psi \right\rangle .
\end{aligned}$$

Because f is Gâteaux differentiable at $(1-t)u + tv$,

$$\lim_{\tau \to 0} \frac{f((1-t)u + tv + (v-u)\tau) - f((1-t)u + tv)}{\tau} = f'((1-t)u + tv)(v-u),$$

so because ψ is continuous,

$$\lim_{\tau \to 0} \frac{h(t+\tau) - h(t)}{\tau} = \langle f'((1-t)u + tv)(v-u), \psi \rangle ,$$

which shows that h is differentiable at t and that

$$h'(t) = \langle f'((1-t)u + tv)(v-u), \psi \rangle .$$

$h : [0,1] \to \mathbb{R}$ is a composition of continuous functions so it is continuous. Applying the mean value theorem, there is some θ, $0 < \theta < 1$, for which

$$h'(\theta) = h(1) - h(0).$$

[9] Antonio Ambrosetti and Giovanni Prodi, *A Primer of Nonlinear Analysis*, p. 13, Theorem 1.8.

[10] Walter Rudin, *Functional Analysis*, second ed., p. 59, Corollary.

On the one hand,

$$h'(\theta) = \langle f'((1-\theta)u + \theta v)(v-u), \psi \rangle \, .$$

On the other hand,

$$h(1) - h(0) = \langle f(v), \psi \rangle - \langle f(u), \psi \rangle = \langle f(v) - f(u), \psi \rangle = \| f(v) - f(u) \| \, .$$

Therefore

$$\begin{aligned} \| f(v) - f(u) \| &= | \langle f'((1-\theta)u + \theta v)(v-u), \psi \rangle | \\ &\leq \| \psi \| \, \| f'((1-\theta)u + \theta v)(v-u) \| \\ &= \| f'((1-\theta)u + \theta v)(v-u) \| \\ &\leq \| f'((1-\theta)u + \theta v) \| \, \| v - u \| \\ &\leq \sup_{w \in [u,v]} \| f'(w) \| \, \| v - u \| \, . \end{aligned}$$

□

7 Antiderivatives

Suppose that X is a Banach space and that $f : [a,b] \to X$ be continuous. Define $F : [a,b] \to X$ by

$$F(x) = \int_a^x f.$$

Let $x_0 \in (a,b)$. For $x \in (a,b)$, we have

$$F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f = \int_{x_0}^x f = f(x_0)(x - x_0) + \int_{x_0}^x (f - f(x_0)),$$

from which it follows that F is Fréchet differentiable at x_0, and that

$$DF(x_0)(x - x_0) = f(x_0)(x - x_0).$$

If we identify $f(x_0) \in X$ with the map $x \mapsto f(x_0)x$, namely if we say that $X = \mathscr{B}(\mathbb{R}, X)$, then $DF(x_0) = f(x_0)$.

Let X be a normed space, let Y be a Banach space, let U be an open subset of X, and let $f \in C^1(U, Y)$. Suppose that $u, v \in U$ satisfy $[u, v] \subset U$. Write $I = (0,1)$ and define $\gamma : I \to U$ by $\gamma(t) = (1-t)u + tv$. We have

$$D\gamma(t) = v - u, \qquad t \in I,$$

and thus by Theorem 6,

$$D(f \circ \gamma)(t) = Df(\gamma(t)) \circ D\gamma(t), \qquad t \in I,$$

that is,

$$D(f \circ \gamma)(t) = Df(\gamma(t)) \circ (v - u), \qquad t \in I,$$

i.e.

$$D(f \circ \gamma)(t) = Df(\gamma(t))(v-u), \qquad t \in I.$$

If $t \in I$ and $t+h \in I$, then

$$\begin{aligned} D(f\circ\gamma)(t+h) - D(f\circ\gamma)(t) &= Df(\gamma(t+h))(v-u) - Df(\gamma(t))(v-u) \\ &= \left(Df(\gamma(t+h)) - Df(\gamma(t))\right)(v-u), \end{aligned}$$

and hence

$$\|D(f\circ\gamma)(t+h) - D(f\circ\gamma)(t)\| \le \|Df(\gamma(t+h)) - Df(\gamma(t))\| \, \|v-u\| \, .$$

Because $Df : U \to \mathscr{B}(X,Y)$ is continuous, it follows that

$$\|D(f\circ\gamma)(t+h) - D(f\circ\gamma)(t)\| \to 0$$

as $h \to 0$, i.e. that $D(f\circ\gamma)$ is continuous at t, and thus that

$$D(f\circ\gamma) : I \to \mathscr{B}(\mathbb{R}, Y)$$

is continuous. If we identify $\mathscr{B}(\mathbb{R}, Y)$ with Y, then

$$D(f\circ\gamma) : I \to Y.$$

On the one hand,

$$\int_0^1 D(f\circ\gamma) = (f\circ\gamma)(1) - (f\circ\gamma)(0) = f(v) - f(u).$$

On the other hand,

$$\int_0^1 D(f\circ\gamma) = \int_0^1 Df(\gamma(t))(v-u)dt = \left(\int_0^1 Df((1-t)u+tv)dt\right)(v-u);$$

here,

$$\int_0^1 Df((1-t)u+tv)dt \in \mathscr{B}(X,Y).$$

Therefore

$$f(v) - f(u) = \left(\int_0^1 Df((1-t)u+tv)dt\right)(v-u).$$