The Dirac delta distribution and Green's functions

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1 $u_s(x) = |x|^s$

If $s \in \mathbb{C}$ and $\Re s \geq 2$, then $u_s(x) = |x|^s$ is in $C^2_{\text{loc}}(\mathbb{R}^n)$.[1] $\Delta : C^2_{\text{loc}}(\mathbb{R}^n) \to C^0_{\text{loc}}(\mathbb{R}^n)$ and, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $|x|^2 = x_1^2 + \cdots + x_n^2$,

$$\begin{aligned}
(\Delta u_s)(x) &= \Delta |x|^s \\
&= \sum_{i=1}^n \frac{\partial}{\partial x_i} \frac{s}{2} \cdot 2x_i \cdot (|x|^2)^{\frac{s}{2}-1} \\
&= \sum_{i=1}^n \left(\frac{s}{2} \cdot 2 \cdot (|x|^2)^{\frac{s}{2}-1} + \frac{s}{2}\left(\frac{s}{2} - 1\right) \cdot (2x_i)^2 \cdot (|x|^2)^{\frac{s}{2}-2} \right) \\
&= ns \cdot |x|^{s-2} + s(s-2)|x|^{s-2} \\
&= s(s+n-2) \cdot |x|^{s-2} \\
&= s(s+n-2) \cdot u_{s-2}.
\end{aligned}$$

We take $n > 2$ in the following.

Typically we talk about functions $\mathbb{C} \to \mathbb{C}$ that are holomorphic (or meromorphic if they are defined on a subset of $\mathbb{C}$). But we can also talk about functions $\mathbb{C} \to V$ that are holomorphic/meromorphic for certain types of topological vector spaces over $\mathbb{C}$. In particular, we can talk about holomorphic/meromorphic functions that take values in the tempered distributions on $\mathbb{R}^n$. If $\Re s > -n$, then u_s is locally integrable (for any point in $\mathbb{R}^n$, there is a neighborhood of the point on which u_s is L^1), and hence it is a tempered distribution for $\Re s > -n$. Thus for $\Re s > -n$, Δu_s is a tempered distribution.

For $\Re s \geq 2$ we have

$$u_{s-2} = \frac{\Delta u_s}{s(s+n-2)},$$

and hence for $\Re s \geq 0$ we have

$$u_s = \frac{\Delta u_{s+2}}{(s+2)(s+n)}.$$

[1] This is all an expansion and gloss on Paul Garrett's note *Meromorphic continuations of distributions*, which is on his homepage.

As u_0 is a constant, $\Delta u_0 = 0$, and so $s - 2$ is a removable singularity of the right-hand side. It follows that u_s is meromorphic and that its only possible pole is at $s = -n$. One iterates this argument and obtains that u_s is meromorphic on $\mathbb{C}$, with at most simple poles at $s = -n, -n-2, -n-4, \ldots$.

Let $\gamma = e^{-|x|^2}$ and let f be a Schwartz function on $\mathbb{R}^n$. For $\Re s > -n-1$, we have $u_s \cdot (f - f(0)\gamma) \in L^1(\mathbb{R}^n)$ (the term $f - f(0)\gamma$ is certainly integrable at infinity and will still be integrable at infinity after being multiplied by $|x|^s$, and while $|x|^s$ might not be integrable at 0, the term $f - f(0)\gamma$ goes to 0 like $|x|^2$). The tempered distribution u_s maps the Schwartz function $f - f(0)\gamma$ to

$$u_s(f - f(0)\gamma) = \int_{\mathbb{R}^n} |x|^s \cdot (f(x) - f(0)\gamma(x))dx.$$

In the above equation (for fixed f), the right-hand side is holomorphic for $\Re s > -n-1$, thus so is the left. Hence the residue of the left side at $s = -n$ is 0:

$$\text{Res}_{s=-n} u_s(f - f(0)\gamma) = 0.$$

Thus

$$\begin{aligned}
\text{Res}_{s=-n} u_s(f) &= \text{Res}_{s=-n} u_s(f - f(0)\gamma) + \text{Res}_{s=-n} u_s(f(0)\gamma) \\
&= \text{Res}_{s=-n} u_s(f(0)\gamma) \\
&= f(0)\text{Res}_{s=-n} u_s(\gamma) \\
&= \delta(f)\text{Res}_{s=-n} u_s(\gamma).
\end{aligned}$$

Using polar coordinates, with $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$,

$$\begin{aligned}
\text{Res}_{s=-n} u_s(\gamma) &= \text{Res}_{s=-n} \int_{\mathbb{R}^n} |x|^s e^{-|x|^2} dx \\
&= \text{Res}_{s=-n} \int_0^\infty \int_{S^{n-1}} |rx'|^s e^{-|rx'|^2} r^{n-1} d\sigma(x') dr \\
&= \sigma(S^{n-1}) \text{Res}_{s=-n} \int_0^\infty r^{s+n-1} e^{-r^2} dr \\
&= \frac{\sigma(S^{n-1})}{2} \text{Res}_{s=-n} \int_0^\infty t^{\frac{s+n-2}{2}} e^{-t} dt \\
&= \frac{\sigma(S^{n-1})}{2} \text{Res}_{s=-n} \Gamma(\frac{s+n}{2}).
\end{aligned}$$

As $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned}
\text{Res}_{s=-n} u_s(\gamma) &= \frac{\sigma(S^{n-1})}{2} \text{Res}_{s=-n} \frac{2}{s+n} \\
&= \frac{\sigma(S^{n-1})}{2} \cdot 2 \\
&= \sigma(S^{n-1}).
\end{aligned}$$

Therefore for any Schwartz function f, we have

$$\mathrm{Res}_{s=-n} u_s(f) = \sigma(S^{n-1}) \cdot \delta(f),$$

hence

$$\mathrm{Res}_{s=-n} u_s = \sigma(S^{n-1}) \cdot \delta.$$

We know that u_s has poles at most at $s = -n, -n-2, -n-4, \ldots$, and we have just explicitly found its residue at $s = -n$.

This fact has an important consequence. As u_s has a simple pole at $s = -n$, the value of $(s+n)u_s$ at $s = -n$ is $\mathrm{Res}_{s=-n} u_s$. But

$$u_s = \frac{\Delta u_{s+2}}{(s+2)(s+n)},$$

so

$$\Delta u_{-n+2} = (-n+2) \cdot \sigma(S^{n-1}) \cdot \delta,$$

i.e.

$$\Delta \frac{1}{|x|^{n-2}} = (-n+2) \cdot \sigma(S^{n-1}) \cdot \delta,$$

with $\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. Recall that we have assumed $n > 2$. In other words, we have just determined the Green's function of the Laplace operator on $\mathbb{R}^n$, $n > 2$.

2 $w_s(x) = |x|^s \cdot \log|x|$

If $\Re s > 2$, then $w_s(x) = |x|^s \cdot \log|x| \in C^2_{\text{loc}}(\mathbb{R}^2)$. Let $u_s(x) = |x|^s$. We have

$$\begin{aligned}
(\Delta w_s)(x) &= \frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\Big((x_1^2+x_2^2)^{\frac{s}{2}}\log(x_1^2+x_2^2)\Big) \\
&= \frac{1}{2}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\Big(\frac{s}{2}(x_1^2+x_2^2)^{\frac{s}{2}-1}\cdot 2x_i\cdot\log(x_1^2+x_2^2) \\
&\quad +(x_1^2+x_2^2)^{\frac{s}{2}}\cdot\frac{2x_i}{x_1^2+x_2^2}\Big) \\
&= \sum_{i=1}^{2}\frac{\partial}{\partial x_i}\Big(\frac{s}{2}(x_1^2+x_2^2)^{\frac{s}{2}-1}\cdot x_i\cdot\log(x_1^2+x_2^2)+(x_1^2+x_2^2)^{\frac{s}{2}-1}\cdot x_i\Big) \\
&= \sum_{i=1}^{2}\frac{s}{2}\left(\frac{s}{2}-1\right)(x_1^2+x_2^2)^{\frac{s}{2}-2}\cdot 2x_i^2\cdot\log(x_1^2+x_2^2) \\
&\quad +\frac{s}{2}(x_1^2+x_2^2)^{\frac{s}{2}-1}\cdot\log(x_1^2+x_2^2)+\frac{s}{2}(x_1^2+x_2^2)^{\frac{s}{2}-1}\frac{2x_i^2}{x_1^2+x_2^2} \\
&\quad +\left(\frac{s}{2}-1\right)(x_1^2+x_2^2)^{\frac{s}{2}-2}\cdot 2x_i^2+(x_1^2+x_2^2)^{\frac{s}{2}-1} \\
&= \sum_{i=1}^{2}s(s-2)|x|^{s-4}\cdot x_i^2\cdot\log|x|+s|x|^{s-2}\cdot\log|x|+s|x|^{s-4}\cdot x_i^2 \\
&\quad +(s-2)|x|^{s-4}\cdot x_i^2+|x|^{s-2} \\
&= s(s-2)|x|^{s-2}\log|x|+2s|x|^{s-2}\log|x|+s|x|^{s-2}+(s-2)|x|^{s-2}+2|x|^{s-2} \\
&= s^2 w_{s-2}(x)+2su_{s-2}(x).
\end{aligned}$$

Hence

$$\Delta w_s = s^2 w_{s-2} + 2su_{s-2},$$

and so

$$(s+2)^2 w_s = -2(s+2)u_s + \Delta w_{s+2}.$$

We calculate

$$\begin{aligned}
\int_{\mathbb{R}^2}|x|^s\log|x|e^{-|x|^2}dx &= \int_0^\infty\int_{S^1}|rx'|^s\log|rx'|e^{-|rx'|^2}rd\sigma(x')dr \\
&= 2\pi\int_0^\infty r^{s+1}\cdot\log r\cdot e^{-r^2}dr \\
&= 2\pi\cdot\frac{1}{4}\Gamma\left(1+\frac{s}{2}\right)\psi\left(1+\frac{s}{2}\right),
\end{aligned}$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, namely the digamma function. Using $\Gamma(z+1) = z\Gamma(z)$ and

$\psi(z+1) = \psi(z) + \frac{1}{z}$, with $\gamma(x) = e^{-x^2}$,

$$\begin{aligned}
\mathrm{Res}_{s=-2}(s+2)w_s(\gamma) &= \frac{\pi}{2} \cdot \mathrm{Res}_{s=-2}(s+2)\frac{1}{1+\frac{s}{2}}\left(\psi\left(1+\frac{s}{2}+1\right) - \frac{1}{1+\frac{s}{2}}\right) \\
&= \pi \cdot \mathrm{Res}_{s=-2}\left(-C - \frac{2}{s+2}\right) \\
&= -2\pi,
\end{aligned}$$

where C is Euler's constant; it is a fact that $\psi(1) = -C$.[2] Thus like in the previous section, if f is a Schwartz function then

$$\mathrm{Res}_{s=-2}(s+2)w_s(f) = \delta(f)\mathrm{Res}_{s=-2}(s+2)w_s(\gamma) = -2\pi \cdot \delta(f).$$

Because

$$(s+2)^2 w_s = -2(s+2)u_s + \Delta w_{s+2},$$

the value of $(s+2)^2 w_s$ at $s=-2$ is $\Delta w_0 - 2 \cdot \mathrm{Res}_{s=-2} u_s$. On the other hand, the value of $(s+2)^2 w_s$ at $s=-2$ is $\mathrm{Res}_{s=-2}(s+2)w_s = -2\pi \cdot \delta$, hence

$$\Delta w_0 = -2\pi \cdot \delta + 2 \cdot \mathrm{Res}_{s=-2} u_s.$$

We can calculate $\mathrm{Res}_{s=-2} u_s$ just like in the previous section. If f is a Schwartz function and $\gamma = e^{-|x|^2}$, then

$$\begin{aligned}
\mathrm{Res}_{s=-2} u_s(f) &= \delta(f)\mathrm{Res}_{s=-2} u_s(\gamma) \\
&= \delta(f)\mathrm{Res}_{s=-2}\frac{1}{2}\Gamma\left(1+\frac{s}{2}\right) \\
&= 2\pi \cdot \delta(f).
\end{aligned}$$

Therefore

$$\Delta w_0 = 2\pi \cdot \delta,$$

i.e.,

$$\Delta \log|x| = 2\pi \cdot \delta.$$

Recall that here $n = 2$. In other words, we have just determined the Green's function of the Laplace operator on $\mathbb{R}^2$.

3 Dirac comb

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. On $\mathbb{R}^n$, tempered distributions integrate against a larger class of functions than do distributions, so it's stronger to be a tempered distribution. But on $\mathbb{T}$, any Schwartz function has compact support, and moreover, any C^∞

[2] Historical note: In the papers of Euler's that I've seen where he mentions the Euler constant, the notation he uses is either C or O, not once the modern γ.

function on $\mathbb{T}$ is a Schwartz function. Thus distributions on $\mathbb{T}$ integrate smooth functions on $\mathbb{T}$. For $\Re s > 2$, define the following distribution on $\mathbb{T}$:

$$u_s = \sum_{\substack{0<\frac{p}{q}\le 1\\ \gcd(p,q)=1}} \frac{1}{q^s}\cdot \delta_{p/q}.$$

Why is this in fact a distribution? If $f \in C^\infty(\mathbb{T})$ then f is certainly bounded (indeed, u_s can take any continuous function on $\mathbb{T}$ as an argument, not just smooth functions). Let $|f(t)| \le K$ for all $t \in \mathbb{T}$. Then,

$$|u_s(f)| \le K \sum_{\substack{0<\frac{p}{q}\le 1\\ \gcd(p,q)=1}} \frac{1}{q^{\Re s}} < K \sum_{q=1}^{\infty} \frac{q}{q^{\Re s}}.$$

Since $\Re s > 2$, this series converges.

Doing some series manipulations we get (probably the hardest step to see is that summing over the products of d and q is the same as summing over q and then over those d that divide it)

$$\begin{aligned}
\zeta(s)\cdot u_s &= \sum_{d\ge 1}\frac{1}{d^s}\sum_{q\ge 1}\frac{1}{q^s}\sum_{\substack{0<p\le q\\ \gcd(p,q)=1}} \delta_{p/q}\\
&= \sum_{d\ge 1}\sum_{q\ge 1}\frac{1}{(qd)^s}\sum_{\substack{0<pd\le qd\\ \gcd(pd,qd)=d}} \delta_{\frac{dp}{dq}}\\
&= \sum_{q=1}^{\infty}\frac{1}{q^s}\sum_{\substack{d|q\\ d\ge 1}}\sum_{\substack{0<p\le q\\ \gcd(p,q)=d}} \delta_{p/q}\\
&= \sum_{q=1}^{\infty}\frac{1}{q^s}\sum_{0<p\le q} \delta_{p/q}\\
&= v_s.
\end{aligned}$$

(The last equality is a definition.) To summarize: $\zeta(s)\cdot u_s = v_s$.

Supposing we are interested in u_s, using the above formula we can instead investigate v_s, which for some purposes is more analytically tractable. We shall determine the Fourier series of v_s. For $\Re s > 2$ and for $n \in \mathbb{Z}$ (recalling that v_s is a distribution, i.e. it integrates functions)

$$\begin{aligned}
\widehat{v_s}(n) &= v_s(e^{-2\pi i n x})\\
&= \sum_{q=1}^{\infty}\frac{1}{q^s}\sum_{0<p\le q}\delta_{p/q}(e^{-2\pi i n x})\\
&= \sum_{q=1}^{\infty}\frac{1}{q^s}\sum_{0<p\le q} e^{-2\pi i n p/q}.
\end{aligned}$$

$p \mapsto e^{-2\pi i n p/q}$, $\mathbb{Z}/q \to \mathbb{C}$, is a character, and, unless it is the trivial character, the sum over $\mathbb{Z}/q$ is equal to 0. So if $q \not| n$ then the inner sum is 0, and if $q|n$ then the inner sum is equal to q. (If the language of characters of $\mathbb{Z}/q$ isn't familiar, you can check this fact directly; to show the inner sum is 0, you show that the inner sum is equal to itself times something that is nonzero.) Thus

$$\widehat{v}_s(n) = \sum_{\substack{q|n \\ q \geq 1}} \frac{1}{q^{s-1}}.$$

For $n = 0$, we get

$$\widehat{v}_s(0) = \zeta(s-1).$$

Otherwise, the above can be written using a standard arithmetic function, the sum of powers of positive divisors. Let $\sigma_\alpha(n)$ denote the sum of the αth powers of the positive divisors of n. Thus for $n \neq 0$ we have

$$\widehat{v}_s(n) = \sigma_{1-s}(n).$$

Using $\zeta(s) \cdot u_s = v_s$ we get

$$\widehat{u}_s(n) = \begin{cases} \frac{\zeta(s-1)}{\zeta(s)} & n = 0, \\ \frac{\sigma_{1-s}(n)}{\zeta(s)} & n \neq 0. \end{cases}$$

The expression on the right-hand side has poles at $s = 2$ and at the zeros of the Riemann zeta function. Otherwise, for a fixed s, the right-hand side has at most polynomial growth in n, and therefore it is the Fourier series of a distribution on $\mathbb{T}$ (see Katznelson, p. 48, Chapter 1, Exercise 7.5), and for $\Re s \leq 2$ we shall define u_s to be this distribution. In summary: u_s is originally defined as a distribution for $\Re s > 2$, and now we have defined it to be a distribution for $s \neq 2$ and $\zeta(s) \neq 0$. Thus u_s is a meromorphic distribution valued functions on $\mathbb{C}$ with poles at $s = 2$ and at the zeros of the Riemann zeta function.

Since $\zeta(1) = \infty$, if $n \neq 0$ then $\widehat{u}_1(n) = 0$. The only pole of the Riemann zeta function is at $s = 1$, hence $\zeta(0)/\zeta(1) = 0$. Thus $\widehat{u}_1(0) = 0$, and it follows that as a distribution on $\mathbb{T}$,

$$u_1 = 0$$

(although the distribution u_1 is 0, this doesn't mean that we can put $s = 1$ into the original definition of u_s and assert that this is 0, as the original definition of u_s was only for $\Re s > 2$, and we have analytically continued u_s as a meromorphic distribution valued function on $\mathbb{C}$. Likewise, although $\zeta(0) = -\frac{1}{2}$, it is incorrect to conclude that $1 + 1 + 1 + \cdots = -\frac{1}{2}$, although for certain formal arguments this may be a correct interpretation.).