Denjoy's theorem on circle diffeomorphisms

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In this note I'm just presenting the proof of Denjoy's theorem in Michael Brin and Garrett Stuck's *Introduction to dynamical systems*, Cambridge University Press, 2002.

Let $S^1 = \mathbb{R}/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, define $R_\alpha : S^1 \to S^1$ by $R_\alpha(x) = x + \alpha + \mathbb{Z}$.

We say that a homeomorphism $f : S^1 \to S^1$ is *orientation preserving* if it lifts to an increasing homeomorphism $F : \mathbb{R} \to \mathbb{R}$: $\pi \circ F = f \circ \pi$.

The *rotation number* of an orientation preserving homeomorphism f is defined by

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

One proves that this is independent both of the lift F of f and the point $x \in \mathbb{R}$. Some facts about the rotation number: it is an invariant of topological conjugacy, and $\rho(f)$ is rational if and only if f has a periodic point. A periodic point is $x \in S^1$ such that $f^n(x) = x$ for some $n \geq 1$.

There are some lemmas in Chapter 7 that I don't want to write out. The important theorem that we're going to use without proof is that if $f : S^1 \to S^1$ is an orientation preserving homeomorphism that is topologically transitive with irrational rotation number $\rho(f)$, then f is topologically conjugate to $R_{\rho(f)}$. This reduces our problem to showing that a map is topologically transitive.

We will use the following lemma in the proof of Denjoy's theorem.

Lemma 1. *Let $f : S^1 \to S^1$ be a C^1 diffeomorphism and let J be an interval in S^1. Let $g = \log f'$. If the interiors of $J, f(J), \ldots, f^{n-1}(J)$ are pairwise disjoint, then for any $x, y \in J$ and any $n \in \mathbb{Z}$ we have*

$$\mathrm{Var}(g) \geq |\log((f^n)'(x)) - \log((f^n)'(y))|.$$

Proof. The intervals $[x, y], [f(x), f(y)], \ldots, [f^{n-1}(x), f^{n-1}(y)]$ are pairwise disjoint, so they are part of a partition of $[0, 1]$. The total variation of g is defined as a supremum over all partitions, so in particular it will be $\geq$ the sum coming from any particular partition or a subset of that partition.

$$
\begin{aligned}
\mathrm{Var}(g) &\geq \sum_{k=0}^{n-1} |g(f^k(y)) - g(f^k(x))| \\
&\geq \Big| \sum_{k=0}^{n-1} g(f^k(y)) - g(f^k(x)) \Big| \\
&= \Big| \log \prod_{k=0}^{n-1} f'(f^k(y)) - \log \prod_{k=0}^{n-1} f'(f^k(x)) \Big| \\
&= |\log((f^n)'(x)) - \log((f^n)'(y))|.
\end{aligned}
$$

□

Now we can prove Denjoy's theorem.

Theorem 2. *If $f : S^1 \to S^1$ is a C^1 diffeomorphism that is orientation preserving, that has irrational rotation number $\rho(f)$, and whose derivative $f' : S^1 \to \mathbb{R}$ has bounded variation, then f is topologically conjugate to $R_{\rho(f)}$.*

Proof. Suppose by contradiction that f is not topologically transitive. It's a fact proved in Chapter 7 of Brin and Stuck that this implies that $\omega(x)$ is perfect and nowhere dense, and is independent of the point x. (Recall that $\omega(x) = \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} f^i(x)}$.) It follows that there is an interval $I = (a, b)$ in its complement.

The intervals $f^n(I)$, $n \in \mathbb{Z}$, are pairwise disjoint, for otherwise f would have a periodic point. Let μ be Haar measure on S^1. Then

$$
\sum_{n\in\mathbb{Z}} \mu(f^n(I)) \leq 1.
$$

Let $x \in S^1$. Suppose for the moment that there are infinitely $n \geq 1$ such that the intervals $(x, f^{-n}(x)), (f(x), f^{1-n}(x)), \ldots, (f^n(x), x)$ are pairwise disjoint; we shall prove that this is true later. By applying the lemma we proved with $y = f^{-n}(x)$ we get

$$
\mathrm{Var}(g) \geq \left| \log \frac{(f^n)'(x)}{(f^n)'(y)} \right| = |\log((f^n)'(x)(f^{-n})'(x)|.
$$

To see the equality in the above line it helps to write out what $(f^{-n})'(x)$ is.

Then for infinitely many n we have

$$\begin{aligned}
\mu(f^n(I)) + \mu(f^{-n}(I)) &= \int_I (f^n)'(x)dx + \int_I (f^{-n})'(x)dx \\
&= \int_I ((f^n)'(x) + (f^{-n})'(x))dx \\
&\geq \int_I \sqrt{(f^n)'(x)(f^{-n})'(x)}dx \\
&= \int_I \sqrt{\exp\log((f^n)'(x)(f^{-n})'(x))}dx \\
&\geq \int_I \sqrt{\exp(-|\log((f^n)'(x)(f^{-n})'(x))|)}dx \\
&\geq \int_I \sqrt{\exp(-\operatorname{Var}(g))}dx \\
&= \exp\Big(-\frac{1}{2}\operatorname{Var}(g)\Big)\mu(I).
\end{aligned}$$

Since $\mu(I) > 0$ this implies that $\sum_{n\in\mathbb{Z}} \mu(f^n(I)) = \infty$, a contradiction. Therefore f is topologically transitive, and so it is topologically conjugate to the $R_{\rho(f)}$. $\square$

It is indeed necessary that f' has bounded variation. Brin and Stuck give an example on p. 161 that they attribute to Denjoy: for any irrational number $\rho \in (0, 1)$, there is a nontransitive orientation preserving C^1 diffeomorphism of S^1 with rotation number ρ. The only condition of Denjoy's theorem that isn't satisfied here is that f' have bounded variation.