# Decomposition of the spectrum of a bounded linear operator

Jordan Bell

April 3, 2014

## 1 Definitions

Let $H$ be a complex Hilbert space. If $\lambda \in \mathbb{C}$, we also write $\lambda$ to denote $\lambda \cdot \mathrm{id}_H \in B(H)$.

For $T \in B(H)$, the *spectrum* $\sigma(T)$ of $T$ is the set of those $\lambda \in \mathbb{C}$ such that the map $T - \lambda$ is not bijective.[1] It happens that it is useful for some purposes to write $\sigma(T)$ as a union of three particular disjoint subsets of itself.

- The *point spectrum* $\sigma_{\mathrm{point}}(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not injective. Equivalently, $\lambda \in \sigma_{\mathrm{point}}(T)$ if $\lambda$ is an *eigenvalue* of $T$.[2]
- The *continuous spectrum* $\sigma_{\mathrm{cont}}(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective, has dense image, and is not surjective.
- The *residual spectrum* $\sigma_{\mathrm{res}}(T)$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda$ is injective and does not have dense image.

It is apparent that the sets $\sigma_{\mathrm{point}}(T)$, $\sigma_{\mathrm{cont}}(T)$, and $\sigma_{\mathrm{res}}(T)$ are disjoint, and that

$$\sigma(T) = \sigma_{\mathrm{point}}(T) \cup \sigma_{\mathrm{cont}}(T) \cup \sigma_{\mathrm{res}}(T).$$

If $T \in B(H)$ then $\sigma(T) \neq \emptyset$, but any of the above three sets may be empty; they merely can't all be empty for a given operator.

[1] $\sigma(T)$ is defined to be the set of $\lambda \in \mathbb{C}$ such that $v \mapsto Tv - \lambda v$ is not a bijection. It is a fact that if $T \in B(H)$ and $v \mapsto Tv - \lambda v$ is a bijection then it is an element of $B(H)$. That it is linear can be proved quickly. The fact that it is bounded is proved using the *open-mapping theorem*, which states that a surjective bounded linear map from one Banach space to another is an open map, from which it follows that a bijective bounded linear map from one Banach space to another has a bounded inverse.

[2] The point spectrum is often called the *discrete spectrum*. From the definition by itself it is not apparent what $\sigma_{\mathrm{point}}(T)$ has to do either with points or discreteness.

## 2 Residual spectrum

If $T \in B(H)$ is a normal operator then $\sigma_{\mathrm{res}}(T) = \emptyset$.[3] We prove this. Suppose that $T-\lambda$ is injective. We have to show that $\mathrm{im}\,(T-\lambda)$ is dense in $H$, and thus that $\lambda \notin \sigma_{\mathrm{res}}(T)$. ($\lambda$ might be in $\sigma_{\mathrm{cont}}(T)$ or might not be in $\sigma(T)$; we merely want to show that it is not in $\sigma_{\mathrm{res}}(T)$.) We have

$$H = \overline{\mathrm{im}\,(T-\lambda)} \oplus (\mathrm{im}\,(T-\lambda))^{\perp}.$$

Let $w \in (\mathrm{im}\,(T-\lambda))^{\perp}$; we have to show that $w = 0$. For all $v \in H$,

$$\langle (T-\lambda)v, w\rangle = 0,$$

so for all $v \in H$ we have $\langle v, (T-\lambda)^* w\rangle = 0$ and therefore $(T-\lambda)^* w = 0$, so $w \in \ker(T-\lambda)^* = \ker(T-\lambda)$.[4] As $T-\lambda$ is injective, $w = 0$, completing the proof.

## 3 Point spectrum

If $A \in B(H)$ is normal then it is straightforward to show that $\ker A = \ker A^*$. Also, if $T \in B(H)$ is normal then for any $z \in \mathbb{C}$, $T - z$ is normal. Thus, $\ker(T-z) = \{0\}$ if and only if $\ker((T-z)^*) = \{0\}$. That is, $\lambda \in \sigma_{\mathrm{point}}(T)$ if and only if $\overline{\lambda} \in \sigma_{\mathrm{point}}(T^*)$. For $X \subseteq \mathbb{C}$ we define $X^* = \{\overline{z} : z \in X\}$. We have shown that if $T \in B(H)$ is normal then

$$\sigma_{\mathrm{point}}(T)^* = \sigma_{\mathrm{point}}(T^*).$$

## 4 Continuous spectrum

If $\lambda \in \sigma_{\mathrm{cont}}(T)$, then $\mathrm{im}\,(T-\lambda)$ is dense in $H$. Also,

$$(T-\lambda)^{-1} : \mathrm{im}\,(T-\lambda) \to H$$

is a surjective linear map (the inverse of a linear map is itself a linear map) that is not continuous. For $\mathrm{im}\,(T-\lambda)$ is dense in $H$, so if $(T-\lambda)^{-1}$ were continuous then it would have a unique extension to a continuous, hence bounded, map $H \to H$. Using this and the fact that $T-\lambda$ is not surjective will give a contradiction.

## 5 Approximate point spectrum

Let $\lambda \in \mathbb{C} \setminus (\sigma_{\mathrm{point}}(T) \cup \sigma_{\mathrm{res}}(T))$. I claim that $\lambda \in \sigma_{\mathrm{cont}}(T)$ if and only if $\lambda$ is in the *approximate point spectrum* of $T$, the set of those $\lambda \in \mathbb{C}$ such that there

[3] $T \in B(H)$ is *normal* if $T^*T = TT^*$; in particular, a self-adjoint operator is normal. Equivalently, $T \in B(H)$ is normal if and only if $\|Tv\| = \|T^*v\|$ for all $v \in H$.

[4] If $S$ is normal then $\ker S = \ker S^*$. Proof: If $v \in \ker S$ then $\langle Sv, Sv\rangle = 0$, hence $\langle S^*Sv, v\rangle = 0$, hence $\langle SS^*v, v\rangle = 0$, hence $\langle S^*v, S^*v\rangle = 0$, hence $S^*v = 0$, hence $v \in \ker S^*$.

is is a sequence $v_n \in H$ with $\|v_n\| = 1$ and $\|(T-\lambda)v_n\| \to 0$ as $n \to \infty$. It is a fact that for $T \in B(H)$, $T$ is invertible if and only if $T(H)$ is dense in $H$ and there is some $\alpha > 0$ such that $\|Tv\| \geq \alpha \|v\|$ for all $v \in H$.[5] If $\lambda \in \sigma_{\mathrm{cont}}(T)$, then $T - \lambda$ is not invertible but the image of $T - \lambda$ is dense in $H$, then it must therefore be that $T - \lambda$ is not bounded below. That is, there is no $\alpha > 0$ such that for every $w \in H$ we have $\|(T-\lambda)w\| \geq \alpha \|w\|$. Then for each $n$ there is some $w \in H$ such that $\|(T-\lambda)w_n\| < \frac{1}{n}\|w_n\|$. Let $v_n = \frac{w_n}{\|w_n\|}$. We have $\|v_n\| = 1$ and $\|(T-\lambda)v_n\| < \frac{1}{n}$, showing that $\lambda \in \sigma_{\mathrm{ap}}(T)$.

On the other hand, if $\lambda \in \sigma_{\mathrm{ap}}(T)$ then there is a sequence $v_n \in H$ such that $\|(T-\lambda)v_n\| \to 0$ as $n \to \infty$. Then for any $\alpha > 0$, there is some $v_n$ such that $\|(T-\lambda)v_n\| < \alpha = \alpha \|v_n\|$, so $T - \lambda$ is not invertible and hence $\lambda \in \sigma(T)$. Since we assumed that $\lambda \in \mathbb{C} \setminus (\sigma_{\mathrm{point}}(T) \cup \sigma_{\mathrm{res}}(T))$, we then have $\lambda \in \sigma_{\mathrm{cont}}(T)$.

Halmos shows in Problem 62 of his *Hilbert Space Problem Book* that $\sigma_{\mathrm{ap}}(T)$ is a closed subset of $\mathbb{C}$, and proves in Problem 63 that $\partial\sigma(T) \subseteq \sigma_{\mathrm{ap}}(T)$: the boundary of the spectrum of $T$ is contained in the approximate point spectrum of $T$.

# 6 Normal operators

We showed earlier that if $T \in B(H)$ is normal then $\sigma_{\mathrm{point}}(T)^* = \sigma_{\mathrm{point}}(T^*)$, where, for $X \subseteq \mathbb{C}$, $X^* = \{\overline{z} : z \in X\}$. This is one reason why it can be helpful to know that an operator is normal. Using this we can show something more about normal operators. Let $T \in B(H)$ be normal and suppose that $\lambda, \mu \in \sigma_{\mathrm{point}}(T)$ are distinct. Then there are nonzero $v, w \in H$ with $Tv = \lambda v$ and $Tw = \mu w$. Using $T^*w = \overline{\mu}w$ we get

$$\begin{aligned}
\lambda \langle v, w \rangle &= \langle \lambda v, w \rangle \\
&= \langle Tv, w \rangle \\
&= \langle v, T^*w \rangle \\
&= \langle v, \overline{\mu}w \rangle \\
&= \mu \langle v, w \rangle .
\end{aligned}$$

As $\lambda \neq \mu$, this means that $\langle v, w \rangle = 0$. In words, if $T \in B(H)$ is normal, then its eigenspaces are mutually orthogonal.

# 7 Compact operators

Let $K(H)$ be the closure in $B(H)$ of the set of finite-rank operators. We call the elements of $K(H)$ *compact operators*. The following are equivalent ways to state that an operator is compact.

[5] In words, $T \in B(H)$ is invertible if and only if it has dense image and is *bounded below*. This result is proved in Paul Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, p. 38, §21, Theorem 3.

- $T \in B(H)$ is compact if and only if for every bounded subset $S$ of $H$, the closure of the image $T(S)$ is compact.

- $T \in B(H)$ is compact if and only if for every sequence $v_n \in H$ with $\|v_n\| = 1$, $T(v_n)$ has a convergent subsequence.

- Let $B$ be the closed unit ball in $H$, and let $B$ be a topological space with the *weak topology*: a net $v_\alpha \in B$ converges weakly to $v \in B$ if for all $w \in H$ we have $\langle v_\alpha, w\rangle \to \langle v, w\rangle$. If $H$ is separable, then the weak topology on $B$ is metrizable and thus can be characterized using merely sequences instead of nets.[6] $T \in B(H)$ is compact if and only if the restriction of $T$ to $B$ is continuous $B \to H$, where $B$ has the weak topology and $H$ has the norm topology.

If $K \in K(H)$ and $V$ is a closed subspace of $V$ such that $K(V) \subseteq V$, then the restriction of $K$ to $V$ is an element of $K(V)$.

$B(H)$ is a $C^*$-algebra, and $K(H)$ is a $C^*$-subalgebra of $B(H)$. If $T \in K(H)$ and $S \in B(H)$, then

$$ST, TS \in K(H).$$

Hence $K(H)$ is an *ideal* of the $C^*$-algebra $B(H)$.[7]

Useful facts about compact operators are proved in Yuri A. Abramovich and Charalambos D. Aliprantis, *An Invitation to Operator Theory*, p. 272, §7.1.

# 8 Fredholm alternative

The *Fredholm alternative* states that if $K \in K(H)$, $\lambda \neq 0$, and $\ker(K-\lambda) = \{0\}$ then $(K-\lambda)^{-1} \in B(H)$.[8] Equivalently, if $K \in K(H)$, $\lambda \neq 0$, and $\lambda \notin \sigma_{\mathrm{point}}(K)$ then $\lambda \notin \sigma(K)$. Equivalently, if $K \in K(H)$, then

$$\sigma(K) \subseteq \sigma_{\mathrm{point}}(K) \cup \{0\}.$$

The above forms are the ones that we want to use. The following is the one that we want to prove, which is equivalent because a nonzero multiple of a compact operator is compact: If $K \in K(H)$ and $\ker(\mathrm{id}_H - K) = \{0\}$ then $(\mathrm{id}_H - K)^{-1} \in B(H)$.

We prove two standalone lemmas that we then use to prove the Fredholm alternative.

---

[6] This is proved in Paul Halmos, *Hilbert Space Problem Book*, Problem 18.

[7] The $C^*$-algebra $B(H)/K(H)$ is called the *Calkin algebra* of $H$. $T \in B(H)$ is called a *Fredholm operator* if $T + K(H)$ is an invertible element of the Calkin algebra. In particular, if $T \in K(H)$ then $\mathrm{id}_H - T$ is a Fredholm operator.

$T \in B(H)$ is a Fredholm operator if and only if the following three conditions holds: $\operatorname{im} T$ is closed in $H$, $\ker T$ is finite dimensional, and $\ker T^*$ is finite dimensional. This equivalence is called *Atkinson's theorem*. The *index* of a Fredholm operator is $\dim\ker T - \dim\ker T^*$. If $T \in K(H)$, then $\mathrm{id}_H - T$ has index 0.

[8] We are following Paul Halmos, *Hilbert Space Problem Book*, p. 293, Problem 140.

- Let $K \in K(H)$ let $A = \mathrm{id}_H - K \in B(H)$, and suppose that $A(H) = H$. Define $K_n = \ker(A^n)$. We have $K_1 \subseteq K_2 \subseteq \cdots$. Assume by contradiction that $K_1 \neq \{0\}$. Then there is some nonzero $f_1 \in K_1$. As $A(H) = H$, there is some $f_2 \in H$ with $Af_2 = f_1$; but $A^2 f_2 = Af_1 = 0$ and $Af_2 = f_1 \neq 0$, so $f_2 \in K_2 \setminus K_1$. Let $f_{n+1} \in K_{n+1} \setminus K_n$ with $Af_{n+1} = f_n$. Therefore $K_1, K_2, \dots$ are a strictly increasing sequence of subspaces of $H$. Using Gram-Schmidt, there is an orthonormal sequence $e_1, e_2, \dots$ with $e_n \in K_n$ for all $n$; we caution that we do not necessarily have $Ae_{n+1} = e_n$. As $Ae_{n+1} \in K_n$, $\langle Ae_{n+1}, e_{n+1} \rangle$, giving

  $$\|Ke_{n+1}\|^2 = \|e_{n+1} - Ae_{n+1}\|^2 = \|e_{n+1}\|^2 + \|Ae_{n+1}\|^2 \geq 1 \|e_{n+1}\|^2 = 1.$$

  Each $e_n$ is an element of the closed unit ball $B$, and $e_n \to 0$ weakly (this is the case for any orthonormal sequence in $H$, basis or not, and is proved using Bessel's inequality). Since $K$ is compact, it is continuous $B \to H$ where $B$ has the weak topology and $H$ has the norm topology; but $e_n \to 0$, $K(0) = 0$, and $\|Ke_n\| \geq 1$, so $Ke_n$ does not converge to 0 in $H$, a contradiction. Therefore $K_1 = \{0\}$, that is, $\ker A = \{0\}$.

- Let $K \in K(H)$ and let $A = \mathrm{id}_H - K \in B(H)$. Suppose by contradiction that $A$ is not bounded below on $(\ker A)^\perp$. So for every $\alpha > 0$ there is some $w \in (\ker A)^\perp$ such that $\|Aw\| \geq \alpha \|w\|$. Then for all $n$ there is some $w_n \in (\ker A)^\perp$ with $\|Aw_n\| < \frac{1}{n} \|w_n\|$. Let $v_n = \frac{w_n}{\|w_n\|} \in (\ker A)^\perp$. Then

  $$\|Av_n\| < \frac{1}{n}.$$

  As $K$ is compact, there is some subsequence $v_{a(n)}$ such that $Kv_{a(n)}$ converges to some $v \in H$. $Av_n \to Av$ and $Av_n \to 0$, so $v \in \ker A$. On the other hand, because $v_n = Av_n + Kv_n \to v$ and $(\ker A)^\perp$ is closed, we have $v \in (\ker A)^\perp$, so $v \in \ker A \cap (\ker A)^\perp = \{0\}$. But as $\|v_n\| = 1$ for each $n$, we have $\|v\| = 1$, a contradiction. Therefore, $A$ is bounded below on $(\ker A)^\perp$.

If $K \in K(H)$ and $A = \mathrm{id}_H - K$, then by the second of the two lemmas, we have that $A$ is bounded below on $(\ker A)^\perp$: there is some $\alpha > 0$ such that $\|Av\| \geq \alpha \|v\|$ for all $v \in (\ker A)^\perp$. If

$$w_n \in A(H) = A(\ker A \oplus (\ker A)^\perp) = A((\ker A)^\perp)$$

with $w_n \to w \in H$, then there are $v_n \in (\ker A)^\perp$ such that $Av_n = w_n$. As $w_n$ converges, for all $\epsilon > 0$ there is some $N$ such that if $n, m \geq N$ then $\|w_n - w_m\| \leq \epsilon$, so $\|A(v_n - v_m)\| = \|Av_n - Av_m\| \leq \epsilon$. But

$$\|A(v_n - v_m)\| \geq \alpha \|v_n - v_m\|,$$

so

$$\|v_n - v_m\| \leq \frac{\epsilon}{\alpha},$$

so $v_n$ is a Cauchy sequence and hence converges, say to $v$. $v_n \in (\ker A)^\perp$, which is closed, so $v \in (\ker A)^\perp$. Then $w_n = Av_n \to Av \in A(H)$. Therefore, if $K \in K(H)$ and $A = \mathrm{id}_H - K$, then $A(H)$ is closed in $H$.

Let $K \in K(H)$ and $A = \mathrm{id}_H - K$, and suppose that $\ker A = \{0\}$. By the above paragraph, $A(H)$ is closed in $H$. $K^* \in K(H)$ and $A^* = \mathrm{id}_H - K^*$,[9] so by the above paragraph we also get that $A^*(H)$ is closed in $H$. It is a fact that if $T \in B(H)$ then $\ker T^* = (T(H))^\perp$, so using this with $T = A^*$ we get

$$\ker A = (A^*(H))^\perp;$$

taking orthogonal complements and using the fact that the double orthogonal complement of a subspace is its closure and that $A^*(H)$ is closed, we obtain

$$(\ker A)^\perp = A^*(H).$$

Since $\ker A = \{0\}$, we have $A^*(H) = H$. Then we can apply the first of the two lemmas: as $A^* = \mathrm{id}_H - K^*$, $K^* \in K(H)$, and $A^*(H) = H$, we have $\ker A^* = \{0\}$. We now apply the second of the two lemmas: $A^*$ is bounded below on $(\ker A^*)^\perp = H$. Using the fact that if $T \in B(H)$ is bounded below and has dense image then $T^{-1} \in B(H)$, we get $(A^*)^{-1} \in B(H)$ ($A^*(H) = H$ so $A^*$ certainly has dense image). Taking adjoints commutes with taking inverses, so $A^{-1} \in B(H)$. This completes the proof of the Fredholm alternative.

# 9 Compact self-adjoint operators

It is a fact that if $T \in B(H)$ is self-adjoint then[10]

$$\|T\| = \sup_{\|v\| \le 1} |\langle Tv, v\rangle| = \sup_{\|v\|=1} |\langle Tv, v\rangle|.$$

Let $T \in B(H)$ be compact and self-adjoint and $T \neq 0$. Since $T$ is self-adjoint, $\langle Tv, v\rangle \in \mathbb{R}$, so either $\|T\| = \sup_{\|v\|=1} \langle Tv, v\rangle$ or $\|T\| = -\inf_{\|v\|=1} \langle Tv, v\rangle$. Say the first is the case. Let $\|v_n\| = 1$ and $\langle Tv_n, v_n\rangle \to \|T\|$ as $n \to \infty$. Then, as $T = T^*$,

$$\begin{aligned}
\langle Tv_n - \|T\| v_n, Tv_n - \|T\| v_n\rangle &= \langle Tv_n, Tv_n\rangle - \langle Tv_n, \|T\| v_n\rangle - \langle \|T\| v_n, Tv_n\rangle \\
&\quad + \langle \|T\| v_n, \|T\| v_n\rangle \\
&= \|Tv_n\|^2 - 2\|T\| \langle Tv_n, v_n\rangle + \|T\|^2 \|v_n\|^2 \\
&\le \|T\|^2 \|v_n\|^2 - 2\|T\| \langle Tv_n, v_n\rangle + \|T\|^2 \|v_n\|^2 \\
&= 2\|T\|^2 - 2\|T\| \langle Tv_n, v_n\rangle .
\end{aligned}$$

[9]The adjoint of a compact operator is itself a compact operator. This is true even for Banach spaces, and a proof of this is given by Paul Garrett in his note *Compact operators on Banach spaces: Fredholm-Riesz*. A bounded linear map $K : X \to Y$, where $X$ and $Y$ are Banach spaces, is said to be *compact* if for every bounded subset $S$ of $X$, the closure of $K(S)$ in $Y$ is compact.

[10]This is proved in Anthony W. Knapp, *Advanced Real Analysis*, p. 37, Proposition 2.2.

Thus $\|Tv_n - \|T\| v_n\| \to 0$ as $n \to \infty$, that is, $Tv_n - \|T\| v_n \to 0$. On the other hand, as $\|v_n\| = 1$ for each $n$, there is some subsequence $v_{a(n)}$ such that $Tv_{a(n)}$ converges, say to $v$. Together with $Tv_n - \|T\| v_n \to 0$ this gives $\|T\| v_{a(n)} \to v$ as $n \to \infty$, from which we get $\|v\| = \frac{1}{\|T\|} > 0$. Thus $v \neq 0$. And

$$(T - \|T\|)v = (T - \|T\|) \lim_{n\to\infty} v_{a(n)} = \lim_{n\to\infty} (T - \|T\|)v_{a(n)} = 0,$$

which means that $\|T\| \in \sigma_{\text{point}}(T)$. Likewise, in the case $\|T\| = -\inf_{\|v\|=1} \langle Tv, v\rangle$ we get $-\|T\| \in \sigma_{\text{point}}(T)$.

# 10 Multiplication operators

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space. $L^2(X)$ is a Hilbert space[11] with inner product

$$\langle f, g\rangle = \int fg^* d\mu,$$

where $g^*(x) = \overline{g(x)}$.

A *multiplication operator* on $L^2(X)$ is an operator $M_\phi : L^2(X) \to L^2(X)$, $\phi \in L^\infty(X)$, of the form

$$(M_\phi f)(x) = \phi(x)f(x).$$

As

$$\|M_\phi f\|^2 = \int_X \phi(x)f(x)\overline{\phi(x)f(x)}d\mu(x) \leq \|\phi\|_\infty^2 \|f\|^2,$$

where $\|\phi\|_\infty$ is the essential supremum of $\phi(x)$ for $x \in X$, we have $\|M_\phi\| \leq \|\phi\|$ and so $M_\phi \in B(H)$. $L^\infty(X)$ is a $C^*$-algebra, and so is $B(L^2(X))$. If $X$ is $\sigma$-finite then I claim that

$$\phi \mapsto M_\phi, \qquad L^\infty(X) \to B(L^2(X))$$

is an injective homomorphism of $C^*$-algebras. It is straightforward to show that this map is a homomorphism of $C^*$-algebras, and this does not use the assumption that $X$ is $\sigma$-finite. For our benefit, we shall show that $\phi \mapsto M_\phi$ is injective. If $\phi \neq 0$, then $\|\phi\|_\infty > 0$, so for

$$E = \{x \in X : |\phi(x)| \geq \frac{1}{2} \|\phi\|_\infty\}$$

[11] Whether $L^2(X)$ is separable depends on the measure space $(X, \mu)$. Let $\mathscr{S}$ be the set of all measurable subsets of $X$ with finite measure, and let $\rho(A, B) = \mu(A \cup B \setminus A \cap B)$, the measure of the symmetric difference of $A$ and $B$. One shows that $\mathscr{S}$ is a pseudometric space with pseudometric $\rho$. It is a fact that $L^2(X)$ is separable if and only if $\mathscr{S}$ is separable; cf. Paul Halmos, *Measure Theory*, p. 177, §42. For this to be the case, it suffices that $X$ is $\sigma$-finite and that its $\sigma$-algebra is countably generated.

we have $0 < \mu(E) \le \infty$. Because $(X, \mu)$ is $\sigma$-finite, there is some subset $F$ of $E$ with $0 < \mu(F) < \infty$. As $f = \phi^* \cdot \chi_F \in L^2(X)$, we have

$$\begin{aligned} M_\phi f &= \int_F \phi(x)\overline{\phi(x)} d\mu(x) \\ &\ge \int_F \frac{1}{4} \|\phi\|_\infty^2 \, d\mu(x) \\ &= \frac{1}{4} \|\phi\|_\infty^2 \cdot \mu(F) \\ &> 0, \end{aligned}$$

so $M_\phi \neq 0$. Generally, an injective homomorphism of $C^*$-algebras is an isometry, so $\|M_\phi\| = \|\phi\|_\infty$.

As $M_{\phi\phi^*} = M_\phi M_{\phi^*} = M_\phi M_\phi^*$ and $M_{\phi\phi^*} = M_{\phi^*\phi}$, we have $M_\phi M_\phi^* = M_\phi^* M_\phi$, namely, a multiplication operator is a normal operator. Since residual spectrum of a normal operator is empty, the residual spectrum of a multiplication operator is empty.

For $\phi \in L^\infty(X)$, we define the *essential range* of $\phi$ to be the set

$$\Big\{ z \in \mathbb{C} : \text{if } \epsilon > 0 \text{ then } \mu(\{x \in X : |f(x) - z| < \epsilon\}) > 0 \Big\}.$$

Equivalently, the essential range of $\phi$ is the set of those $z \in \mathbb{C}$ such that for all $\epsilon > 0$,

$$\mu(\phi^{-1}(D_\epsilon(z))) > 0,$$

in words, those $z \in \mathbb{C}$ such that the inverse image of every $\epsilon$-disc about $z$ has positive measure. Equivalently, the essential range of $\phi$ is the intersection of all closed subsets $K$ of $\mathbb{C}$ such that for almost all $x \in X$, $\phi(x) \in K$. It is a fact that if $\phi \in L^\infty(X)$ then the essential range of $\phi$ is a compact subset of $\mathbb{C}$.

Let $\phi \in L^\infty(X)$. If $\lambda$ is not in the essential range of $\phi$, then there is some $\epsilon > 0$ such that $\mu(\phi^{-1}(D_\epsilon(\lambda))) = 0$, which means that for almost all $x \in X$ we have $|\phi(x) - \lambda| \ge \epsilon$. Define $\psi(x) = \frac{1}{\phi(x) - \lambda}$. For almost all $x \in X$,

$$|\psi(x)| = \frac{1}{|\phi(x) - \lambda|} \le \frac{1}{\epsilon},$$

hence $\psi \in L^\infty(X)$. Then

$$M_\psi M_{\phi-\lambda} = M_{\phi-\lambda} M_\psi = M_{(\phi-\lambda)\cdot\psi} = M_1 = \mathrm{id}_{L^2(X)},$$

so $M_{\phi-\lambda}$ is invertible. But $M_{\phi-\lambda} = M_\phi - \lambda$, so $M_\phi - \lambda$ is invertible and hence $\lambda \notin \sigma(M_\phi)$.

If $\lambda$ is in the essential range of $\phi$, then for each $n$ we have $0 < \mu(\phi^{-1}(D_{1/n}(\lambda))) \le \infty$; since $\Sigma$ is $\sigma$-finite, for each $n$ there is a subset $E_n$ of $\phi^{-1}(D_{1/n}(\lambda))$ with $0 < \mu(E_n) < \infty$, and so $\chi_{E_n} \in L^2(X)$. We have, since $|\phi(x) - \lambda| < \frac{1}{n}$ for

$x \in E_n$,

$$\begin{aligned}
\|(M_\phi - \lambda)\chi_{E_n}\|^2 &= \int_X |(\phi(x) - \lambda)\chi_{E_n}(x)|^2 d\mu(x) \\
&= \int_{E_n} |\phi(x) - \lambda|^2 d\mu(x) \\
&\leq \frac{1}{n^2} \int_{E_n} d\mu(x) \\
&= \frac{1}{n^2} \int_X \chi_{E_n} d\mu(x) \\
&= \frac{1}{n^2} \|\chi_{E_n}\|^2,
\end{aligned}$$

so for each $n$,

$$\|(M_\phi - \lambda)\chi_{E_n}\| \leq \frac{1}{n} \|\chi_{E_n}\|.$$

It follows that $M_\phi - \lambda$ is not invertible, as it is not bounded below. Therefore $\lambda \in \sigma(M_\phi)$. Therefore the essential range of $\phi \in L^\infty(X)$ is equal to the spectrum of $M_\phi \in B(L^2(X))$.

We say that $\phi \in L^\infty(X)$ is *invertible* if there is some $\psi \in L^\infty(X)$ such that $\phi(x)\psi(x) = 1$ for almost all $x \in X$. (It would not make sense to demand that $\phi(x)\psi(x) = 1$ for all $x \in X$.) For $\phi \in L^\infty(X)$ to be invertible, it is necessary and sufficient that there is some $\alpha > 0$ such that $|\phi(x)| \geq \alpha$ for almost all $x \in X$ (lest its inverse not have an essential supremum).

If $\lambda$ is not just an element of the essential range $\phi$ but is an isolated element of the essential range, then we can say more than just that $\lambda \in \sigma(M_\phi)$. In this case, there is some $\epsilon > 0$ such that the intersection of $D_\epsilon(\lambda)$ and the essential range of $\phi$ is equal to the singleton $\{\lambda\}$. Let $E$ be a subset of $\phi^{-1}(D_\epsilon(\lambda))$ with $0 < \mu(E) < \infty$. For almost all $x \in X$, $\phi(x)$ is an element of the essential range of $\phi$, hence for almost all $x \in E$ we have $\phi(x) = \lambda$. Therefore, for almost all $x \in X$ we have

$$(M_\phi \chi_E)(x) = \phi(x)\chi_E(x) = \lambda\chi_E(x).$$

Hence $(M_\phi - \lambda)\chi_E = 0$, and as $\mu(E) > 0$ we have $\chi_E \neq 0$. Therefore, if $\lambda$ is an isolated element of the essential range of $\phi$ then $\lambda \in \sigma_{\text{point}}(M_\phi)$.[12]

# 11 Functional calculus

Let $T \in B(H)$ be self-adjoint. The spectrum $\sigma(T)$ is a compact subset of $\mathbb{R}$, and one checks that the set $C(\sigma(T))$ of continuous functions $\sigma(T) \to \mathbb{C}$ is a $C^*$-algebra, with norm $\|f\| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$.

[12] If $\lambda$ is an isolated element of the essential range of $\phi$ then one finds that the inverse image of the singleton $\{\lambda\}$ has positive measure. I would be surprised if this were not the origin of the term *point spectrum*. Being isolated corresponds to being discrete.

Let $\mathbb{C}[x]$ be the set of polynomials with complex coefficients. For $T \in B(H)$ self-adjoint and $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{C}[x]$, we define

$$p(T) \in B(H)$$

by

$$p(T) = \sum_{k=0}^{n} a_k T^k.$$

$T \mapsto p(T)$ is a homomorphism of $C^*$-algebras, where[13]

$$\left( \sum_{k=0}^{n} a_k x^k \right)^* = \sum_{k=0}^{n} \overline{a_k} x^k.$$

It is a fact that[14]

$$\sigma(p(T)) = p(\sigma(T)),$$

where for $M \subseteq \mathbb{C}$ we define $p(M) = \{p(z) : z \in M\}$. It is also a fact that

$$\|p(T)\| = \|p\| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|;$$

this is proved using the result that the norm of a normal operator $T$ is equal to its *spectral radius*, which is given by the two following expressions that one proves are equal:

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} = \max_{\lambda \in \sigma(T)} |\lambda|.$$

The above is used to define $f(T)$ for any continuous function $f : \sigma(T) \to \mathbb{C}$. This map $C(\sigma(T)) \to B(H)$ is called the *continuous functional calculus*. It is an isometric homomorphism of $C^*$-algebras. The continuous functional calculus can be used to prove things about the spectrum of self-adjoint operators that do not obviously have to do with making sense of continuous functions applied to these operators.

Let $T \in B(H)$ be self-adjoint and let $\lambda \in \sigma(T)$ be an isolated point in $\sigma(T)$. I will show that that $\lambda \in \sigma_{\text{point}}(T)$. Since $\lambda$ is isolated in $\sigma(T)$, the function $f : \sigma(T) \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 1 & z = \lambda, \\ 0 & z \neq \lambda, \end{cases}$$

is continuous. Since $f$ is continuous, $f(T) \in B(H)$, and because $f = f^*$, $f(T)$ is self-adjoint. Let $P = f(T)$. As $\|P\| = \|f\| = 1$, $P \neq 0$. Define $g \in C(\sigma(T))$ by $g(x) = (x - \lambda) f(x)$. Then $g(x) = 0$ for all $x \in \sigma(T)$, so $g(T) = 0$, i.e.

$$(T - \lambda) P = 0.$$

Hence $\operatorname{im} P \subseteq \ker(T - \lambda)$. As $P \neq 0$, there is some $v \in H$ with $Pv \neq 0$. Then $(T - \lambda) P v = 0$, $Pv \neq 0$, so $\lambda \in \sigma_{\text{point}}(T)$.

[13] If we had not stipulated that $T$ be self-adjoint then we would have to define the conjugation of polynomials as conjugation of polynomial functions: for $p$ defined by $p(z) = \sum_{k=0}^{n} a_k z^k$, then $p^*$ is defined by $p^*(z) = \sum_{k=0}^{n} \overline{a_k} (\overline{z})^k$.

[14] See Paul Halmos, *Hilbert Space Problem Book*, p. 62, Problem 97.

## 12 Spectral measures

It is a fact that if $f \geq 0$ then $f(T) \geq 0$, where, for a self-adjoint operator $T$, $T \geq 0$ means $\langle Tv, v\rangle \geq 0$ for all $v \in H$. For $T \in B(H)$ self-adjoint and $v \in H$, using the continuous functional calculus talked about in the previous section we define $\phi : C(\sigma(T)) \to \mathbb{C}$ by

$$\phi(f) = \langle v, f(T)v\rangle .$$

$\phi$ is a *positive linear functional*: if $f$ is real-valued and $f(x) \geq 0$ for all $x \in \sigma(T)$, then $\phi(f) \geq 0$. $\sigma(T)$ is indeed a locally compact Hausdorff space and since $\sigma(T)$ is compact the continuous functions of compact support on $\sigma(T)$ are precisely the continuous functions on $\sigma(T)$, so we satisfy the conditions of the Riesz-Markov theorem. Thus there exists a unique regular Borel measure $\mu$ on the Borel $\sigma$-algebra of $\sigma(T)$ such that, for all $f \in C(\sigma(T))$,

$$\langle v, f(T)v\rangle = \phi(f) = \int_{\sigma(T)} f(x)d\mu(x).$$

*Lebesgue's decomposition theorem* states that

$$\mu = \mu_{\text{ac}} + \mu_{\text{sing}} + \mu_{\text{pp}},$$

where

- $\mu_{\text{ac}}$ is absolutely continuous with respect to Lebesuge measure: if $A$ is a measurable subset of $\sigma(T)$ and its Lebesgue measure is 0, then $\mu_{\text{ac}}(A) = 0$.
- $\mu_{\text{sing}}$ and Lebesgue measure are mutually singular,[15] and if $\lambda \in \sigma(T)$ then $\mu_{\text{sing}}(\{\lambda\}) = 0$.
- There is a countable subset $J$ of $\sigma(T)$ such that

$$\mu_{\text{pp}} = \sum_{\lambda \in J} a_\lambda \delta_\lambda, \qquad a_\lambda \in \mathbb{C}.$$

We define $H_{\text{ac}}$ to be the set of those $v \in H$ such that $\mu$ is equal to the absolutely continuous part of its Lebesgue decomposition, i.e. the other two parts are 0, and we define $H_{\text{sing}}$ and $H_{\text{pp}}$ likewise. (Note that we first took $v \in H$ and then defined $\mu$ using $v$.) One proves that $H_{\text{ac}}$, $H_{\text{sing}}$ and $H_{\text{pp}}$ are closed subspaces of $H$ and that they are invariant under $T$, and defines the *absolutely continuous spectrum* of $T$ to be the spectrum of the restriction of $T$ to $H_{\text{ac}}$; the *singular spectrum* of $T$ to be the spectrum of the restriction of $T$ to $H_{\text{sing}}$; and the *pure point spectrum* of $T$ to be the spectrum of the restriction of $T$ to $H_{\text{pp}}$. It is a fact that[16]

$$\sigma(T) = \sigma_{\text{ac}}(T) \cup \sigma_{\text{sing}}(T) \cup \overline{\sigma_{\text{pp}}(T)},$$

but these three sets might not be disjoint.

[15]There are disjoint measurable sets $A$ and $B$ with $A \cup B = \sigma(T)$ such that $\mu_{\text{sing}}(A) = 0$ and the Lebesgue measure of $B$ is 0.

[16]See Reed and Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis*, revised and enlarged ed., p. 231, §VII.2.