The cross-polytope, the ball, and the cube

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1 ℓ^q norms and volume of the unit ball

For $x, y \in \mathbb{R}^n$,

$$\langle x, y\rangle = \sum_{j=1}^n x_j y_j.$$

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$.

$$x = \sum_{j=1}^n x_j e_j = \sum_{j=1}^n \langle x, e_j\rangle e_j.$$

For $1 \leq q < \infty$ let

$$|x|_q = \left(\sum_{j=1}^n |x_j|^q\right)^{1/q}$$

and for $q = \infty$ let

$$|x|_\infty = \max_{1 \leq j \leq n} |x_j|.$$

Then for for $1 \leq q \leq \infty$ let

$$B_q^n = \{x \in \mathbb{R}^n : |x|_q \leq 1\}.$$

For $0 \leq k \leq n$ let λ_k be k-dimensional Lebesgue measure on $\mathbb{R}^n$. We calculate the volume of the unit ball with the ℓ^q norm for $1 \leq q < \infty$.

Theorem 1. For $n \geq 1$ and for $1 \leq q < \infty$,

$$\lambda_n(B_q^n) = \frac{(2\Gamma(\frac{1}{q}+1))^n}{\Gamma(\frac{n}{q}+1)}.$$

Proof. For $R \geq 0$ let $V_q^n(R) = \lambda_n(R \cdot B_q^n)$. For $n = 1$,

$$V_q^1(R) = \lambda_1(R \cdot B_q^1) = \int_{-R \leq x_1 \leq R} d\lambda_1(x_1) = 2R.$$

By induction, suppose for some n that

$$V_q^n(R) = \frac{(2R\Gamma(\frac{1}{q}+1))^n}{\Gamma(\frac{n}{q}+1)}.$$

Using Fubini's theorem and the induction hypothesis and doing the change of variable $x_{n+1} = Rt$ we calculate

$$\begin{aligned}
V_q^{n+1}(R) &= \int_{|x_1|^q+\cdots+|x_n|^q+|x_{n+1}|^q \le R^q} d\lambda_{n+1}(x) \\
&= \int_{-R\le x_{n+1}\le R} \left(\int_{|x_1|^q+\cdots+|x_n|_q \le R^q - |x_{n+1}|^q} d\lambda_n(x_1,\dots,x_n) \right) d\lambda_1(x_{n+1}) \\
&= \int_{-R\le x_{n+1}\le R} V_q^n((R^q - |x_{n+1}|^q)^{1/q}) d\lambda_1(x_{n+1}) \\
&= \int_{-R\le x_{n+1}\le R} \frac{(2(R^q - |x_{n+1}|^q)^{1/q}\Gamma(\frac{1}{q}+1))^n}{\Gamma(\frac{n}{q}+1)} d\lambda_1(x_{n+1}) \\
&= \frac{(2\Gamma(\frac{1}{q}+1))^n}{\Gamma(\frac{n}{q}+1)} \int_{-1\le t\le 1} (R^q - |Rt|^q)^{n/q} \cdot R d\lambda_1(t) \\
&= \frac{(2\Gamma(\frac{1}{q}+1))^n}{\Gamma(\frac{n}{q}+1)} \cdot R^{n+1} \cdot 2 \int_{0\le t\le 1} (1-t^q)^{n/q} d\lambda_1(t).
\end{aligned}$$

Now, doing the change of variable $u = t^q$, namely $t = u^{1/q}$ with $t' = \frac{1}{q}u^{\frac{1}{q}-1}$ and using the beta function $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1} d\lambda_1(u)$,

$$\begin{aligned}
\int_{0\le t\le 1} (1-t^q)^{n/q} d\lambda_1(t) &= \int_{0\le u\le 1} (1-u)^{n/q} \cdot \frac{1}{q} u^{\frac{1}{q}-1} d\lambda_1(u) \\
&= \frac{1}{q} B\left(\frac{1}{q}, \frac{n}{q}+1\right).
\end{aligned}$$

But $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, and using $\Gamma(a+1) = a\Gamma(a)$,

$$\frac{1}{q} B\left(\frac{1}{q}, \frac{n}{q}+1\right) = \frac{1}{q} \frac{\Gamma\left(\frac{1}{q}\right)\Gamma\left(\frac{n}{q}+1\right)}{\Gamma\left(\frac{1}{q}+\frac{n}{q}+1\right)} = \frac{\Gamma\left(\frac{1}{q}+1\right)\Gamma\left(\frac{n}{q}+1\right)}{\Gamma\left(\frac{n+1}{q}+1\right)}.$$

Therefore

$$\begin{aligned}
V_q^{n+1}(R) &= \frac{(2\Gamma(\frac{1}{q}+1))^n}{\Gamma(\frac{n}{q}+1)} \cdot R^{n+1} \cdot 2 \cdot \frac{\Gamma\left(\frac{1}{q}+1\right)\Gamma\left(\frac{n}{q}+1\right)}{\Gamma\left(\frac{n+1}{q}+1\right)} \\
&= \frac{(2R\Gamma(\frac{1}{q}+1))^{n+1}}{\Gamma\left(\frac{n+1}{q}+1\right)},
\end{aligned}$$

which proves the claim. □

B_1^n is an n-dimensional **cross-polytope**, B_2^n is an n-dimensional **Euclidean ball**, and B_∞^n is an n-dimensional **cube**.

$$\lambda_n(B_1^n) = \frac{2^n}{n!}, \quad \lambda_n(B_2^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \quad \lambda_n(B_\infty^n) = 2^n,$$

using $\Gamma(n+1) = n!$ and $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$.

2 Intersection of a hyperplane and the cube

Let $\xi \in S^{n-1}$ and $t \in \mathbb{R}$, and define

$$P_{\xi,t} = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = t\}.$$

In particular,

$$\xi^\perp = P_{\xi,0}.$$

Let

$$A_\xi(t) = \lambda_{n-1}(P_{\xi,t} \cap B_\infty^n) = \int_{P_{\xi,t}} 1_{B_\infty^n}(x) d\lambda_{n-1}(x).$$

Theorem 2. For $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,

$$A_\xi(t) = \frac{2^n}{\pi} \int_0^\infty \cos tr \cdot \prod_{i=1}^n \frac{2}{\xi_i r} \sin \xi_i r dr.$$

Proof. Then by Fubini's theorem,

$$\begin{aligned} \widehat{A}_\xi(\tau) &= \int_{\mathbb{R}} A_\xi(t) e^{-2\pi i t\tau} d\lambda_1(t) \\ &= \int_{\mathbb{R}} \left(\int_{P_{\xi,t}} 1_{B_\infty^n}(x) e^{-2\pi i \langle x,\xi\rangle \tau} d\lambda_{n-1}(x) \right) d\lambda_1(t) \\ &= \int_{\mathbb{R}^n} 1_{B_\infty^n}(x) e^{-2\pi i \langle x,\xi\rangle \tau} d\lambda_n(x). \end{aligned}$$

Now,

$$1_{B_\infty^n}(x) = (1_{B_\infty^1} \otimes \cdots \otimes 1_{B_\infty^1})(x) = \prod_{i=1}^n 1_{B_\infty^1}(x_i),$$

whence, by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} 1_{B_\infty^n}(x) e^{-2\pi i \langle x,\xi\rangle \tau} d\lambda_n(x) &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^n 1_{B_\infty^1}(x_i) e^{-2\pi i x_i \xi_i \tau} \right) d\lambda_n(x) \\ &= \prod_{i=1}^n \int_{\mathbb{R}} 1_{B_\infty^1}(x_i) e^{-2\pi i x_i \xi_i \tau} d\lambda_1(x_i). \end{aligned}$$

But, when $\xi_i\tau \neq 0$,

$$\int_{\mathbb{R}} 1_{B^1_\infty}(x_i) e^{-2\pi i x_i \xi_i \tau} d\lambda_1(x_i) = \int_{-1}^{1} e^{-2\pi i x_i \xi_i \tau} d\lambda_1(x_i) = \frac{1}{\pi \xi_i \tau} \sin 2\pi \xi_i \tau,$$

thus

$$\widehat{A}_\xi(\tau) = \prod_{i=1}^{n} \frac{1}{\pi \xi_i \tau} \sin 2\pi \xi_i \tau.$$

By the Fourier inversion theorem, using that $\widehat{A}_\xi$ is an even function,

$$\begin{aligned}
A_\xi(t) &= \int_{\mathbb{R}} \widehat{A}_\xi(\tau) e^{2\pi i t \tau} d\lambda_1(\tau) \\
&= \int_{\mathbb{R}} \widehat{A}_\xi(\tau) \cos 2\pi t \tau d\lambda_1(\tau) \\
&= 2 \int_0^\infty \widehat{A}_\xi(\tau) \cos 2\pi t \tau d\tau \\
&= 2 \int_0^\infty \cos 2\pi t \tau \cdot \prod_{i=1}^{n} \frac{1}{\pi \xi_i \tau} \sin 2\pi \xi_i \tau d\tau \\
&= \frac{2^n}{\pi} \int_0^\infty \cos tr \cdot \prod_{i=1}^{n} \frac{2}{\xi_i r} \sin \xi_i r dr.
\end{aligned}$$

□

3 Schwartz functions

Let $\mathscr{S}$ be the Fréchet space of Schwartz function $\mathbb{R}^n \to \mathbb{C}$ and let $\mathscr{S}'$ be the locally convex space of tempered distributions $\mathscr{S} \to \mathbb{C}$. If $f : \mathbb{R}^n \to \mathbb{C}$ is locally integrable and there is some N such that

$$\int_{|x|_2 \leq R} |f(x)| d\lambda_n(x) = O(R^N), \qquad R \to \infty,$$

it is a fact that

$$\phi \mapsto \langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x) d\lambda_n(x), \qquad \phi \in \mathscr{S},$$

is a tempered distribution.

Lemma 3. For $1 \leq q \leq \infty$ and for $0 < h < n$, $|x|_q^{-h}$ is a tempered distribution.

Proof. For $1 \leq q \leq 2$,

$$|x|_2 \leq |x|_q \leq n^{\frac{1}{q} - \frac{1}{2}} |x|_2,$$

and for $2 \leq q \leq \infty$,

$$|x|_q \leq |x|_2 \leq n^{\frac{1}{2} - \frac{1}{q}} |x|_q.$$

Then for $1 \le q \le 2$ and for $0 < h < n$, using polar coordinates and as $\sigma(S^{n-1}) = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$,

$$\begin{aligned}
\int_{|x|_2 \le R} |x|_q^{-h} d\lambda_n(x) &\le \int_{|x|_2 \le R} |x|_2^{-h} d\lambda_n(x) \\
&= \int_{S^{n-1}} \left(\int_0^\infty r^{-h} \cdot r^{n-1} dr \right) d\sigma \\
&= \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \cdot \int_0^R r^{-h+n-1} dr \\
&= \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \cdot \frac{r^{-h+n}}{-h+n} \Big|_0^R \\
&= \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \cdot \frac{R^{-h+n}}{-h+n} \\
&= O(R^{-h+n}).
\end{aligned}$$

For $2 \le q \le \infty$ and for $0 < r < n$,

$$\begin{aligned}
\int_{|x|_2 \le R} |x|_q^{-h} d\lambda_n(x) &\le \int_{|x|_2 \le R} (n^{-\frac{1}{2}+\frac{1}{q}} |x|_2)^{-h} d\lambda_n(x) \\
&= n^{\frac{h}{2}-\frac{h}{q}} \int_{|x|_2 \le R} |x|_2^{-h} d\lambda_n(x) \\
&= n^{\frac{h}{2}-\frac{h}{q}} \cdot \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} \cdot \frac{R^{-h+n}}{-h+n} \\
&= O(R^{-h+n}).
\end{aligned}$$

□

For $\phi \in \mathscr{S}$ let

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-2\pi i \langle x, \xi \rangle} d\lambda_n(x).$$

For $1 \le q < \infty$ define $c_q : \mathbb{R} \to \mathbb{R}$ by

$$c_q(z) = e^{-|z|^q}, \qquad z \in \mathbb{R},$$

which belongs to $\mathscr{S}(\mathbb{R})$, and let $\gamma_q = \widehat{c}_q$.

For a tempered distribution T,

$$\left\langle \widehat{T}, \phi \right\rangle = \left\langle T, \widehat{\phi} \right\rangle, \qquad \phi \in \mathscr{S}.$$

Define $f_{q,h}(x) = |x|_q^{-h}$. We calculate the Fourier transform of the tempered distribution $f_{q,h}$.[1]

[1] Alexander Koldobsky and Vladyslav Yaskin, *The Interface between Convex Geometry and Harmonic Analysis*, p. 9, Lemma 2.1.

Theorem 4. Let $0 < h < n$. For $1 \leq q < \infty$,

$$\widehat{f}_{q,h}(\xi) = \frac{q}{\Gamma(h/q)} \int_0^\infty t^{n-h-1} \prod_{j=1}^n \gamma_q(t\xi_j) dt,$$

and for $q = \infty$,

$$\widehat{f}_{\infty,h}(\xi) = 2^n h \int_0^\infty t^{n-h-1} \prod_{j=1}^n \frac{\sin t\xi_j}{t\xi_j} dt$$

Proof. Suppose that $1 \leq q < \infty$. For $x \neq 0$, doing the change of variable $z = t^{1/q}|x|_q^{-1}$,

$$\begin{aligned}
\int_0^\infty z^{h-1} e^{-z^q |x|_q^q} dz &= \int_0^\infty (t^{1/q}|x|_q^{-1})^{h-1} e^{-t} |x|_q^{-1} \frac{1}{q} t^{\frac{1}{q}-1} dt \\
&= \frac{|x|_q^{-h}}{q} \int_0^\infty t^{\frac{h}{q}-1} e^{-t} dt \\
&= \frac{|x|_q^{-h}}{q} \cdot \Gamma(h/q),
\end{aligned}$$

i.e. $f_{q,h}(x) = \frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1} e^{-z^q|x|_q^q} dz$.

For $z > 0$ define $F_{q,z} : \mathbb{R}^n \to \mathbb{R}$ by

$$F_{q,z}(x) = e^{-|zx|_q^q}, \qquad x \in \mathbb{R}^n,$$

which is a Schwartz function. Doing the change of variable $y = z \cdot x$ and using Fubini's theorem,

$$\begin{aligned}
\widehat{F}_{q,z}(\xi) &= \int_{\mathbb{R}^n} e^{-|zx|_q^q} e^{-2\pi i \langle x, \xi \rangle} d\lambda_n(x) \\
&= \int_{\mathbb{R}^n} e^{-|y_1|^q - \cdots - |y_n|^q} e^{-2\pi i \left\langle y, z^{-1}\xi \right\rangle} \cdot z^{-n} d\lambda_n(y) \\
&= z^{-n} \prod_{j=1}^n \int_{\mathbb{R}} e^{-|y_j|^q} e^{-2\pi i y_j \cdot z^{-1} \xi_j} d\lambda_1(y_j) \\
&= z^{-n} \prod_{j=1}^n \gamma_q(z^{-1}\xi_j).
\end{aligned}$$

Then for $\phi \in \mathscr{S}$,

$$\begin{aligned}
\left\langle \widehat{f}_{q,h}, \phi \right\rangle &= \int_{\mathbb{R}^n} f_{q,h}(\xi)\widehat{\phi}(\xi) d\lambda_n(\xi) \\
&= \int_{\mathbb{R}^n} \left(\frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1} e^{-z^q |\xi|_q^q} dz \right) \widehat{\phi}(\xi) d\lambda_n(\xi) \\
&= \frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1} \left(\int_{\mathbb{R}^n} e^{-|z\xi|_q^q} \widehat{\phi}(\xi) d\lambda_n(\xi) \right) dz \\
&= \frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1} \left\langle F_{q,z}, \widehat{\phi} \right\rangle dz \\
&= \frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1} \left\langle \widehat{F}_{q,z}, \phi \right\rangle dz \\
&= \frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1-n} \left(\int_{\mathbb{R}^n} \prod_{j=1}^n \gamma_q(z^{-1}\xi_j) \cdot \phi(\xi) d\lambda_n(\xi) \right) dz \\
&= \int_{\mathbb{R}^n} \left(\frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1-n} \prod_{j=1}^n \gamma_q(z^{-1}\xi_j) dz \right) \phi(\xi) d\lambda_n(\xi).
\end{aligned}$$

This implies, doing the change of variable $z = t^{-1}$,

$$\begin{aligned}
\widehat{f}_{q,h}(\xi) &= \frac{q}{\Gamma(h/q)} \int_0^\infty z^{h-1-n} \prod_{j=1}^n \gamma_q(z^{-1}\xi_j) dz \\
&= \frac{q}{\Gamma(h/q)} \int_0^\infty t^{n-h-1} \prod_{j=1}^n \gamma_q(t\xi_j) dt.
\end{aligned}$$

□

4 Fourier transform

We remind ourselves that $c_q(z) = e^{-|z|^q}$, $z \in \mathbb{R}$, and $\gamma_q = \widehat{c}_q$. We prove that γ_q is positive and logconvex.[2]

Theorem 5. For $1 \le q \le 2$, $\gamma_q(\sqrt{z}) > 0$, and $z \mapsto \log \gamma_q(\sqrt{z})$ is convex on $\mathbb{R}_{\ge 0}$.

Proof. Let $0 < \alpha \le 1$, and for $z \in [0, \infty)$ let $f(z) = \exp z$ and $g(z) = -z^\alpha$. Then for $k \in \mathbb{Z}_{\ge 0}$ and $z \in (0, \infty)$,

$$g^{(k)}(z) = -k! \binom{\alpha}{k} z^{\alpha-k}, \qquad \operatorname{sgn} g^{(k)}(z) = (-1)^k.$$

[2] Alexander Koldobsky and Vladyslav Yaskin, *The Interface between Convex Geometry and Harmonic Analysis*, p. 4, Lemma 1.4.

For $n \geq 1$, **Faà di Bruno's formula** tells us

$$(f \circ g)^{(n)}(z) = \sum_{(m_1,\ldots,m_n),1\cdot m_1+\cdots+n\cdot m_n=n} \frac{n!}{m_1!\cdots m_n!}(f^{(m_1+\cdots+m_n)} \circ g)(z) \cdot \prod_{k=1}^{n} \left(\frac{g^{(k)}(z)}{k!}\right)^{m_k}.$$

Then

$$\begin{aligned}(-1)^n(f \circ g)^{(n)}(z) &= \sum_{(m_1,\ldots,m_n),1\cdot m_1+\cdots+n\cdot m_n=n} \frac{n!}{m_1!\cdots m_n!}(\exp \circ g)(z) \\ &\quad \cdot \prod_{k=1}^{n} \left((-1)^k \frac{g^{(k)}(z)}{k!}\right)^{m_k} \\ &\geq 0.\end{aligned}$$

This shows that $f \circ g$ is **completely monotone**. Furthermore, $(f \circ g)(0) = 1$, so by the **Bernstein-Widder theorem** there is a Borel probability measure μ on $[0, \infty)$ such that

$$(f \circ g)(z) = \int_{[0,\infty)} e^{-zt} d\mu(t), \qquad z \in [0, \infty).$$

With $\alpha = \frac{q}{2}$, there is thus a Borel probability measure μ_q on $[0, \infty)$ such that

$$\exp(-z^{q/2}) = \int_{[0,\infty)} e^{-zt} d\mu_q(t), \qquad z \in [0, \infty).$$

Then for $z \in \mathbb{R}$,

$$c_q(z) = \exp(-|z|^q) = \exp(-(z^2)^{q/2}) = \int_{[0,\infty)} e^{-z^2 t} d\mu_{q/2}(t).$$

For $w \in \mathbb{R}$ we calculate, using the Fourier transform of a Gaussian,

$$\begin{aligned}\gamma_q(w) &= \int_{\mathbb{R}} e^{-2\pi i w z} c_q(z) d\lambda_1(z) \\ &= \int_{\mathbb{R}} e^{-2\pi i w z} \left(\int_{[0,\infty)} e^{-z^2 t} d\mu_{q/2}(t)\right) d\lambda_1(z) \\ &= \int_{[0,\infty)} \left(\int_{\mathbb{R}} e^{-tz^2} e^{-2\pi i w z} d\lambda_1(z)\right) d\mu_{q/2}(t) \\ &= \int_{[0,\infty)} \sqrt{\frac{\pi}{t}} \exp\left(-\frac{(\pi w)^2}{t}\right) d\mu_{q/2}(t) \\ &= \pi^{1/2} \int_{[0,\infty)} t^{-1/2} e^{-\pi^2 w^2/t} d\mu_{q/2}(t).\end{aligned}$$

From the final expression it is evident that $\gamma_k(w) > 0$. Furthermore, for $w_1, w_2 \in (0, \infty)$, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\log \gamma_q \left(\sqrt{\frac{w_1 + w_2}{2}} \right) &= \frac{1}{2} \log \left(\pi^{1/2} \int_{[0,\infty)} t^{-1/4} e^{-\pi^2 \cdot \frac{w_1}{2t}} \cdot t^{-1/4} e^{-\pi^2 \cdot \frac{w_2}{2t}} \, d\mu_{q/2}(t) \right)^2 \\
&\leq \frac{1}{2} \log \left(\pi \int_{[0,\infty)} t^{-1/2} e^{-\pi^2 \cdot \frac{w_1}{t}} \, d\mu_{q/2}(t) \right. \\
&\left. \cdot \int_{[0,\infty)} t^{-1/2} e^{-\pi^2 \cdot \frac{w_2}{t}} \, d\mu_{q/2}(t) \right) \\
&= \frac{1}{2} \log \left(\pi^{1/2} \int_{[0,\infty)} t^{-1/2} e^{-\pi^2 \cdot \frac{w_1}{t}} \, d\mu_{q/2}(t) \right) \\
&+ \frac{1}{2} \log \left(\pi^{1/2} \int_{[0,\infty)} t^{-1/2} e^{-\pi^2 \cdot \frac{w_2}{t}} \, d\mu_{q/2}(t) \right) \\
&= \frac{1}{2} \log \gamma_q(\sqrt{w_1}) + \frac{1}{2} \log \gamma_q(\sqrt{w_2}).
\end{aligned}$$

Because $w \mapsto \log \gamma_q(\sqrt{w})$ is continuous, this suffices to prove that it is convex.

□