Complexification, complex structures, and linear ordinary differential equations

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1 Motivation

The solution of the initial value problem

$$x'(t) = Ax(t), \qquad x(0) = x_0 \in \mathbb{R}^n,$$

where A is an $n \times n$ matrix over $\mathbb{R}$, is $x(t) = \exp(At)x_0$. If we want to compute the solution and if A is diagonalizable, say $A = PDP^{-1}$, we use

$$\exp(At) = \exp((PDP^{-1})t) = P\exp(Dt)P^{-1}.$$

Thus if the matrix A has complex eigenvalues, then although $\exp(At)x_0 \in \mathbb{R}^n$, it may not be the case that $P^{-1}x_0 \in \mathbb{R}^n$. For example, if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$D = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}.$$

For $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$P^{-1}x_0 = \frac{1}{2}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i \\ -i \end{pmatrix}.$$

This is similar to how Cardano's formula, which expresses the roots of a real cubic polynomial in terms of its coefficients, involves complex numbers and yet the final result may still be real.

In the following, unless I specify the dimension of a vector space, any statement about real vector spaces is about real vector spaces of finite or infinite dimension, and any statement about complex vector spaces is about complex vector spaces of finite or infinite dimension.

2 Direct sums

If V is a real vector space, a *complex structure* for V is an $\mathbb{R}$-linear map $J : V \to V$ such that $J^2 = -\mathrm{id}_V$.

If V is a real vector space and $J : V \to V$ is a complex structure, define a complex vector space V_J in the following way: let the set of elements of V_J be V, let addition in V_J be addition in V, and define scalar multiplication in V_J by

$$(a + ib)v = av + bJ(v).$$

One checks that for $\alpha, \beta \in \mathbb{C}$ and $v \in V_J$ we have $(\alpha\beta)v = \alpha(\beta v)$, and thus that V_J is indeed a complex vector space with this definition of scalar multiplication.[1]

Let V be a real vector space, and define the $\mathbb{R}$-linear map $J : V \oplus V \to V \oplus V$ by

$$J(v, w) = (-w, v).$$

$J^2 = -\mathrm{id}_{V\oplus V}$. J is a complex structure on the real vector space $V \oplus V$. The *complexification* of V is the complex vector space $V^{\mathbb{C}} = (V \oplus V)_J$. Thus, $V^{\mathbb{C}}$ has the same set of elements as $V \oplus V$, the same addition as $V \oplus V$, and scalar multiplication

$$(a + ib)(v, w) = a(v, w) + bJ(v, w),$$

which gives

$$(a+ib)(v, w) = a(v, w) + b(-w, v) = (av, aw) + (-bw, bv) = (av - bw, aw + bv).$$

If the real vector space V has dimension n and if $\{e_1, \ldots, e_n\}$ is a basis for V, then

$$\{(e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_n)\}$$

is a basis for the real vector space $V \oplus V$. Let $v \in V^{\mathbb{C}}$. Using the basis for the real vector space $V \oplus V$, there exist

$$a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$$

such that

$$\begin{aligned} v &= a_1(e_1, 0) + \cdots a_n(e_n, 0) + b_1(0, e_1) + \cdots + b_n(0, e_n) \\ &= a_1(e_1, 0) + \cdots + a_n(e_n, 0) + b_1 J(e_1, 0) + \cdots + b_n J(e_n, 0) \\ &= (a_1 + ib_1)(e_1, 0) + \cdots + (a_n + ib_n)(e_n, 0), \end{aligned}$$

where in the last line we used the definition of scalar multiplication in $V^{\mathbb{C}}$. One checks that the set $\{(e_1, 0), \ldots, (e_n, 0)\}$ is linearly independent over $\mathbb{C}$, and therefore it is a basis for $V^{\mathbb{C}}$. Hence

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V.$$

[1] One should also verify that distributivity holds with this definition of scalar product; the other properties of a vector space are satisfied because V_J has the same addition as the real vector space V.

3 Complexification is a functor

If V, W are real vector spaces and $T : V \to W$ is an $\mathbb{R}$-linear map, we define

$$T^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$$

by

$$T^{\mathbb{C}}(v_1, v_2) = (Tv_1, Tv_2);$$

this is a $\mathbb{C}$-linear map. Setting $\iota_V(v_1, v_2) = (v_1, 0)$ and $\iota_W(w_1, w_2) = (w_1, 0)$, $T^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$ is the unique $\mathbb{C}$-linear map such that $T^{\mathbb{C}} \circ \iota_V = \iota_W \circ T$.[2]

Complexification is a *functor* from the category of real vector spaces to the category of complex vector spaces:

$$(\mathrm{id}_V)^{\mathbb{C}}(v_1, v_2) = (\mathrm{id}_V v_1, \mathrm{id}_V v_2) = (v_1, v_2) = \mathrm{id}_{V^{\mathbb{C}}}(v_1, v_2),$$

so $(\mathrm{id}_V)^{\mathbb{C}} = \mathrm{id}_{V^{\mathbb{C}}}$, and if $S : U \to V$ and $T : V \to S$ are $\mathbb{R}$-linear maps, then

$$(T \circ S)^{\mathbb{C}}(v_1, v_2) = (T(Sv_1), T(Sv_2)) = T^{\mathbb{C}}(Sv_1, Sv_2) = T^{\mathbb{C}}(S^{\mathbb{C}}(v_1, v_2)),$$

so $(T \circ S)^{\mathbb{C}} = T^{\mathbb{C}} \circ S^{\mathbb{C}}$.

4 Complexifying a complex structure

If V is a real vector space and $J : V \to V$ is a complex structure, then

$$\begin{aligned} (J^{\mathbb{C}})^2(v_1, v_2) &= J^{\mathbb{C}}(Jv_1, Jv_2) \\ &= (J^2 v_1, J^2 v_2) \\ &= (-v_1, -v_2) \\ &= -(v_1, v_2), \end{aligned}$$

so $(J^{\mathbb{C}})^2 = -\mathrm{id}_{V^{\mathbb{C}}}$. Let

$$E_i = \{w \in V^{\mathbb{C}} : J^{\mathbb{C}} w = iw\}, \qquad E_{-i} = \{w \in V^{\mathbb{C}} : J^{\mathbb{C}} w = -iw\}.$$

If $w \in V^{\mathbb{C}}$, then one checks that

$$w - iJ^{\mathbb{C}} w \in E_i, \qquad w + iJ^{\mathbb{C}} w \in E_{-i},$$

and

$$w = \frac{1}{2}\left(w - iJ^{\mathbb{C}} w\right) + \frac{1}{2}\left(w + iJ^{\mathbb{C}} w\right).$$

It follows that

$$V^{\mathbb{C}} = E_i \oplus E_{-i}.$$

[2] See Keith Conrad's, `https://kconrad.math.uconn.edu/blurbs/linmultialg/complexification.pdf`

5 Complex structures, inner products, and symplectic forms

If V is a real vector space of odd dimension, then one can show that there is no linear map $J : V \to V$ satisfying $J^2 = -\mathrm{id}_V$, i.e. there does not exist a complex structure for it. On the other hand, if V has even dimension, let

$$\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$$

be a basis for the real vector space V, and define $J : V \to V$ by

$$Je_j = f_j, \qquad Jf_j = -e_j.$$

Then $J : V \to V$ is a complex structure.

If V is a real vector space of dimension $2n$ with a complex structure J, let $e_1 \neq 0$. Check that $Je_1 \notin \mathrm{span}\{e_1\}$. If $n > 1$, let

$$e_2 \notin \mathrm{span}\{e_1, Je_1\}.$$

Check that the set $\{e_1, e_2, Je_1, Je_2\}$ is linearly independent. If $n > 2$, let

$$e_3 \notin \mathrm{span}\{e_1, e_2, Je_1, Je_2\}.$$

Check that the set $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ is linearly independent. If $2i < 2n$ then there is some

$$e_{i+1} \notin \mathrm{span}\{e_1, \ldots, e_i, Je_1, \ldots, Je_i\}.$$

I assert that

$$\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$$

is a basis for V.

Using the above basis $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ for V, let $f_i = Je_i$, and define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ by

$$\langle e_i, e_j \rangle = \delta_{i,j}, \qquad \langle f_i, f_j \rangle = \delta_{i,j}, \qquad \langle e_i, f_j \rangle = 0, \qquad \langle f_i, e_j \rangle = 0.$$

Check that this is an inner product on the real vector space V. Moreover,

$$\langle Je_i, Je_j \rangle = \langle f_i, f_j \rangle = \delta_{i,j} = \langle e_i, e_j \rangle ,$$

and

$$\langle Jf_i, Jf_j \rangle = \left\langle J^2 e_i, J^2 e_j \right\rangle = \langle -e_i, -e_j \rangle = \langle e_i, e_j \rangle = \delta_{i,j} = \langle f_i, f_j \rangle ,$$

and

$$\langle Je_i, Jf_j \rangle = \langle f_i, -e_j \rangle = - \langle f_i, e_j \rangle = 0 = \langle e_i, f_j \rangle ,$$

and

$$\langle Jf_i, Je_j \rangle = \langle -e_i, f_j \rangle = - \langle e_i, f_j \rangle = 0 = \langle f_i, e_j \rangle .$$

Hence for any $v, w \in V$,

$$\langle Jv, Jw\rangle = \langle v, w\rangle .$$

We say that the complex structure J is *compatible* with the inner product $\langle\cdot,\cdot\rangle$, i.e. $J : (V, \langle\cdot,\cdot\rangle) \to (V, \langle\cdot,\cdot\rangle)$ is an *orthogonal transformation*.

A *symplectic form* on a real vector space V is a bilinear form $\omega : V \times V \to \mathbb{R}$ such that $\omega(v, w) = -\omega(w, v)$, and such that if $\omega(v, w) = 0$ for all w then $v = 0$; we say respectively that ω is *skew-symmetric* and *non-degenerate*. If a real vector space V has a complex structure J, and $\langle\cdot,\cdot\rangle$ is an inner product on V that is compatible with J, define ω by

$$\omega(v, w) = \left\langle v, J^{-1}w\right\rangle = \langle v, -Jw\rangle = -\langle v, Jw\rangle ,$$

which is equivalent to

$$\omega(v, Jw) = \langle v, w\rangle .$$

Using that the inner product is compatible with J and that it is symmetric,

$$\omega(v, w) = -\langle v, Jw\rangle = -\left\langle Jv, J^2w\right\rangle = -\langle Jv, -w\rangle = \langle w, Jv\rangle = -\omega(w, v),$$

so ω is skew-symmetric. If $w \in V$ and $\omega(v, w) = 0$ for all $v \in V$, then

$$-\langle v, Jw\rangle = 0$$

for all $v \in V$, and thus $Jw = 0$. Since J is invertible, $w = 0$. Thus ω is nondegenerate. Therefore ω is a symplectic form on V.[3] We have

$$\omega(Jv, Jw) = -\left\langle Jv, J^2w\right\rangle = -\left\langle J^2v, J^3w\right\rangle = -\langle -v, -Jw\rangle = -\langle v, Jw\rangle = \omega(v, w).$$

We say that J is *compatible* with the sympletic form ω, namely, $J : (V, \omega) \to (V, \omega)$ is a *symplectic transformation*.

On the other hand, if V is a real vector space with symplectic form ω and J is a compatible complex structure, then $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ defined by

$$\langle v, w\rangle = \omega(v, Jw)$$

is an inner product on V that is compatible with the complex structure J.

Suppose V is a real vector space with complex structure $J : V \to V$ and that $h : V_J \times V_J \to \mathbb{C}$ is an inner product on the complex vector space V_J. Define $g : V \times V \to \mathbb{R}$ by[4]

$$g(v_1, v_2) = \frac{1}{2}\left(h(v_1, v_2) + \overline{h(v_1, v_2)}\right) = \frac{1}{2}\left(h(v_1, v_2) + h(v_2, v_1)\right).$$

[3] Using the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for V, $f_i = Je_i$, we have

$$\omega(e_i, f_j) = -\langle e_i, Jf_j\rangle = -\left\langle e_i, J^2e_j\right\rangle = -\langle e_i, -e_j\rangle = \langle e_i, e_j\rangle = \delta_{i,j},$$

and

$$\omega(e_i, e_j) = -\langle e_i, Je_j\rangle = -\langle e_i, f_j\rangle = 0, \qquad \omega(f_i, f_j) = 0.$$

A basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for a symplectic vector space that satisfies these three conditions is called a *Darboux basis*.

[4] The letter h refers to a Hermitian form, i.e. an inner product on a complex vector space, and the letter g refers to the usual notation for a metric on a Riemannian manifold.

It is straightforward to check that g is an inner product on the real vector space V. Similarly, define $\omega : V \times V \to \mathbb{R}$ by

$$\omega(v_1, v_2) = \frac{i}{2}\left(h(v_1, v_2) - \overline{h(v_1, v_2)}\right) = \frac{i}{2}\left(h(v_1, v_2) - h(v_2, v_1)\right).$$

It is apparent that ω is skew-symmetric. If $\omega(v_1, v_2) = 0$ for all v_1, then in particular $\omega(iv_1, v_1) = 0$, and so

$$h(iv_1, v_1) - h(v_1, iv_1) = 0.$$

As h is a complex inner product,

$$ih(v_1, v_1) - \bar{i}h(v_1, v_1) = 0,$$

i.e.

$$2ih(v_1, v_1) = 0,$$

and thus $h(v_1, v_1) = 0$, which implies that $v_1 = 0$. Therefore ω is nondegenerate, and thus ω is a symplectic form on the real vector space V. With these definitions of g and ω, for $v_1, v_2 \in V_J$ we have

$$h(v_1, v_2) = g(v_1, v_2) - i\omega(v_1, v_2),$$

which writes the inner product on the complex vector space V_J using an inner product on the real vector space V and a symplectic form on the real vector space V; note that V_J has the same set of elements as V. Moreover, for $v_1, v_2 \in V$ we have

$$\begin{aligned}
\omega(v_1, Jv_2) &= \frac{i}{2}\left(h(v_1, Jv_2) - h(Jv_2, v_1)\right) \\
&= \frac{i}{2}\left(h(v_1, iv_2) - h(iv_2, v_1)\right) \\
&= \frac{i}{2}\left(-ih(v_1, v_2) - ih(v_2, v_1)\right) \\
&= g(v_1, v_2).
\end{aligned}$$

5.1 Tensor products

Here we give another presentation of the complexification of a real vector space, this time using tensor products of real vector spaces. If you were satisfied by the first definition you don't need to read this one; read this either if you are curious about another way to define complexification, if you want to see a pleasant application of tensor products, or if you didn't like the first definition. Let V be a real vector space of dimension n. $\mathbb{C}$ is a real vector space of dimension 2, and

$$V \otimes_{\mathbb{R}} \mathbb{C}$$

is a real vector space of dimension $2n$. If V has basis $\{e_1, \ldots, e_n\}$, then $V \otimes_{\mathbb{R}} \mathbb{C}$ has basis $\{e_1 \otimes 1, \ldots, e_n \otimes 1, e_1 \otimes i, \ldots, e_n \otimes i\}$. Since every element of $V \otimes_{\mathbb{R}} \mathbb{C}$ can be written uniquely in the form

$$v_1 \otimes 1 + v_2 \otimes i, \qquad v_1, v_2 \in V,$$

one often writes

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus iV;$$

here iV is a real vector space that is isomorphic to V.

The *complexification* of V is the complex vector space $V^{\mathbb{C}}$ whose set of elements is $V \otimes_{\mathbb{R}} \mathbb{C}$, with the same addition as the real vector space $V \otimes_{\mathbb{R}} \mathbb{C}$, and with scalar multiplication defined by

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta), \qquad v \in V, \alpha, \beta \in \mathbb{C}.$$

Let $v \in V^{\mathbb{C}}$. Using the basis of the real vector space $V \otimes_{\mathbb{R}} \mathbb{C}$, there exist some

$$a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$$

such that

$$\begin{aligned} v &= a_1 e_1 \otimes 1 + \cdots + a_n e_n \otimes 1 + b_1 e_1 \otimes i + \cdots + b_n e_n \otimes i \\ &= e_1 \otimes (a_1 + ib_1) + \cdots + e_n \otimes (a_n + ib_n) \\ &= (a_1 + ib_1) e_1 \otimes 1 + \cdots + (a_n + ib_n) e_n \otimes 1, \end{aligned}$$

where in the last line we used the definition of scalar multiplication in $V^{\mathbb{C}}$. One checks that the $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ is linearly independent over $\mathbb{C}$, and hence that it is a basis for the complex vector space $V^{\mathbb{C}}$, so $V^{\mathbb{C}}$ has dimension n over $\mathbb{C}$.

If V and W are real vector spaces and $T : V \to W$ is a linear map, define $T^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$ by

$$T^{\mathbb{C}}(v \otimes z) = (Tv) \otimes z.$$

With this definition of $T^{\mathbb{C}}$, one can check that complexification is a functor from the category of real vector spaces to the category of complex vector spaces.

6 Decomplexification

If V is a complex vector space, let $V^{\mathbb{R}}$ be the real vector space whose set of elements is V, in which addition is the same as addition in V, and in which scalar multiplication is defined by

$$av = (a + 0i)v, \qquad a \in \mathbb{R}.$$

Because V is a complex vector space, it is apparent that $V^{\mathbb{R}}$ is a real vector space with this scalar multiplication. We call $V^{\mathbb{R}}$ the *decomplexification* of the complex vector space V.

If V has basis $\{e_1, \ldots, e_n\}$ and $v \in V$, then there are $a_1 + ib_1, \ldots, a_n + ib_n \in \mathbb{C}$ such that

$$v = (a_1 + ib_1)e_1 + \cdots + (a_n + ib_n)e_n = a_1 e_1 + \cdots + a_n e_n + b_1(ie_1) + \cdots + b_n(ie_n).$$

One checks that

$$e_1, \ldots, e_n, ie_1, \ldots, ie_n$$

are linearly independent over $\mathbb{R}$, and hence are a basis for the real vector space $V^{\mathbb{R}}$. Thus,

$$\dim_{\mathbb{R}} V^{\mathbb{R}} = 2 \dim_{\mathbb{C}} V.$$

If V is a complex vector space and $T : V \to V$ is a $\mathbb{C}$-linear map, define $T^{\mathbb{R}} : V^{\mathbb{R}} \to V^{\mathbb{R}}$ by

$$T^{\mathbb{R}} v = Tv.$$

Because T is $\mathbb{C}$-linear it follows that $T^{\mathbb{R}}$ is $\mathbb{R}$-linear. Decomplexification is a functor from the category of complex vector spaces to the category of real vector spaces. Since decomplexification is defined simply by ignoring the fact that V is closed under multiplication by complex scalars and only using real scalars, decomplexification is called a forgetful functor

7 Complex conjugation in complexified vector spaces

If V is a real vector space, define $\sigma : V^{\mathbb{C}} \to V^{\mathbb{C}}$ by

$$\sigma(v_1, v_2) = (v_1, -v_2).$$

We call σ *complex conjugation* in $V^{\mathbb{C}}$. We have $\sigma \circ \sigma = \mathrm{id}_{V^{\mathbb{C}}}$. If $T : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is a $\mathbb{C}$-linear map, define $T^{\sigma} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ by

$$T^{\sigma}(w) = \sigma(T\sigma(w)).$$

T^{σ} is a $\mathbb{C}$-linear map. It is a fact that if $T : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is $\mathbb{C}$-linear, then $T^{\sigma} = T$ if and only if there is some $\mathbb{R}$-linear $S : V \to V$ such that $T = S^{\mathbb{C}}$. In words, a linear map on the complexification of a real vector space is equal to its own conjugate if and only if it is the complexification of a linear map on the real vector space.

The following are true statements:[5] $(\mathbb{C}^n = (\mathbb{R}^n)^{\mathbb{C}})$

- If $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map, then

$$\exp(T)^{\mathbb{R}} = \exp(T^{\mathbb{R}}),$$

and

$$\exp(T)^{\sigma} = \exp(T^{\sigma}).$$

[5] These are exercises from V. I. Arnold's *Ordinary differential equations*, p. 122, §18.4, in Richard A. Silverman's translation.

- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then

$$\exp(T)^{\mathbb{C}} = \exp(T^{\mathbb{C}}).$$

- If $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map, then

$$\det T^{\mathbb{R}} = |\det T|^2,$$

and

$$\det T^{\sigma} = \overline{\det T}.$$

- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then

$$\det T^{\mathbb{C}} = \det T.$$

- If $T : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map, then

$$\operatorname{Tr}(T^{\mathbb{R}}) = \operatorname{Tr} T + \operatorname{Tr} T^{\sigma},$$

and

$$\operatorname{Tr} T^{\sigma} = \overline{\operatorname{Tr} T}.$$

- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, then

$$\operatorname{Tr} T^{\mathbb{C}} = \operatorname{Tr} T.$$

8 Linear ordinary differential equations over $\mathbb{C}$

Let A be an $n \times n$ matrix over $\mathbb{C}$. The solution of the initial value problem

$$z'(t) = Az(t), \qquad z(0) = z_0 \in \mathbb{C}^n,$$

is $z(t) = \exp(At)$.

If A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then, with

$$E_\lambda = \{z \in \mathbb{C}^n : Az = \lambda z\},$$

we have

$$\mathbb{C}^n = E_{\lambda_1} \oplus + \cdots + \oplus E_{\lambda_n},$$

where each E_{λ_k} has dimension 1. For $z \in E_{\lambda_k}$,

$$\exp(At)z = \sum_{m=0}^{\infty} t^m \frac{A^m z}{m!} = \sum_{m=0}^{\infty} t^m \frac{\lambda_k^m z}{m!} = e^{\lambda_k t} z.$$

Let $\xi_k \in E_{\lambda_k}$ be nonzero, $1 \le k \le n$. They are a basis for $\mathbb{C}^n$, so there are $c_k \in \mathbb{C}$ such that

$$z_0 = \sum_{k=1}^{n} c_k \xi_k.$$

Then

$$z(t) = \exp(At)z_0 = \exp(At)\sum_{k=1}^{n} c_k\xi_k = \sum_{k=1}^{n} c_k \exp(At)\xi_k = \sum_{k=1}^{n} c_k e^{\lambda_k t}\xi_k.$$

Suppose that A is an $n \times n$ matrix over $\mathbb{C}$, that $z_0 \in \mathbb{C}^n$, that $A^\sigma = A$ and that $\sigma(z_0) = z_0$. The solution of the initial value problem

$$z'(t) = Az(t), \qquad z(0) = z_0,$$

is $z(t) = \exp(At)z_0$. We have, as $\exp(At)^\sigma = \exp((At)^\sigma) = \exp(At)$,

$$\sigma(z(t)) = \sigma(\exp(At)z_0) = \sigma(\exp(At)\sigma(z_0)) = \exp(At)^\sigma z_0 = \exp(At)z_0 = z(t).$$

Therefore, if $A^\sigma = A$ and $\sigma(z_0) = z_0$, then $\sigma(z(t)) = z(t)$ for all t.

9 Linear ordinary differential equations over $\mathbb{R}$

Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $x_0 \in \mathbb{R}^n$. Let $z_0 = (x_0, 0) \in \mathbb{C}^n = (\mathbb{R}^n)^{\mathbb{C}}$, and let $z(t) = (z_1(t), z_2(t))$ be the solution of the initial value problem

$$z'(t) = A^{\mathbb{C}}z(t), \qquad z(0) = z_0 \in \mathbb{C}^n.$$

As $A^{\mathbb{C}}$ is the complexification of a real linear map, $(A^{\mathbb{C}})^\sigma = A^{\mathbb{C}}$, and

$$\sigma(z_0) = \sigma(x_0, 0) = (x_0, -0) = (x_0, 0) = z_0,$$

so $\sigma(z(t)) = z(t)$, i.e. $(z_1(t), z_2(t)) = (z_1(t), -z_2(t))$, so $z_2(t) = 0$ for all t. But $z'(t) = (z_1'(t), z_2'(t))$ and $A^{\mathbb{C}}z(t) = (Az_1(t), Az_2(t))$, so

$$z_1'(t) = Az_1(t)$$

for all t. Also, $z(0) = (z_1(0), z_2(0))$ and $z(0) = z_0 = (x_0, 0)$, so $z_1(0) = x_0$. Therefore, $x(t) = z_1(t)$ is the solution of the initial value problem

$$x'(t) = Ax(t), \qquad x(0) = x_0.$$

Thus, to solve an initial value problem in $\mathbb{R}^n$ we can complexify it, solve the initial value problem in $\mathbb{C}^n$, and take the first entry of the solution of the complex initial value problem.

If A is an $n \times n$ matrix over $\mathbb{R}$, let

$$\det(A - xI) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \in \mathbb{R},$$

its characteristic polynomial. The Cayley-Hamilton theorem states that

$$\sum_{k=0}^{n} a_k A^k = 0.$$

Taking the complexification of this gives

$$\sum_{k=0}^{n} a_k (A^{\mathbb{C}})^k = 0.$$

It follows that the roots of $\det(A - xI)$ are the same as the roots of $\det(A^{\mathbb{C}} - xI^{\mathbb{C}})$. A complex root of $\det(A - xI)$ is not an eigenvalue of $A : \mathbb{R}^2 \to \mathbb{R}^2$, but is indeed an eigenvalue of $A^{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}^2$, so the roots of the characteristic polynomial of A are the eigenvalues of $A^{\mathbb{C}}$.

10 Linear ordinary differential equations in $\mathbb{R}^2$

Let A be a 2×2 matrix over $\mathbb{R}$.[6] Suppose that the roots of the characteristic polynomial

$$\det(A - xI) = \det A - x\mathrm{Tr}\, A + x^2$$

are $\lambda, \overline{\lambda}$, i.e. that the roots of the characteristic polynomial are complex conjugate. Let $\lambda = \alpha + i\omega$, $\omega \neq 0$.[7] λ is an eigenvalue for $A^{\mathbb{C}}$, so let $A^{\mathbb{C}}(v_1, v_2) = \lambda(v_1, v_2)$, $(v_1, v_2) \neq 0$. Furthermore,

$$\sigma(A^{\mathbb{C}}(v_1, v_2)) = \sigma(\lambda(v_1, v_2)),$$

so

$$(A^{\mathbb{C}})^{\sigma}\sigma(v_1, v_2) = \overline{\lambda}\sigma(v_1, v_2),$$

hence, as $(A^{\mathbb{C}})^{\sigma} = A^{\mathbb{C}}$,

$$A^{\mathbb{C}}(v_1, -v_2) = \overline{\lambda}(v_1, -v_2).$$

Therefore $(v_1, -v_2)$ is an eigenvector of $A^{\mathbb{C}}$ with eigenvalue $\overline{\lambda} \neq \lambda$, so $(v_1, -v_2)$ and (v_1, v_2) are linearly independent over $\mathbb{C}$. If $a_1 v_1 + a_2 v_2 = 0$, $a_1, a_2 \in \mathbb{R}$, then

$$\left(\frac{a_1}{2} - i\frac{a_2}{2}\right)(v_1, v_2) + \left(\frac{a_1}{2} + i\frac{a_2}{2}\right)(v_1, -v_2) = 0,$$

[6] This section follows Arnold, p. 132, §20.3.

[7] Define $J : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$J = \frac{1}{\omega}(A - \alpha I).$$

We have $J^2 = \frac{1}{\omega^2}(A^2 - 2\alpha A + \alpha^2 I)$. By the Cayley-Hamilton theorem,

$$I \det A - A\mathrm{Tr}\, A + A^2 = 0,$$

so

$$I\lambda\overline{\lambda} - A(\lambda + \overline{\lambda}) + A^2 = 0,$$

and written using $\lambda = \alpha + i\omega$ this is

$$I(\alpha^2 + \omega^2) - 2\alpha A + A^2 = 0.$$

Hence

$$J^2 = -I,$$

so $J = \frac{1}{\omega}(A - \alpha I)$ is a complex structure on $\mathbb{R}^2$.

from which it follows that $a_1, a_2 = 0$. Therefore $v_1, v_2 \in \mathbb{R}^2$ are linearly independent over $\mathbb{R}$.

We have

$$(\alpha + i\omega)(v_1, v_2) = (\alpha v_1 - \omega v_2, \alpha v_1 + \omega v_2),$$

and

$$A^{\mathbb{C}}(v_1, v_2) = (Av_1, Av_2),$$

so

$$Av_1 = \alpha v_1 - \omega v_2, \qquad Av_2 = \alpha v_1 + \omega v_2,$$

and hence

$$A \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 - \omega v_2 & \alpha v_1 + \omega v_2 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}.$$

Therefore

$$A = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}.$$