

Compact operators on Banach spaces

Jordan Bell

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1 Introduction

In this note I prove several things about compact linear operators from one Banach space to another, especially from a Banach space to itself. Some of these may things be simpler to prove for compact operators on a Hilbert space, but since often in analysis we deal with compact operators from one Banach space to another, such as from a Sobolev space to an L^p space, and since the proofs here are not absurdly long, I think it's worth the extra time to prove all of this for Banach spaces. The proofs that I give are completely detailed, and one should be able to read them without using a pencil and paper. When I want to use a fact that is not obvious but that I do not wish to prove, I give a precise statement of it, and I verify that its hypotheses are satisfied.

2 Preliminaries

If X and Y are normed spaces, let $\mathcal{B}(X, Y)$ be the set of bounded linear maps $X \rightarrow Y$. It is straightforward to check that $\mathcal{B}(X, Y)$ is a normed space with the operator norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|.$$

If X is a normed space and Y is a Banach space, one proves that $\mathcal{B}(X, Y)$ is a Banach space.¹ Let $\mathcal{B}(X) = \mathcal{B}(X, X)$. If X is a Banach space then so is $\mathcal{B}(X)$, and it is straightforward to verify that $\mathcal{B}(X)$ is a Banach algebra.

To say $T \in \mathcal{B}(X)$ is **invertible** means that there is some $S \in \mathcal{B}(X)$ such that $ST = \text{id}_X$ and $TS = \text{id}_X$, and we write $T^{-1} = S$. It follows from the **open mapping theorem** that if $T \in \mathcal{B}(X)$, $\ker T = \{0\}$, and $T(X) = X$, then T is invertible (i.e. if a bounded linear map is bijective then its inverse is also a bounded linear map, where we use the open mapping theorem to show that the inverse is continuous).

The **spectrum** $\sigma(T)$ of $T \in \mathcal{B}(X)$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda \text{id}_X$ is not invertible. If $T - \lambda \text{id}_X$ is not injective, we say that λ is an **eigenvalue** of T , and then there is some nonzero $x \in \ker(T - \lambda \text{id}_X)$, which thus satisfies

¹Walter Rudin, *Functional Analysis*, second ed., p. 92, Theorem 4.1.

$Tx = \lambda x$; we call any nonzero element of $\ker(T - \lambda \text{id}_X)$ an **eigenvector** of T . The **point spectrum** of T is the set of eigenvalues of T .

We say that a subset of a topological space is **precompact** if its closure is compact. The **Heine-Borel theorem** states that a subset S of a complete metric space M is precompact if and only if it is **totally bounded**: to be totally bounded means that for every $\epsilon > 0$ there are finitely many points $x_1, \dots, x_r \in S$ such that $S \subseteq \bigcup_{k=1}^r B_\epsilon(x_i)$, where $B_\epsilon(x)$ is the open ball of radius ϵ and center x .

If X and Y are Banach spaces and $B_1(0)$ is the open unit ball in X , a linear map $T : X \rightarrow Y$ is said to be **compact** if $T(B_1(0))$ is precompact; equivalently, if $T(B_1(0))$ is totally bounded. Check that a linear map $T : X \rightarrow Y$ is compact if and only if the image of *every* bounded set is precompact. Thus, if we want to prove that a linear map is compact we can show that the image of the open unit ball is precompact, while if we know that a linear map is compact we can use that the image of every bounded set is precompact. It is straightforward to prove that a compact linear map is bounded. Let $\mathcal{B}_0(X, Y)$ denote the set of compact linear maps $X \rightarrow Y$. It does not take long to prove that $\mathcal{B}_0(X)$ is an ideal in the algebra $\mathcal{B}(X)$.

3 Basic facts about compact operators

Theorem 1. Let X and Y be Banach spaces. If $T : X \rightarrow Y$ is linear, then T is compact if and only if $x_n \in X$ being a bounded sequence implies that there is a subsequence $x_{a(n)}$ such that $Tx_{a(n)}$ converges in Y .

Proof. Suppose that T is compact and let $x_n \in X$ be bounded, with

$$M = \sup_n \|x_n\| < \infty.$$

Let V be the closed ball in X of radius M and center 0. V is bounded in X , so $N = \overline{T(V)}$ is compact in Y . As $Tx_n \in N$, there is some convergent subsequence $Tx_{a(n)}$ that converges to some $y \in N$.

Suppose that if x_n is a bounded sequence in X then there is a subsequence such that $Tx_{a(n)}$ is convergent, let U be the open unit ball in X , and let $y_n \in T(U)$ be a sequence. It is a fact that a subset of a metric space is precompact if and only if every sequence has a subsequence that converges to some element in the space; this is not obvious, but at least we are only taking as given a fact about metric spaces. (What we have asserted is that a set in a metric space is precompact if and only if it is **sequentially precompact**.) As y_n are in the image of T , there is a subsequence such that $y_{a(n)}$ is convergent and this implies that $T(U)$ is precompact, and so T is a compact operator. \square

Theorem 2. Let X and Y be Banach spaces. If $T \in \mathcal{B}_0(X, Y)$, then $T(X)$ is separable.

Proof. Let U_n be the closed ball of radius n in X . As $\overline{T(U_n)}$ is a compact metric space it is separable, and hence $T(U_n)$, a subset of it, is separable too, say with dense subset L_n . We have

$$T(X) = \bigcup_{n=1}^{\infty} T(U_n),$$

and one checks that $\bigcup_{n=1}^{\infty} L_n$ is a dense subset of the right-hand side, showing that $T(X)$ is separable. \square

The following theorem gathers some important results about compact operators.²

Theorem 3. Let X and Y be Banach spaces.

- If $T \in \mathcal{B}(X, Y)$, T is compact, and $T(X)$ is a closed subset of Y , then $\dim T(X) < \infty$.
- $\mathcal{B}_0(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$.
- If $T \in \mathcal{B}(X)$, T is compact, and $\lambda \neq 0$, then $\dim \ker(T - \lambda \text{id}_X) < \infty$.
- If $\dim X = \infty$, $T \in \mathcal{B}(X)$, and T is compact, then $0 \in \sigma(T)$.

Proof. If $T \in \mathcal{B}(X, Y)$ is compact and $T(X)$ is closed then as a closed subspace of a Banach space, $T(X)$ is itself a Banach space. Of course $T : X \rightarrow T(X)$ is surjective, and X is a Banach space so by the open mapping theorem $T : X \rightarrow T(X)$ is an open map. Let $Tx \in T(X)$. As T is an open map, $T(B_1(x))$ is open, and hence $\overline{T(B_1(x))}$ is a neighborhood of Tx . But because T is compact and $B_1(x)$ is bounded, $\overline{T(B_1(x))}$ is compact. Hence $\overline{T(B_1(x))}$ is a compact neighborhood of Tx . As every element of $T(X)$ has a compact neighborhood, $T(X)$ is locally compact. But a locally compact topological vector space is finite dimensional, so $\dim T(X) < \infty$.

It is straightforward to check that $\mathcal{B}_0(X, Y)$ is linear subspace of $\mathcal{B}(X, Y)$. Let T be in the closure of $\mathcal{B}_0(X, Y)$ and let U be the open unit ball in X . We wish to show that $T(U)$ is totally bounded. Let $\epsilon > 0$. As T is in the closure of $\mathcal{B}_0(X, Y)$, there is some $S \in \mathcal{B}_0(X, Y)$ with $\|S - T\| < \epsilon$. As S is compact, its image $S(U)$ is totally bounded, so there are finitely many $Sx_1, \dots, Sx_r \in S(U)$, with $x_1, \dots, x_r \in U$, such that $S(U) \subseteq \bigcup_{k=1}^r B_\epsilon(Sx_k)$. If $x \in U$, then

$$\|Sx - Tx\| = \|(S - T)x\| \leq \|S - T\| \|x\| < \|S - T\| < \epsilon.$$

Let $x \in U$. Then there is some k such that $Sx \in B_\epsilon(Sx_k)$, and

$$\|Tx - Tx_k\| \leq \|Tx - Sx\| + \|Sx - Sx_k\| + \|Sx_k - Tx_k\| < 3\epsilon,$$

so $T(U) \subseteq \bigcup_{k=1}^r B_{3\epsilon}(Tx_k)$, showing that $T(U)$ is totally bounded and hence that T is a compact operator.

²Walter Rudin, *Functional Analysis*, second ed., p. 104, Theorem 4.18.

If $T \in \mathcal{B}(X)$ is compact and $\lambda \neq 0$, let $Y = \ker(T - \lambda \text{id}_X)$. (If λ is not an eigenvalue of T , then $Y = \{0\}$.) Y is a closed subspace of X , and hence is itself a Banach space. If $y \in Y$ then $Ty = \lambda y \in Y$. Define $S : Y \rightarrow Y$ by $Sy = Ty = \lambda y$, and as T is compact so is S . Now we use the hypothesis that $\lambda \neq 0$: if $y \in Y$, then $S(\frac{1}{\lambda}y) = y$, so $S : Y \rightarrow Y$ is surjective. We have shown that $S : Y \rightarrow Y$ is compact and that $S(Y)$ is a closed subset of Y (as it is equal to Y), and as a closed image of a compact operator is finite dimensional, we obtain $\dim S(Y) < \infty$, i.e. $\dim Y < \infty$.

If $\dim X = \infty$ and $T \in \mathcal{B}(X)$ is compact, suppose by contradiction that $0 \notin \sigma(T)$. So T is invertible, with $TT^{-1} = \text{id}_X$. As $\mathcal{B}_0(X)$ is an ideal in the algebra $\mathcal{B}(X)$, id_X is compact. Of course $\text{id}_X(X) = X$ is a closed subset of X . But we proved that if the image of a compact linear operator is closed then that image is finite dimensional, contradicting $\dim X = \infty$. \square

4 Dual spaces

If X is a normed space, let $X^* = \mathcal{B}(X, \mathbb{C})$, the set of bounded linear maps $X \rightarrow \mathbb{C}$. X^* is called the **dual space** of X , and is a Banach space since \mathbb{C} is a Banach space. Define $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{C}$ by

$$\langle x, \lambda \rangle = \lambda(x), \quad x \in X, \lambda \in X^*.$$

This is called the **dual pairing** of X and X^* .

The following theorem gives an expression for the norm of an element of the dual space.³

Theorem 4. If X is a normed space and V is the closed unit ball in X^* , then

$$\|x\| = \sup_{\lambda \in V} |\langle x, \lambda \rangle|, \quad x \in X.$$

Proof. It follows from the Hahn-Banach extension theorem that if $x_0 \in X$, then there is some $\lambda_0 \in X^*$ such that $\lambda_0(x_0) = \|x_0\|$ and such that if $x \in X$ then $|\lambda_0(x)| \leq \|x\|$.⁴ That is, that there is some $\lambda_0 \in V$ such that $\lambda_0(x_0) = \|x_0\|$. Hence

$$\sup_{\lambda \in V} |\langle x_0, \lambda \rangle| \geq |\langle x_0, \lambda_0 \rangle| = |\lambda_0(x_0)| = \|x_0\|.$$

If $\lambda \in V$, then

$$|\langle x_0, \lambda \rangle| = |\lambda(x_0)| \leq \|\lambda\| \|x_0\| \leq \|x_0\|,$$

so

$$\sup_{\lambda \in V} |\langle x_0, \lambda \rangle| \leq \|x_0\|.$$

\square

³Walter Rudin, *Functional Analysis*, second ed., p. 94, Theorem 4.3 (b).

⁴Walter Rudin, *Functional Analysis*, second ed., p. 58, Theorem 3.3.

Let X be a Banach space. For $x \in X$, it is apparent that $\lambda \mapsto \lambda(x)$ is a linear map $X^* \rightarrow \mathbb{C}$. From Theorem 4, it is bounded, with norm $\|x\|$. Define $\phi : X \rightarrow X^{**}$ by

$$(\phi x)(\lambda) = \langle x, \lambda \rangle, \quad x \in X, \lambda \in X^*.$$

It is apparent that ϕ is a linear map. By Theorem 4, if $x \in X$ then $\|\phi x\| = \|x\|$, so ϕ is an isometry. Let $\phi x_n \in \phi(X)$ be a Cauchy sequence. $\phi^{-1} : \phi(X) \rightarrow X$ is an isometry, so $\phi^{-1}\phi x_n$ is a Cauchy sequence, i.e. x_n is a Cauchy sequence, and so, as X is a Banach space, x_n converges to some x . Then ϕx_n converges to ϕx , and thus $\phi(X)$ is a complete metric space. But a subset of a complete metric space is closed if and only if it is complete, so $\phi(X)$ is a closed subspace of X^{**} . Hence, $\phi(X)$ is a Banach space and $\phi : X \rightarrow \phi(X)$ is an isometric isomorphism. A Banach space is said to be **reflexive** if $\phi(X) = X^{**}$, i.e. if every bounded linear map $X^* \rightarrow \mathbb{C}$ is of the form $\phi(x)$ for some $x \in X$.

5 Adjoints

If X and Y are normed spaces and $T \in \mathcal{B}(X, Y)$, define $T^* : Y^* \rightarrow X^*$ by $T^*\lambda = \lambda \circ T$; as $T^*\lambda$ is the composition of two bounded linear maps it is indeed a bounded linear map $X \rightarrow \mathbb{C}$. T^* is called the **adjoint** of T . It is straightforward to check that T^* is linear and that it satisfies, for $S = T^*$,

$$\langle Tx, \lambda \rangle = \langle x, S\lambda \rangle, \quad x \in X, \lambda \in Y^*. \quad (1)$$

On the other hand, suppose that $S : Y^* \rightarrow X^*$ is a function that satisfies (1). Let $\lambda \in Y^*$, and let $x \in X$. Then

$$(S\lambda)(x) = \lambda(Tx) = (T^*\lambda)(x).$$

This is true for all x , so $S\lambda = T^*\lambda$, and that is true for all λ , so $S = T^*$. Thus $T^*\lambda = \lambda \circ T$ is the unique function $Y^* \rightarrow X^*$ that satisfies (1), not just the unique bounded linear map that does. (That is, satisfying (1) completely determines a function.)

Using Theorem 4,

$$\begin{aligned} \|T\| &= \sup_{\|x\| \leq 1} \|Tx\| \\ &= \sup_{\|x\| \leq 1} \sup_{\|\lambda\| \leq 1} |\langle Tx, \lambda \rangle| \\ &= \sup_{\|x\| \leq 1} \sup_{\|\lambda\| \leq 1} |\langle x, T^*\lambda \rangle| \\ &= \sup_{\|\lambda\| \leq 1} \sup_{\|x\| \leq 1} |T^*\lambda(x)| \\ &= \sup_{\|\lambda\| \leq 1} \|T^*\lambda\| \\ &= \|T^*\|. \end{aligned}$$

In particular, $T^* \in \mathcal{B}(Y^*, X^*)$.

In the following we prove that the adjoint T^* of a compact operator T is itself a compact operator, and that if the adjoint of a bounded linear operator is compact then the original operator is compact.⁵ In the proof we only show that if we take any sequence λ_n in the closed unit ball then it has a subsequence such that $T\lambda_{a(n)}$ converges. Check that it suffices merely to do this rather than showing that this happens for any bounded sequence.

Theorem 5. If X and Y are Banach spaces and $T \in \mathcal{B}(X, Y)$, then T is compact if and only if T^* is compact.

Proof. Suppose that $T \in \mathcal{B}(X, Y)$ is compact, and let $\lambda_n \in Y^*$, $n \geq 1$, be a sequence in the closed unit ball in Y^* .

If M is a metric space with metric ρ and \mathcal{F} is a set of functions $M \rightarrow \mathbb{C}$, we say that \mathcal{F} is **equicontinuous** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $f \in \mathcal{F}$ and $\rho(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$. We say that \mathcal{F} is **pointwise bounded** if for every $x \in M$ there is some $m(x) < \infty$ such that if $f \in \mathcal{F}$ and $x \in M$ then $|f(x)| \leq m(x)$. The **Arzelà-Ascoli theorem**⁶ states that if (M, ρ) is a separable metric space and \mathcal{F} is a set of functions $M \rightarrow \mathbb{C}$ that is equicontinuous and pointwise bounded, then for every sequence $f_n \in \mathcal{F}$ there is a subsequence that converges uniformly on every compact subset of M .

Let V be the closed unit ball in X . As T is a compact operator, $\overline{T(V)}$ is compact and therefore separable, because any compact metric space is separable. Define $f_n : \overline{T(V)} \rightarrow \mathbb{C}$ by

$$f_n(y) = \langle y, \lambda_n \rangle = \lambda_n(y).$$

For $y_1, y_2 \in \overline{T(V)}$ we have

$$|f_n(y_1) - f_n(y_2)| = |\lambda_n(y_1 - y_2)| \leq \|\lambda_n\| \|y_1 - y_2\| \leq \|y_1 - y_2\|.$$

Hence for $\epsilon > 0$, if $n \geq 1$ and $\|y_1 - y_2\| < \epsilon$ then $|f_n(y_1) - f_n(y_2)| < \epsilon$. This shows that $\{f_n\}$ is equicontinuous. If $y \in \overline{T(V)}$, then, for any $n \geq 1$,

$$|f_n(y)| = |\lambda_n(y)| \leq \|\lambda_n\| \|y\| \leq \|y\|,$$

showing that $\{f_n\}$ is pointwise bounded. Therefore we can apply the Arzelà-Ascoli theorem: there is a subsequence $f_{a(n)}$ such that $\overline{f_{a(n)}}$ converges uniformly on every compact subset of $\overline{T(V)}$, in particular on $\overline{T(V)}$ itself and therefore on any subset of it, in particular $T(V)$. We are done using the Arzelà-Ascoli theorem: we used it to prove that there is a subsequence $f_{a(n)}$ that converges uniformly on $T(V)$.

Let $\epsilon > 0$. As $f_{a(n)}$ converges uniformly on $T(V)$, there is some N such that

⁵Walter Rudin, *Functional Analysis*, second ed., p. 105, Theorem 4.19.

⁶Walter Rudin, *Real and Complex Analysis*, third ed., p. 245, Theorem 11.28.

if $n, m \geq N$ and $y \in T(V)$, then $|f_{a(n)}(y) - f_{a(m)}(y)| < \epsilon$. Thus, if $n, m \geq N$,

$$\begin{aligned}
\|T^* \lambda_{a(n)} - T^* \lambda_{a(m)}\| &= \|\lambda_{a(n)} \circ T - \lambda_{a(m)} \circ T\| \\
&= \sup_{x \in V} |\lambda_{a(n)}(Tx) - \lambda_{a(m)}(Tx)| \\
&= \sup_{x \in V} |\lambda_{a(n)}(Tx) - \lambda_{a(m)}(Tx)| \\
&= \sup_{x \in V} |f_{a(n)}(Tx) - f_{a(m)}(Tx)| \\
&< \epsilon.
\end{aligned}$$

This means that $T^* \lambda_{a(n)} \in X^*$ is a Cauchy sequence. As X^* is a Banach space, this sequence converges, and therefore T^* is a compact operator.

Suppose that $T^* \in \mathcal{B}(Y^*, X^*)$ is compact. Therefore, by what we showed in the first half of the proof we have that $T^{**} : X^{**} \rightarrow Y^{**}$ is compact. If V be the closed unit ball in X^{**} , then $T^{**}(V)$ is totally bounded.

We have seen that $\phi : X \rightarrow X^{**}$ defined by $(\phi x)\lambda = \lambda(x)$, $x \in X$, $\lambda \in X^*$, is an isometric isomorphism $X \rightarrow \phi(X)$. Let $\psi : Y \rightarrow Y^{**}$ be the same for Y , and let U be the closed unit ball in X . If $x \in X$ and $\lambda \in Y^*$ then

$$\langle \lambda, \psi Tx \rangle = \langle Tx, \lambda \rangle = \langle x, T^* \lambda \rangle = \langle T^* \lambda, \phi x \rangle = \langle \lambda, T^{**} \phi x \rangle.$$

Therefore $\psi T = T^{**} \phi$. If $x \in U$ then $\phi x \in V$, as ϕ is an isometry. Hence if $x \in U$ then $\psi Tx = T^{**} \phi x \in T^{**}(V)$, thus

$$\psi T(U) \subseteq T^{**}(V).$$

As $\psi T(U)$ is contained in a totally bounded set it is itself totally bounded, and as ψ is an isometry, it follows that $T(U)$ is totally bounded. Hence T is a compact operator. \square

6 Complemented subspaces

If M is a closed subspace of a topological vector space X and there exists a closed subspace N of X such that

$$X = M + N, \quad M \cap N = \{0\},$$

we say that M is **complemented in** X and that X is the **direct sum** of M and N , which we write as $X = M \oplus N$.

We are going to use the following lemma to prove the theorem that comes after it.⁷

Lemma 6. If X is a locally convex topological vector space and M is a subspace of X with $\dim X < \infty$, then M is complemented in X .

⁷Walter Rudin, *Functional Analysis*, second ed., p. 106, Lemma 4.21.

In particular, a normed space is locally convex so the lemma applies to normed spaces. In the following theorem we prove that if $T \in \mathcal{B}(X)$ is compact and $\lambda \neq 0$ then $T - \lambda \text{id}_X$ has closed image.⁸

Theorem 7. If X is a Banach space, $T \in \mathcal{B}(X)$ is compact, and $\lambda \neq 0$, then the image of $T - \lambda \text{id}_X$ is closed.

Proof. According to Theorem 3, $\dim \ker(T - \lambda \text{id}_X) < \infty$, and we can then use Lemma 6: $\ker(T - \lambda \text{id}_X)$ is a finite dimensional subspace of the locally convex space X , so there is a closed subspace N of X such that $X = \ker(T - \lambda \text{id}_X) \oplus N$.

Define $S : N \rightarrow X$ by $Sx = Tx - \lambda x$, so $S \in \mathcal{B}(N, X)$. It is apparent that $T(X) = S(N)$ and that S is injective, and we shall prove that $S(N)$ is closed. To show that $S(N)$ is closed, check that it suffices to prove that S is **bounded below**: that there is some $r > 0$ such that if $x \in N$ then $\|Sx\| \geq r\|x\|$.⁹

Suppose by contradiction that for every $r > 0$ there is some $x \in N$ such that $\|Sx\| < r\|x\|$. So for each $n \geq 1$, let $x_n \in N$ with $\|Sx_n\| < \frac{1}{n}\|x_n\|$, and put $v_n = \frac{x_n}{\|x_n\|}$, so that $\|v_n\| = 1$ and $\|Sv_n\| < \frac{1}{n}$. As T is compact, there is some subsequence such that $Tv_{a(n)}$ converges, say to v . Combining this with $Sv_n \rightarrow 0$ we get $\lambda v_{a(n)} \rightarrow v$. On the one hand, $\|\lambda v_{a(n)}\| = |\lambda|\|v_{a(n)}\| = |\lambda|$, so $\|v\| = |\lambda|$. On the other hand, since $\lambda v_{a(n)} \in N$ and N is closed, we get $v \in N$. S is continuous and $\lambda v_{a(n)} \rightarrow 0$, so

$$Sv = \lim_{n \rightarrow \infty} S(\lambda v_{a(n)}) = \lambda \lim_{n \rightarrow \infty} Sv_{a(n)} = 0.$$

Because S is injective and $Sv = 0$, we get $v = 0$, contradicting $\|v\| = |\lambda| > 0$. Therefore S is bounded below, and hence has closed image, completing the proof. \square

The following theorem states that the point spectrum of a compact operator is countable and bounded, and that if there is a limit point of the point spectrum it is 0.¹⁰ By countable we mean bijective with a subset of the integers.

Theorem 8. If X is a Banach space, $T \in \mathcal{B}(X)$ is compact, and $r > 0$, then there are only finitely many eigenvalues λ of T such that $|\lambda| > r$.

The following theorem shows that if $T \in \mathcal{B}(X)$ is compact and $\lambda \neq 0$, then the operator $T - \lambda \text{id}_X$ is injective if and only if it is surjective.¹¹ This tells us that if $\lambda \neq 0$ is not an eigenvalue of T , then $T - \lambda \text{id}_X$ is both injective and surjective, and hence is invertible, which means that if $\lambda \neq 0$ is not an eigenvalue of T then $\lambda \notin \sigma(T)$. This is an instance of the **Fredholm alternative**.

Theorem 9 (Fredholm alternative). Let X be a Banach space, $T \in \mathcal{B}(X)$ be compact, and $\lambda \neq 0$. $T - \lambda \text{id}_X$ is injective if and only if it is surjective.

⁸Walter Rudin, *Functional Analysis*, second ed., p. 107, Theorem 4.23.

⁹A common way of proving that a linear operator is invertible is by proving that it has dense image and that it is bounded below: bounded below implies injective and bounded below and dense image imply surjective.

¹⁰Walter Rudin, *Functional Analysis*, second ed., p. 107, Theorem 4.24.

¹¹Paul Garrett, *Compact operators on Banach spaces: Fredholm-Riesz*, <http://www.math.umn.edu/~garrett/m/fun/fredholm-riesz.pdf>

Proof. Suppose that $T - \text{id}_X$ is injective and let $V_n = (T - \text{id}_X)^n X$, $n \geq 1$. If $(T - \text{id}_X)^n x \in V_n$, then, as $(T - \text{id}_X)x \in X$, we have

$$(T - \text{id}_X)^{n-1}(T - \text{id}_X)x \in V_{n-1},$$

so $V_n \supseteq V_{n-1}$. Thus

$$V_1 \supseteq V_2 \supseteq \dots$$

Certainly V_n is a normed vector space. Define $T_n \in \mathcal{B}(V_n)$ by $T_n x = Tx$, namely, T_n is the restriction of T to V_n .

As T is a compact operator, by Theorem 7 we get that $V_1 = (T - \text{id}_X)(X)$ is closed. Hence V_1 is a Banach space, being a closed subspace of a Banach space. Assume as induction hypothesis that V_n is a closed subset of X . Thus V_n is a Banach space, and $T_n \in \mathcal{B}(V_n)$ is a compact operator, as it is the restriction of the compact operator T to V_n . Therefore by Theorem 7, the image of $T_n - \text{id}_{V_n}$ is closed, but this image is precisely V_{n+1} . Therefore, if $n \geq 1$ then V_n is a closed subspace of X .

Suppose by contradiction that there is some $x \notin (T - \text{id}_X)X = V_1$. If $y \in X$ then

$$(T - \text{id}_X)^n x - (T - \text{id}_X)^{n+1} y = (T - \text{id}_X)^n (x - (T - \text{id}_X)y).$$

As $x \notin (T - \text{id}_X)X$, we have $x - (T - \text{id}_X)y \neq 0$. As we have supposed that $T - \text{id}_X$ is injective, any positive power of it is injective, and hence the right hand side of the above equation is not 0. Thus $(T - \text{id}_X)^n x \neq (T - \text{id}_X)^{n+1} y$, and as $y \in X$ was arbitrary,

$$(T - \text{id}_X)^n x \notin (T - \text{id}_X)^{n+1} X.$$

However, of course $(T - \text{id}_X)^n x \in V_n$, so if $n \geq 1$ then V_n strictly contains V_{n+1} .

Riesz's lemma states that if M is a normed space, N is a proper closed subspace of M , and $0 < r < 1$, then there is some $x \in M$ with $\|x\| = 1$ and $\inf_{y \in N} \|x - y\| \geq r$.¹² For each $n \geq 1$, using Riesz's lemma there is some $v_n \in V_n$, $\|v_n\| = 1$, such that

$$\inf_{y \in V_{n+1}} \|v_n - y\| \geq \frac{1}{2};$$

we proved that each V_n is closed and that V_n is a strictly decreasing sequence to allow us to use Riesz's lemma.

If $n, m \geq 1$, then $(T - \text{id}_X)v_m \in V_{m+1}$ and check that $Tv_{m+n} \subseteq V_{m+n}$, so

$$Tv_m - Tv_{m+n} = \lambda v_m + (T - \text{id}_X)v_m - Tv_{m+n} \in \lambda v_m + V_{m+1}.$$

¹²Paul Garrett, *Riesz's lemma*, http://www.math.umn.edu/~garrett/m/fun/riesz_lemma.pdf In this reference, Riesz's lemma is stated for Banach spaces, but the proof in fact works for normed spaces with no modifications.

From this and the definition of the sequence v_m , we get

$$\|Tv_m - Tv_{m+n}\| \geq |\lambda| \cdot \frac{1}{2}.$$

That is, the distance between any two terms in Tv_m is $\geq \frac{|\lambda|}{2}$, which is a fixed positive constant, hence Tv_m has no convergent subsequence. But $\|v_m\| = 1$, so v_m is bounded and therefore, as T is compact, the sequence Tv_m has a convergent subsequence, a contradiction. Therefore $T - \text{id}_X$ is surjective.

Suppose that $T - \text{id}_X$ is surjective. One checks that if a bounded linear operator is surjective then its adjoint is injective. For $x \in X$ and $\mu \in X^*$, $\langle \text{id}_X x, \mu \rangle = \mu(\lambda x) = \lambda \mu(x) = \langle x, \text{id}_{X^*} \mu \rangle$, so $(\text{id}_X)^* = \text{id}_{X^*}$. Hence $(T - \text{id}_X)^* = T^* - \text{id}_{X^*}$. T is compact so T^* is compact. As $T^* - \text{id}_{X^*}$ is injective and T^* is compact, $T^* - \text{id}_{X^*}$ is surjective, whence its adjoint $T^{**} - \text{id}_{X^{**}} : X^{**} \rightarrow X^{**}$ is injective. One checks that if $S \in \mathcal{B}(X)$ and $S^{**} : X^{**} \rightarrow X^{**}$ is injective then S is injective; this is proved using the fact that $\phi : X \rightarrow X^{**}$ defined by $(\phi x)(\lambda) = \lambda(x)$ is an isometric isomorphism $X \rightarrow \phi(X)$. Using this, $T - \text{id}_X$ is injective, completing the proof. \square

7 Compact metric spaces

In the proof of Theorem 5 we stated the Arzelà-Ascoli theorem. First we state definitions again. If M is a metric space with metric ρ and \mathcal{F} is a set of functions $M \rightarrow \mathbb{C}$, we say that \mathcal{F} is **equicontinuous** if for all $\epsilon > 0$ there is some $\delta > 0$ such that $f \in \mathcal{F}$ and $\rho(x, y) < \delta$ imply that $|f(x) - f(y)| < \epsilon$. We say that \mathcal{F} is **pointwise bounded** if for all $x \in M$ there is some $m(x)$ such that if $f \in \mathcal{F}$ then $|f(x)| \leq m(x)$. The **Arzelà-Ascoli theorem** states that if M is a separable metric space and \mathcal{F} is equicontinuous and pointwise bounded, then every sequence in \mathcal{F} has a sequence that converges uniformly on every compact subset of M .¹³

We are going to use a converse of the Arzelà-Ascoli theorem in the case of a compact metric space.¹⁴ Let M be a compact metric space and let $C(M)$ be the set of continuous functions $M \rightarrow \mathbb{C}$. It does not take long to prove that with the norm $\|f\| = \sup_{x \in M} |f(x)|$, $C(M)$ is a Banach space.

Theorem 10. Let (M, ρ) be a compact metric space and let $\mathcal{F} \subseteq C(M)$. \mathcal{F} is precompact in $C(M)$ if and only if \mathcal{F} is bounded and equicontinuous.

Proof. Suppose that \mathcal{F} is bounded and equicontinuous. To say that \mathcal{F} is bounded is to say that there is some C such that if $f \in \mathcal{F}$ then $\|f\| \leq C$, and this implies that \mathcal{F} is pointwise bounded. As M is compact it is separable, so the Arzelà-Ascoli theorem tells us that every sequence in \mathcal{F} has a subsequence that converges on every compact subset of M . To say that a sequence of functions $M \rightarrow \mathbb{C}$ converges uniformly on the compact subsets of M is to say

¹³Walter Rudin, *Real and Complex Analysis*, third ed., p. 245, Theorem 11.28.

¹⁴John B. Conway, *A Course in Functional Analysis*, second ed., p. 175, Theorem 3.8.

that the sequence converges in the norm of the Banach space $C(M)$, and thus if \mathcal{F} is a subset of $C(M)$, then to say that every sequence in \mathcal{F} has a subsequence that converges uniformly on every compact subset of M is to say that \mathcal{F} is precompact in $C(M)$.

In the other direction, suppose that \mathcal{F} is precompact. Hence it is totally bounded in $C(M)$. It is straightforward to verify that \mathcal{F} is bounded. We have to show that \mathcal{F} is equicontinuous. Let $\epsilon > 0$. As \mathcal{F} is totally bounded, there are $f_1, \dots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{k=1}^n B_{\epsilon/3}(f_k)$. As each $f_k : M \rightarrow \mathbb{C}$ is continuous and M is compact, there is some $\delta_k > 0$ such that if $\rho(x, y) < \delta_k$ then $|f_k(x) - f_k(y)| < \frac{\epsilon}{3}$. Let $\delta = \min_{1 \leq k \leq n} \delta_k$. If $f \in \mathcal{F}$ and $\rho(x, y) < \delta$, then, taking k such that $\|f - f_k\| < \frac{\epsilon}{3}$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &< \|f - f_k\| + \frac{\epsilon}{3} + \|f_k - f\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon, \end{aligned}$$

showing that \mathcal{F} is equicontinuous. □

We now show that if M is a compact metric space then the Banach space $C(M)$ has the **approximation property**: every compact linear operator $C(M) \rightarrow C(M)$ is the limit of a sequence of bounded finite rank operators.¹⁵

Theorem 11. If (M, ρ) is a compact metric space, then $\mathcal{B}_{00}(C(M))$ is a dense subset of $\mathcal{B}_0(C(M))$.

Proof. Let $T \in \mathcal{B}_0(C(M))$, let V be the closed unit ball in $C(M)$, and let $\epsilon > 0$. Because $T(V)$ is precompact in $C(M)$, by Theorem 10 it is bounded and equicontinuous. Then there is some $\delta > 0$ such that if $Tf \in T(V)$ and $\rho(x, y) < \delta$ then $|(Tf)(x) - (Tf)(y)| < \epsilon$. M is compact, so there are $x_1, \dots, x_n \in M$ such that $M = \bigcup_{j=1}^n B_\delta(x_j)$. It is a fact that there is a **partition of unity** that is **subordinate** to this open covering of M : there are continuous functions $\phi_1, \dots, \phi_n : M \rightarrow [0, 1]$ such that if $x \in M$ then $\sum_{j=1}^n \phi_j(x) = 1$, and $\phi_j(x) = 0$ if $x \notin B_\delta(x_j)$.¹⁶ Define $T_\epsilon : C(M) \rightarrow C(M)$ by

$$T_\epsilon f = \sum_{j=1}^n (Tf)(x_j) \phi_j.$$

It is apparent that T_ϵ is linear. $\|T_\epsilon f\| \leq \sum_{j=1}^n \|T\| \|f\| = n \|T\| \|f\|$, so $\|T_\epsilon\| \leq n \|T\|$. And the image of T_ϵ is contained in the span of $\{\phi_1, \dots, \phi_n\}$. Therefore $T_\epsilon \in \mathcal{B}_{00}(C(M))$.

¹⁵John B. Conway, *A Course in Functional Analysis*, second ed., p. 176, Theorem 3.11.

¹⁶John B. Conway, *Functional Analysis*, second ed., p. 139, Theorem 6.5.

If $f \in V$ and $x \in M$, then for each j either $x \in B_\delta(x_j)$, in which case $|(Tf)(x) - (Tf)(x_j)| < \epsilon$, or $x \notin B_\delta(x_j)$, in which case $\phi_j(x) = 0$. This gives us

$$\begin{aligned}
|(Tf)(x) - (T_\epsilon f)(x)| &= \left| (Tf)(x) \cdot \sum_{j=1}^n \phi_j(x) - \sum_{j=1}^n (Tf)(x_j) \phi_j(x) \right| \\
&= \left| \sum_{j=1}^n ((Tf)(x) - (Tf)(x_j)) \phi_j(x) \right| \\
&\leq \sum_{j=1}^n |(Tf)(x) - (Tf)(x_j)| \phi_j(x) \\
&< \sum_{j=1}^n \epsilon \phi_j(x) \\
&= \epsilon,
\end{aligned}$$

showing that $\|Tf - T_\epsilon f\| < \epsilon$, and as this is true for all $f \in V$ we get $\|T - T_\epsilon\| < \epsilon$. \square