Bernstein's inequality and Nikolsky's inequality for $\mathbb{R}^d$

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February 16, 2015

1 Complex Borel measures and the Fourier transform

Let $\mathcal{M}(\mathbb{R}^d) = rca(\mathbb{R}^d)$ be the set of complex Borel measures on $\mathbb{R}^d$. This is a Banach algebra with the total variation norm, with convolution as multiplication; for $\mu \in \mathcal{M}(\mathbb{R}^d)$, we denote by $|\mu|$ the **total variation of** μ, which itself belongs to $\mathcal{M}(\mathbb{R}^d)$, and the **total variation norm of** μ is $\|\mu\| = |\mu|(\mathbb{R}^d)$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, it is a fact that the union O of all open sets $U \subset \mathbb{R}^d$ such that $|\mu|(U) = 0$ itself satisfies $|\mu|(O) = 0$. We define $\operatorname{supp} \mu = \mathbb{R}^d \setminus O$, called the **support of** μ.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, we define $\hat{\mu} : \mathbb{R}^d \to \mathbb{C}$ by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d\mu(x), \qquad \xi \in \mathbb{R}^d.$$

It is a fact that $\hat{\mu}$ belongs to $C_u(\mathbb{R})$, the collection of bounded uniformly continuous functions $\mathbb{R}^d \to \mathbb{C}$. For $\xi \in \mathbb{R}^d$,

$$|\hat{\mu}(\xi)| \leq \int_{\mathbb{R}^d} |e^{-2\pi i \xi \cdot x}| d|\mu|(x) = |\mu|(\mathbb{R}^d) = \|\mu\| . \tag{1}$$

Let m_d be Lebesgue measure on $\mathbb{R}^d$. For $f \in L^1(\mathbb{R}^d)$, let

$$\Lambda_f = f m_d,$$

which belongs to $\mathcal{M}(\mathbb{R}^d)$. We define $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ by

$$\hat{f}(\xi) = \widehat{\Lambda_f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d\Lambda_f(x) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dm_d(x), \quad \xi \in \mathbb{R}^d.$$

The following theorem establishes properties of the Fourier transform of a complex Borel measure with compact support.[1]

[1] Thomas H. Wolff, *Lectures on Harmonic Analysis*, p. 3, Proposition 1.3.

Theorem 1. *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\operatorname{supp}\mu$ is compact, then $\hat{\mu} \in C^\infty(\mathbb{R}^d)$ and for any multi-index α,*

$$D^\alpha \hat{\mu} = \mathscr{F}((-2\pi i x)^\alpha \mu).$$

For $R > 0$, if $\operatorname{supp}\mu \subset \overline{B(0,R)}$, then

$$\|D^\alpha \hat{\mu}\|_\infty \leq (2\pi R)^{|\alpha|_1} \|\mu\|.$$

Proof. For $j = 1, \ldots, d$, let e_j be the jth coordinate vector in $\mathbb{R}^d$, with length 1. Let $\xi \in \mathbb{R}^d$, and define

$$\Delta(h) = \frac{\hat{\mu}(\xi + h e_j) - \hat{\mu}(\xi)}{h}, \qquad h \neq 0.$$

We can write this as

$$\Delta(h) = \int_{\mathbb{R}^d} \frac{e^{-2\pi i h x_j} - 1}{h} e^{-2\pi i \xi \cdot x} d\mu(x).$$

For any $x \in \mathbb{R}^d$,

$$\left| \frac{e^{-2\pi i h x_j} - 1}{h} \right| = \frac{|e^{-2\pi i h x_j} - 1|}{|h|} \leq \frac{|-2\pi i h x_j|}{|h|} = 2\pi |x_j|.$$

Because μ has compact support, $2\pi|x_j| \in L^1(\mu)$. Furthermore, for each $x \in \mathbb{R}^d$ we have

$$\frac{e^{-2\pi i h x_j} - 1}{h} \to -2\pi i x_j, \qquad h \to 0.$$

Therefore, the dominated convergence theorem tells us that

$$\lim_{h \to 0} \Delta(h) = \int_{\mathbb{R}^d} -2\pi i x_j e^{-2\pi i \xi \cdot x} d\mu(x).$$

On the other hand, for $\alpha_k = 1$ for $k = j$ and $\alpha_k = 0$ otherwise,

$$(D^\alpha \hat{\mu})(\xi) = \lim_{h \to 0} \Delta(h),$$

so

$$(D^\alpha \hat{\mu})(\xi) = \int_{\mathbb{R}^d} (-2\pi i x)^\alpha e^{-2\pi i \xi \cdot x} d\mu(x) = \mathscr{F}((-2\pi i x)^\alpha \mu)(\xi),$$

and in particular, $\hat{\mu} \in C^1(\mathbb{R}^d)$. (The Fourier transform of a regular complex Borel measure on a locally compact abelian group is bounded and uniformly continuous.[2]) Because μ has compact support so does $(-2\pi i x)^\alpha \mu$, hence we can play the above game with $(-2\pi i x)^\alpha \mu$, and by induction it follows that for any α,

$$D^\alpha \hat{\mu} = \mathscr{F}((-2\pi i x)^\alpha \mu),$$

[2]Walter Rudin, *Fourier Analysis on Groups*, p. 15, Theorem 1.3.3.

and in particular, $\hat{\mu} \in C^\infty(\mathbb{R}^d)$.

Suppose that $\operatorname{supp}\mu \subset \overline{B(0,R)}$. The total variation of the complex measure $(-2\pi i x)^\alpha \mu$ is the positive measure

$$(2\pi)^{|\alpha|_1} |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d} |\mu|,$$

hence

$$\begin{aligned}
\|(-2\pi i x)^\alpha \mu\| &= (2\pi)^{|\alpha|_1} \int_{\mathbb{R}^d} |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d} d|\mu|(x) \\
&= (2\pi)^{|\alpha|_1} \int_{\overline{B(0,R)}} |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d} d|\mu|(x) \\
&\leq (2\pi)^{|\alpha|_1} \int_{\overline{B(0,R)}} R^{\alpha_1} \cdots R^{\alpha_d} d|\mu|(x) \\
&= (2\pi R)^{|\alpha|_1} \int_{\overline{B(0,R)}} d|\mu|(x) \\
&= (2\pi R)^{|\alpha|_1} \int_{\mathbb{R}^d} d|\mu|(x) \\
&= (2\pi R)^{|\alpha|_1} \|\mu\|.
\end{aligned}$$

Then using (1),

$$\|\mathscr{F}((-2\pi i x)^\alpha \mu)\|_\infty \leq \|(-2\pi i x)^\alpha \mu\| \leq (2\pi R)^{|\alpha|_1} \|\mu\|.$$

But we have already established that $D^\alpha \hat{\mu} = \mathscr{F}((-2\pi i x)^\alpha \mu)$, which with the above inequality completes the proof. □

2 Test functions

For an open subset Ω of $\mathbb{R}^d$, we denote by $\mathscr{D}(\Omega)$ the set of those $\phi \in C^\infty(\Omega)$ such that $\operatorname{supp}\phi$ is a compact set. Elements of $\mathscr{D}(\Omega)$ are called **test functions**.

It is a fact that there is a test function ϕ satisfying: (i) $\phi(x) = 1$ for $|x| \leq 1$, (ii) $\phi(x) = 0$ for $|x| \geq 2$, (iii) $0 \leq \phi \leq 1$, and (iv) ϕ is radial. We write, for $k = 1, 2, \ldots$,

$$\phi_k(x) = \phi(k^{-1}x), \qquad x \in \mathbb{R}^d.$$

For any multi-index α,

$$(D^\alpha \phi_k)(x) = k^{-|\alpha|_1} (D^\alpha \phi)(k^{-1}x), \qquad x \in \mathbb{R}^d,$$

hence

$$\|D^\alpha \phi_k\|_\infty = k^{-|\alpha|_1} \|D^\alpha \phi\|_\infty. \tag{2}$$

We use the following lemma to prove the theorem that comes after it.[3]

[3]Thomas H. Wolff, *Lectures on Harmonic Analysis*, p. 4, Lemma 1.5.

Lemma 2. *Suppose that $f \in C^N(\mathbb{R}^d)$ and $D^\alpha f \in L^1(\mathbb{R}^d)$ for each $|\alpha| \leq N$. Then for each $|\alpha| \leq N$, $D^\alpha(\phi_k f) \to D^\alpha f$ in $L^1(\mathbb{R}^d)$ as $k \to \infty$.*

Proof. Let $|\alpha| \leq N$. In the case $\alpha = 0$,

$$\begin{aligned}
\|\phi_k f - f\|_1 &= \int_{\mathbb{R}^d} |\phi_k(x)f(x) - f(x)|dx \\
&= \int_{|x|\geq k} |\phi_k(x)f(x) - f(x)|dx \\
&\leq \int_{|x|\geq k} |f(x)|dx.
\end{aligned}$$

Because $f \in L^1(\mathbb{R}^d)$, this tends to 0 as $k \to \infty$.

Suppose that $\alpha > 0$. The Leibniz rule tells us that with $c_\beta = \binom{\alpha}{\beta}$, we have, for each k,

$$D^\alpha(\phi_k f) = \phi_k D^\alpha f + \sum_{0<\beta\leq\alpha} c_\beta D^{\alpha-\beta} f D^\beta \phi_k.$$

For $C_1 = \max_\beta |c_\beta|$,

$$\begin{aligned}
\|D^\alpha(\phi_k f) - \phi_k D^\alpha f\|_1 &\leq \sum_{0<\beta\leq\alpha} \left\|c_\beta D^{\alpha-\beta} f D^\beta \phi_k\right\|_1 \\
&\leq C_1 \sum_{0<\beta\leq\alpha} \left\|D^\beta \phi_k\right\|_\infty \left\|D^{\alpha-\beta} f\right\|_1.
\end{aligned}$$

Let $C_2 = \max_{0<\beta\leq\alpha} \left\|D^\beta \phi\right\|_\infty$. By (2), for $0 < \beta \leq \alpha$ we have

$$\left\|D^\beta \phi_k\right\|_\infty = k^{-|\beta|_1} \left\|D^\beta \phi\right\|_\infty \leq C_2 k^{-|\beta|_1} \leq C_2 k^{-1}.$$

Thus

$$\|D^\alpha(\phi_k f) - \phi_k D^\alpha f\|_1 \leq C_1 C_2 k^{-1} \sum_{0<\beta\leq\alpha} \left\|D^{\alpha-\beta} f\right\|_1,$$

which tends to 0 as $k \to \infty$. For any k,

$$\begin{aligned}
\|\phi_k D^\alpha f - D^\alpha f\|_1 &= \int_{\mathbb{R}^d} |\phi_k(x)(D^\alpha f)(x) - (D^\alpha f)(x)|dx \\
&= \int_{|x|\geq k} |\phi_k(x)(D^\alpha f)(x) - (D^\alpha f)(x)|dx \\
&\leq \int_{|x|\geq k} |(D^\alpha f)(x)|dx,
\end{aligned}$$

and because $D^\alpha f \in L^1(\mathbb{R}^d)$, this tends to 0 as $k \to \infty$. But

$$\|D^\alpha(\phi_k f) - D^\alpha f\|_1 \leq \|D^\alpha(\phi_k f) - \phi_k D^\alpha f\|_1 + \|\phi_k D^\alpha f - D^\alpha f\|_1,$$

which completes the proof. □

Now we calculate the Fourier transform of the derivative of a function, and show that the smoother a function is the faster its Fourier transform decays.[4]

Theorem 3. *If $f \in C^N(\mathbb{R}^d)$ and $D^\alpha f \in L^1(\mathbb{R}^d)$ for each $|\alpha| \leq N$, then for each $|\alpha| \leq N$,*

$$\widehat{D^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi), \qquad \xi \in \mathbb{R}^d. \tag{3}$$

There is a constant $C = C(f, N)$ such that

$$|\hat{f}(\xi)| \leq C(1 + |\xi|)^{-N}, \qquad \xi \in \mathbb{R}^d.$$

Proof. If $g \in C_c^1(\mathbb{R}^d)$, then for any $1 \leq j \leq d$, integrating by parts,

$$\int_{\mathbb{R}^d} (\partial_j g)(x) e^{-2\pi i\xi\cdot x} dx = 2\pi i\xi_j \int_{\mathbb{R}^d} g(x) e^{-2\pi i\xi\cdot x} dx.$$

It follows by induction that if $g \in C_c^N(\mathbb{R}^d)$, then for each $|\alpha| \leq N$,

$$\widehat{D^\alpha g}(\xi) = (2\pi i\xi)^\alpha \hat{g}(\xi), \qquad \xi \in \mathbb{R}^d.$$

Let $|\alpha| \leq N$. For $k = 1, 2, \ldots$, let $f_k = \phi_k f$. For each k we have $f_k \in C^N(\mathbb{R}^d)$, hence

$$\widehat{D^\alpha f_k}(\xi) = (2\pi i\xi)^\alpha \widehat{f_k}(\xi), \qquad \xi \in \mathbb{R}^d.$$

On the one hand,

$$\left\| \widehat{D^\alpha f_k} - \widehat{D^\alpha f} \right\|_\infty = \|\mathscr{F}(D^\alpha f_k - D^\alpha f)\|_\infty \leq \|D^\alpha f_k - D^\alpha f\|_1,$$

and Lemma 2 tells us that this tends to 0 as $k \to \infty$. On the other hand, for $\xi \in \mathbb{R}^d$,

$$\begin{aligned} |\widehat{D^\alpha f_k}(\xi) - (2\pi i\xi)^\alpha \hat{f}(\xi)| &= |(2\pi i\xi)^\alpha \widehat{f_k}(\xi) - (2\pi i\xi)^\alpha \hat{f}(\xi)| \\ &= |(2\pi i\xi)^\alpha||\mathscr{F}(f_k - f)(\xi)| \\ &\leq |(2\pi i\xi)^\alpha| \|f_k - f\|_1, \end{aligned}$$

which by Lemma 2 tends to 0 as $k \to \infty$. Therefore, for $\xi \in \mathbb{R}^d$,

$$|\widehat{D^\alpha f}(\xi) - (2\pi i\xi)^\alpha \hat{f}(\xi)| \leq \left\| \widehat{D^\alpha f_k} - \widehat{D^\alpha f} \right\|_\infty + |\widehat{D^\alpha f_k}(\xi) - (2\pi i\xi)^\alpha \hat{f}(\xi)|,$$

and because the right-hand side tends to 0 as $k \to \infty$, we get

$$\widehat{D^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi).$$

If $y \in S^{d-1}$ then there is at least one $1 \leq j \leq d$ with $y_j \neq 0$, from which we get

$$\sum_{|\beta|_1 = N} |y^\beta| > 0.$$

[4]Thomas H. Wolff, *Lectures on Harmonic Analysis*, p. 4, Proposition 1.4.

The function $y \mapsto \sum_{|\beta|_1=N} |y^\beta|$ is continuous $S^{d-1} \to \mathbb{R}$, so there is some $C_N > 0$ such that

$$\frac{1}{C_N} \leq \sum_{|\beta|_1=N} |y^\beta|, \qquad y \in S^{d-1}.$$

For nonzero $x \in \mathbb{R}^d$, write $x = |x|y$, with which $\sum_{|\beta|_1=N} |x^\beta| = |x|^N \sum_{|\beta|_1=N} |y^\beta|$. Therefore

$$|x|^N \leq C_N \sum_{|\beta|_1=N} |x^\beta|, \qquad x \in \mathbb{R}^d.$$

For $|\alpha| \leq N$, because the Fourier transform of an element of L^1 belongs to C_0, we have by (3) that $\xi \mapsto \xi^\alpha \hat{f}(\xi)$ belongs to $C_0(\mathbb{R}^d)$, and in particular is bounded. Then for $\xi \in \mathbb{R}^d$,

$$\begin{aligned} |\xi|^N |\hat{f}(\xi)| &\leq C_N \sum_{|\beta|_1=N} |\xi^\beta||\hat{f}(\xi)| \\ &= C_N \sum_{|\beta|_1=N} |\xi^\beta \hat{f}(\xi)| \\ &\leq C_N \sum_{|\beta|_1=N} \left\| \xi^\beta \hat{f}(\xi) \right\|_\infty \\ &= C'. \end{aligned}$$

On the one hand, for $|\xi| \geq 1$ we have

$$1 + |\xi| \leq 2|\xi|,$$

hence

$$|\xi|^{-N} \leq \left(\frac{1 + |\xi|}{2} \right)^{-N} = 2^N (1 + |\xi|)^{-N},$$

giving

$$|\hat{f}(\xi)| \leq C' |\xi|^{-N} \leq C' 2^N (1 + |\xi|)^{-N}.$$

On the other hand, for $|\xi| \leq 1$ we have

$$1 + |\xi| \leq 2,$$

and so

$$|\hat{f}(\xi)| \leq \left\| \hat{f} \right\|_\infty 2^N 2^{-N} \leq \left\| \hat{f} \right\|_\infty 2^N (1 + |\xi|)^{-N}.$$

Thus, for

$$C = \max \left\{ 2^N C', 2^N \left\| \hat{f} \right\|_\infty \right\}$$

we have

$$|\hat{f}(\xi)| \leq C(1 + |\xi|)^{-N}, \qquad \xi \in \mathbb{R}^d,$$

completing the proof. □

3 Bernstein's inequality for L^2

For a Borel measurable function $f : \mathbb{R}^d \to \mathbb{C}$, let O be the union of those open subsets U of $\mathbb{R}^d$ such that $f(x) = 0$ for almost all $x \in U$. In other words, O is the largest open set on which $f = 0$ almost everywhere. The **essential support of** f is the set

$$\operatorname{ess\,supp} f = \mathbb{R}^d \setminus O.$$

The following is **Bernstein's inequality for** $L^2(\mathbb{R}^d)$.[5]

Theorem 4. *If $f \in L^2(\mathbb{R}^d)$, $R > 0$, and*

$$\operatorname{ess\,supp} \hat{f} \subset \overline{B(0,R)}, \tag{4}$$

then there is some $f_0 \in C^\infty(\mathbb{R}^d)$ such that $f(x) = f_0(x)$ for almost all $x \in \mathbb{R}^d$, and for any multi-index α,

$$\|D^\alpha f_0\|_2 \leq (2\pi R)^{|\alpha|_1} \|f\|_2 .$$

Proof. Let χ_R be the indicator function for $\overline{B(0,R)}$. By (4), the Cauchy-Schwarz inequality, and the Parseval identity,

$$\left\|\hat{f}\right\|_1 = \left\|\chi_R \hat{f}\right\|_1 \leq \|\chi_R\|_2 \left\|\hat{f}\right\|_2 = m_d(\overline{B(0,R)})^{1/2} \|f\|_2 < \infty,$$

so $\hat{f} \in L^1(\mathbb{R}^d)$. The Plancherel theorem[6] tells us that if $g \in L^2(\mathbb{R}^d)$ and $\hat{g} \in L^1(\mathbb{R}^d)$, then

$$g(x) = \int_{\mathbb{R}^d} \hat{g}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

for almost all $x \in \mathbb{R}^d$. Thus, for $f_0 : \mathbb{R}^d \to \mathbb{C}$ defined by

$$f_0(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi = \mathscr{F}(\hat{f})(-x), \qquad x \in \mathbb{R}^d,$$

we have $f(x) = f_0(x)$ for almost all $x \in \mathbb{R}^d$. Because $f = f_0$ almost everywhere,

$$\widehat{f_0} = \hat{f}.$$

Applying Theorem 1 to $d\mu(\xi) = \widehat{f_0}(-\xi) d\xi$, we have $f_0 \in C^\infty(\mathbb{R}^d)$ and for any multi-index α,

$$D^\alpha f_0 = \mathscr{F}((-2\pi i \xi)^\alpha \hat{f}(-\xi)).$$

[5] Thomas H. Wolff, *Lectures on Harmonic Analysis*, p. 31, Proposition 5.1.

[6] Walter Rudin, *Real and Complex Analysis*, third ed., p. 187, Theorem 9.14.

By Parseval's identity,

$$\begin{aligned}
\|D^\alpha f_0\|_2 &= \left\|(-2\pi i\xi)^\alpha \hat{f}(-\xi)\right\|_2 \\
&= \left\|(2\pi i\xi)^\alpha \chi_R(\xi)\hat{f}(\xi)\right\|_2 \\
&\le \|(2\pi i\xi)^\alpha \chi_R(\xi)\|_\infty \left\|\hat{f}\right\|_2 \\
&\le (2\pi R)^{|\alpha|_1} \left\|\hat{f}\right\|_2 \\
&= (2\pi R)^{|\alpha|_1} \|f\|_2 ,
\end{aligned}$$

proving the claim. □

4 Nikolsky's inequality

Nikolsky's inequality tells us that if the Fourier transform of a function is supported on a ball centered at the origin, then for $1 \le p \le q \le \infty$, the L^q norm of the function is bounded above in terms of its L^p norm.[7]

Theorem 5. *There is a constant C_d such that if $f \in \mathscr{S}(\mathbb{R}^d)$, $R > 0$,*

$$\operatorname{supp} \hat{f} \subset \overline{B(0,R)},$$

and $1 \le p \le q \le \infty$, then

$$\|f\|_q \le C_d R^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_p .$$

Proof. Let $g = f_R$, i.e.

$$g(x) = R^{-d} f(R^{-1}x), \qquad x \in \mathbb{R}^d.$$

Then for $\xi \in \mathbb{R}^d$,

$$\hat{g}(\xi) = \int_{\mathbb{R}^d} g(x)e^{-2\pi i\xi\cdot x}dx = \int_{\mathbb{R}^d} R^{-d} f(R^{-1}x)e^{-2\pi i\xi\cdot x}dx = \hat{f}(R\xi),$$

showing that $\operatorname{supp}\hat{g} = R^{-1}\operatorname{supp}\hat{f} \subset \overline{B(0,1)}$. Let $\chi \in \mathscr{D}(\mathbb{R}^d)$ with $\chi(\xi) = 1$ for $|\xi| \le 1$, with which

$$\hat{g} = \chi\hat{g}.$$

Then $g = (\mathscr{F}^{-1}\chi) * g$, and using Young's inequality, with $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$,

$$\|g\|_q \le \left\|\mathscr{F}^{-1}\chi\right\|_r \|g\|_q = \|\hat{\chi}\|_r \|g\|_q . \tag{5}$$

[7] Camil Muscalu and Wilhelm Schlag, *Classical and Multilinear Harmonic Analysis*, volume I, p. 83, Lemma 4.13.

Moreover,

$$\begin{aligned}\|g\|_a &= \left(\int_{\mathbb{R}^d} |R^{-d} f(R^{-1}x)|^a dx\right)^{1/a} \\ &= \left(\int_{\mathbb{R}^d} R^{-da+d} |f(y)|^a dy\right)^{1/a} \\ &= R^{d\left(\frac{1}{a}-1\right)} \|f\|_a ,\end{aligned}$$

so (5) tells us

$$R^{d\left(\frac{1}{q}-1\right)} \|f\|_q \leq \|\hat{\chi}\|_r R^{d\left(\frac{1}{p}-1\right)} \|f\|_p ,$$

i.e.

$$\|f\|_q \leq \|\hat{\chi}\|_r R^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_p .$$

Now, $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$, so $0 \leq \frac{1}{r} \leq 1$ because $1 \leq p \leq q \leq \infty$, namely, $1 \leq r \leq \infty$. By the log-convexity of L^r norms, for $\frac{1}{r} = 1 - \theta$ we have

$$\|\hat{\chi}\|_r \leq \|\hat{\chi}\|_1^{1-\theta} \|\hat{\chi}\|_\infty^{\theta} .$$

Thus with

$$C_d = \max\{\|\hat{\chi}\|_1 , \|\hat{\chi}\|_\infty\}$$

we have proved the claim. □

5 The Dirichlet kernel and Fejér kernel for R

The function $D_M \in C_0(\mathbb{R})$ defined by

$$D_M(x) = \frac{\sin 2\pi M x}{\pi x}, \qquad x \neq 0$$

and $D_M(0) = 2M$, is called the **Dirichlet kernel**. Let χ_M be the indicator function for the set $[-M, M]$. We have, for $x \neq 0$,

$$\begin{aligned}\widehat{\chi_R}(x) &= \int_{\mathbb{R}} \chi_R(\xi) e^{-2\pi i x \xi} d\xi \\ &= \int_{-M}^{M} e^{-2\pi i x \xi} d\xi \\ &= \left.\frac{e^{-2\pi i x \xi}}{-2\pi i x}\right|_{-M}^{M} \\ &= \frac{e^{-2\pi i M x}}{-2\pi i x} + \frac{e^{2\pi i M x}}{2\pi i x} \\ &= \frac{1}{\pi x} \frac{e^{2\pi i M x} - e^{-2\pi i M x}}{2i} \\ &= \frac{\sin 2\pi M x}{\pi x}.\end{aligned}$$

For $x = 0$, $\widehat{\chi_R}(0) = 2M = D_M(0)$. Thus,

$$D_M = \widehat{\chi_R}.$$

For $f \in L^1(\mathbb{R})$ and $M > 0$, we define

$$(S_M f)(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i \xi x} d\xi, \qquad x \in \mathbb{R}.$$

It is straightforward to check that

$$(S_M f)(x) = \int_{\mathbb{R}} \frac{\sin 2\pi M t}{\pi t} f(x-t) dt = (D_M * f)(x), \qquad x \in \mathbb{R}.$$

For $f \in L^1(\mathbb{R})$, $M > 0$, and $x \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{M}\int_0^M (S_m f)(x) dm &= \frac{1}{M}\int_0^M \left(\int_{-m}^{m} \hat{f}(\xi) e^{2\pi i \xi x} d\xi\right) dm \\
&= \frac{1}{M}\int_0^M \left(\int_{-m}^{m} \left(\int_{\mathbb{R}} f(y) e^{-2\pi i \xi y} dy\right) e^{2\pi i \xi x} d\xi\right) dm \\
&= \frac{1}{M}\int_{\mathbb{R}} f(y) \left(\int_0^M \left(\int_{-m}^{m} e^{-2\pi i \xi (y-x)} d\xi\right) dm\right) dy \\
&= \frac{1}{M}\int_{\mathbb{R}} f(y) \left(\int_0^M D_m(y-x) dm\right) dy \\
&= \frac{1}{M}\int_{\mathbb{R}} f(y) \left(\int_0^M \frac{\sin 2\pi m (y-x)}{\pi (y-x)} dm\right) dy \\
&= \frac{1}{M}\int_{\mathbb{R}} f(y) \left(-\frac{\cos 2\pi m (y-x)}{2\pi^2 (y-x)^2}\bigg|_0^M\right) dy \\
&= \frac{1}{M}\int_{\mathbb{R}} f(y) \left(\frac{1}{2\pi^2 (y-x)^2} - \frac{\cos 2\pi M (y-x)}{2\pi^2 (y-x)^2}\right) dy.
\end{aligned}$$

We define the **Fejér kernel** $K_M \in C_0(\mathbb{R})$ by

$$K_M(x) = \frac{1 - \cos 2\pi M x}{2M\pi^2 x^2}, \qquad x \neq 0,$$

and $K_M(0) = M$. Thus, because K_M is an even function,

$$\frac{1}{M}\int_0^M (S_m f)(x) dm = (K_M * f)(x).$$

One proves that K_M is an **approximate identity**: $K_M \geq 0$,

$$\int_{\mathbb{R}} K_M(x) dx = 1,$$

and for any $\delta > 0$,

$$\lim_{M\to\infty} \int_{|x|>\delta} K_M(x)dx = 0.$$

The fact that K_M is an approximate identity implies that for any $f \in L^1(\mathbb{R})$, $K_M * f \to f$ in $L^1(\mathbb{R})$ as $M \to \infty$.

We shall use the Fejér kernel to prove Bernstein's inequality for $\mathbb{R}$.[8]

Theorem 6. *If $\mu \in \mathcal{M}(\mathbb{R})$, $M > 0$, and*

$$\operatorname{supp} \mu \subset [-M, M],$$

then

$$\|\hat{\mu}'\|_\infty \leq 4\pi M \|\hat{\mu}\|_\infty .$$

Proof. For $x_0 \in \mathbb{R}$, let $d\mu_{x_0}(t) = e^{-2\pi i x_0 t} d\mu(t)$. μ_{x_0} has the same support has μ, and

$$\widehat{\mu_{x_0}}(x) = \int_{\mathbb{R}} e^{-2\pi i x t} d\mu_{x_0}(t) = \int_{\mathbb{R}} e^{-2\pi i x t} e^{-2\pi i x_0 t} d\mu(t) = \hat{\mu}(x + x_0).$$

It follows that to prove the claim it suffices to prove that $|\hat{\mu}'(0)| \leq 4\pi M \|\hat{\mu}\|_\infty$.

Write $f = \hat{\mu} \in C_u(\mathbb{R})$. Define $\Delta_M \in C_c(\mathbb{R})$ by

$$\Delta_M(t) = \begin{cases} M - |t| & |t| < M \\ 0 & |t| \geq M, \end{cases} \qquad t \in \mathbb{R}.$$

We calculate, for $x \neq 0$,

$$\begin{aligned} \int_{\mathbb{R}} \Delta_M(t) e^{-2\pi i x t} dt &= -\frac{e^{-2\pi i M x}(-1 + e^{2\pi i M x})^2}{4\pi^2 x^2} \\ &= \frac{(\sin \pi M x)^2}{\pi^2 x^2} \\ &= \frac{1 - \cos 2\pi M x}{2\pi^2 x^2}. \end{aligned}$$

so

$$\widehat{\Delta_M}(x) = M K_M(x).$$

Then for $t \in [-M, M]$,

$$\begin{aligned} \int_{\mathbb{R}} (e^{2\pi i M \xi} - e^{-2\pi i M \xi}) K_M(\xi) e^{-2\pi i \xi t} d\xi &= \widehat{K_M}(t - M) - \widehat{K_M}(t + M) \\ &= \frac{\Delta_M(-t + M) - \Delta_M(-t - M)}{M} \\ &= \frac{t}{M}. \end{aligned}$$

[8] Mark A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, p. 122, Theorem 2.3.17.

On the one hand, the integral of the left-hand side with respect to μ is

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}(e^{2\pi i M\xi} - e^{-2\pi i M\xi})K_M(\xi)e^{-2\pi i\xi t}d\xi d\mu(t) \\
&= \int_{\mathbb{R}}(e^{2\pi i M\xi} - e^{-2\pi i M\xi})K_M(\xi)f(\xi)d\xi.
\end{aligned}$$

On the other hand, the integral of the right-hand side with respect to μ is

$$\begin{aligned}
\int_{\mathbb{R}}\frac{t}{M}d\mu(t) &= \frac{1}{-2\pi i M}\int_{\mathbb{R}} -2\pi i t d\mu(t) \\
&= \frac{1}{-2\pi i M}\mathscr{F}((-2\pi i t)\mu)(0) \\
&= \frac{1}{-2\pi i M}f'(0).
\end{aligned}$$

Hence

$$\frac{1}{-2\pi i M}f'(0) = \int_{\mathbb{R}}(e^{2\pi i M\xi} - e^{-2\pi i M\xi})K_M(\xi)f(\xi)d\xi,$$

giving

$$|f'(0)| \leq 4\pi M \left\|f\right\|_{\infty}\left\|K_M\right\|_1 = 4\pi M\left\|f\right\|_{\infty},$$

proving the claim. □