Bernoulli polynomials

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1 Bernoulli polynomials

For $k \geq 0$, the **Bernoulli polynomial** $B_k(x)$ is defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}, \qquad |z| < 2\pi. \tag{1}$$

The **Bernoulli numbers** are $B_k = B_k(0)$, the constant terms of the Bernoulli polynomials. For any x, using L'Hospital's rule the left-hand side of (1) tends to 1 as $z \to 0$, and the right-hand side tends to $B_0(x)$, hence $B_0(x) = 1$. Differentiating (1) with respect to x,

$$\sum_{k=0}^{\infty} B_k'(x) \frac{z^k}{k!} = \frac{z^2 e^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^{k+1}}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x) \frac{z^k}{(k-1)!},$$

so $B_0'(x) = 0$ and for $k \geq 1$ we have $\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!}$, i.e. $B_k'(x) = kB_{k-1}(x)$. Furthermore, for $k \geq 1$, integrating (1) with respect to x on $[0, 1]$ produces

$$1 = \sum_{k=0}^{\infty} \left(\int_0^1 B_k(x) dx \right) \frac{z^k}{k!}, \qquad |z| < 2\pi,$$

hence $\int_0^1 B_0(x) dx = 1$ and for $k \geq 1$,

$$\int_0^1 B_k(x) dx = 0.$$

The first few Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

The Bernoulli polynomials satisfy the following:

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k(x+1)\frac{z^k}{k!} &= \frac{ze^{(x+1)z}}{e^z-1} \\
&= \frac{ze^{xz}(e^z-1+1)}{e^z-1} \\
&= ze^{xz} + \frac{ze^{xz}}{e^z-1} \\
&= \sum_{k=0}^{\infty} \frac{x^k z^{k+1}}{k!} + \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!} \\
&= \sum_{k=1}^{\infty} \frac{x^{k-1} z^k}{(k-1)!} + \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!},
\end{aligned}$$

hence for $k \geq 1$ it holds that $B_k(x+1) = kx^{k-1} + B_k(x)$. In particular, for $k \geq 2$, $B_k(1) = B_k(0)$.

Using (1),

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k(1-x)\frac{z^k}{k!} &= \frac{ze^{(1-x)z}}{e^z-1} \\
&= \frac{ze^z e^{-xz}}{e^z-1} \\
&= \frac{ze^{-xz}}{1-e^{-z}} \\
&= \frac{-ze^{-xz}}{e^{-z}-1} \\
&= \sum_{k=0}^{\infty} B_k(x)\frac{(-z)^k}{k!},
\end{aligned}$$

hence for $k \geq 0$,

$$B_k(1-x) = (-1)^k B_k(x).$$

Finally, it is a fact that for $k \geq 2$,

$$\sup_{0 \leq x \leq 1} |B_k(x)| \leq \frac{2\zeta(k)k!}{(2\pi)^k}. \qquad (2)$$

2 Periodic Bernoulli functions

For $x \in \mathbb{R}$, let $[x]$ be the greatest integer $\leq x$, and let $R(x) = x - [x]$, called the fractional part of x. Write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and define the **periodic Bernoulli functions** $P_k : \mathbb{T} \to \mathbb{R}$ by

$$P_k(t) = B_k(R(t)), \qquad t \in \mathbb{T}.$$

For $k \geq 2$, because $B_k(1) = B_k(0)$, the function P_k is continuous. For $f : \mathbb{T} \to \mathbb{C}$ define its **Fourier transform** $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\widehat{f}(n) = \int_{\mathbb{T}} f(t) e^{-2\pi i n t} dt, \qquad n \in \mathbb{Z}.$$

For $k \geq 1$, one calculates $\widehat{P}_k(0) = 0$ and using integration by parts,

$$\widehat{P}_k(n) = -\frac{1}{(2\pi i n)^k}$$

for $n \neq 0$. Thus for $k \geq 1$, the Fourier series of P_k is[1]

$$P_k(t) \sim \sum_{n \in \mathbb{Z}} \widehat{P}_k(n) e^{2\pi i n t} = -\frac{1}{(2\pi i)^k} \sum_{n \neq 0} n^{-k} e^{2\pi i n t}.$$

For $k \geq 2$, $\sum_{n \in \mathbb{Z}} |\widehat{P}_k(n)| < \infty$, from which it follows that $\sum_{|n| \leq N} \widehat{P}_k(n) e^{2\pi i n t}$ converges to $P_k(t)$ uniformly for $t \in \mathbb{T}$. Furthermore, for $t \notin \mathbb{Z}$,[2]

$$P_1(t) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2\pi n t.$$

For $f, g \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
\widehat{f * g}(n) &= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(x - y) g(y) dy \right) e^{-2\pi i n x} dx \\
&= \int_{\mathbb{T}} g(y) \left(\int_{\mathbb{T}} f(x - y) e^{-2\pi i n x} dx \right) dy \\
&= \int_{\mathbb{T}} g(y) \left(\int_{\mathbb{T}} f(x) e^{-2\pi i n x} e^{-2\pi i n y} dx \right) dy \\
&= \widehat{f}(n) \widehat{g}(n).
\end{aligned}$$

For $k, l \geq 1$ and for $n \neq 0$,

$$\begin{aligned}
\widehat{P_k * P_l}(n) &= \widehat{P_k}(n) \widehat{P_l}(n) \\
&= -(2\pi i n)^{-k} \cdot -(2\pi i n)^{-l} \\
&= (2\pi i n)^{-k-l} \\
&= -\widehat{P_{k+l}}(n),
\end{aligned}$$

and $\widehat{P_k * P_l}(0) = 0 = -\widehat{P_{k+l}}(0)$, so $P_k * P_l = -P_{k+l}$.

[1] cf. `http://www.math.umn.edu/~garrett/m/mfms/notes_c/bernoulli.pdf`

[2] Hugh L. Montgomery and Robert C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, p. 499, Theorem B.2.

3 Euler-Maclaurin summation formula

The **Euler-Maclaurin summation formula** is the following.[3] If $a < b$ are real numbers, K is a positive integer, and f is a C^K function on an open set that contains $[a, b]$, then

$$\sum_{a<m\le b} f(m) = \int_a^b f(x)dx + \sum_{k=1}^{K} \frac{(-1)^k}{k!}(P_k(b)f^{(k-1)}(b) - P_k(a)f^{(k-1)}(a)) - \frac{(-1)^K}{K!}\int_a^b P_K(x)f^{(K)}(x)dx.$$

Applying the Euler-Maclaurin summation formula with $a = 1, b = n, K = 2, f(x) = \log x$ yields[4]

$$\sum_{1\le m\le n} \log n = n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log 2\pi + O(n^{-1}).$$

Since $e^{1+O(n^{-1})} = 1 + O(n^{-1})$,

$$n! = n^n e^{-n}\sqrt{2\pi n}(1 + O(n^{-1})),$$

Stirling's approximation.

Write $a_n = -\log n + \sum_{1\le m\le n} \frac{1}{m}$. Because $\log(1-x)$ is concave,

$$a_n - a_{n-1} = \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) \le 1 + 1 - \frac{1}{n} = 0,$$

which means that the sequence a_n is nonincreasing. For $f(x) = \frac{1}{x}$, because f is positive and nonincreasing,

$$\sum_{1\le m\le n} f(m) \ge \int_1^{n+1} f(x)dx = \log(n+1) > \log n,$$

hence $a_n > 0$. Because a_n is positive and nonincreasing, there exists some nonnegative limit, γ, called **Euler's constant**. Using the Euler-Maclaurin summation formula with $a = 1, b = n, K = 1, f(x) = \frac{1}{x}$, as $P_1(x) = [x] - \frac{1}{2}$,

$$\sum_{1<m\le n} \frac{1}{m} = \log n + \frac{1}{2n} - \frac{1}{2} + \frac{1}{2}\int_1^n \frac{1}{x^2}dx - \int_1^n R(x)\frac{1}{x^2}dx,$$

which is

$$\sum_{1<m\le n} \frac{1}{m} = \log n - \int_1^\infty \frac{R(x)}{x^2}dx + \int_n^\infty \frac{R(x)}{x^2}dx;$$

[3] Hugh L. Montgomery and Robert C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, p. 500, Theorem B.5.

[4] Hugh L. Montgomery and Robert C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, p. 503, Eq. B.25.

as $0 \le R(x)x^{-2} \le x^{-2}$, the function $x \mapsto R(x)x^{-2}$ is integrable on $[1,\infty)$. Since $0 \le \int_n^\infty R(x)x^{-2}dx \le \int_n^\infty x^{-2}dx = n^{-1}$,

$$\sum_{1 \le m \le n} \frac{1}{m} = \log n + C + O(n^{-1})$$

for $C = 1 - \int_1^\infty R(x)x^{-2}$. But $-\log n + \sum_{1 \le m \le n} \frac{1}{m} \to \gamma$ as $n \to \infty$, from which it follows that $C = \gamma$, and thus

$$\sum_{1 \le m \le n} \frac{1}{m} = \log n + \gamma + O(n^{-1}).$$

4 Hurwitz zeta function

For $0 < \alpha \le 1$ and $\operatorname{Re} s > 1$, define the **Hurwitz zeta function** by

$$\zeta(s,\alpha) = \sum_{n \ge 0} (n+\alpha)^{-s}.$$

For $\operatorname{Re} s > 0$,

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt,$$

and for $n \ge 0$ do the change of variable $t = (n+\alpha)u$,

$$\begin{aligned}\Gamma(s) &= \int_0^\infty (n+\alpha)^{s-1}u^{s-1}e^{-(n+\alpha)u}(n+\alpha)du \\ &= (n+\alpha)^s \int_0^\infty u^{s-1}e^{-nu}e^{-\alpha u}du.\end{aligned}$$

For real $s > 1$,

$$(n+\alpha)^{-s}\Gamma(s) = \int_0^\infty u^{s-1}e^{-nu}e^{-\alpha u}du.$$

Then

$$\sum_{0 \le n \le N} (n+\alpha)^{-s}\Gamma(s) = \sum_{0 \le n \le N} \int_0^\infty u^{s-1}e^{-nu}e^{-\alpha u}du = \int_0^\infty f_N(s,u)du,$$

where

$$f_N(s,u) = \begin{cases} u^{s-1}e^{-\alpha u}\frac{1-e^{-(N+1)u}}{1-e^{-u}} & u > 0 \\ 0 & u = 0. \end{cases}$$

$f_N(s,u) \ge 0$ and the sequence $f_N(s,u)$ is pointwise nondecreasing, and

$$\lim_{N \to \infty} f_N(s,u) = f(s,u) = \begin{cases} u^{s-1}e^{-\alpha u}\frac{1}{1-e^{-u}} & u > 0 \\ 0 & u = 0. \end{cases}$$

By the **monotone convergence theorem**,

$$\int_0^\infty f_N(s,u)du \to \int_0^\infty f(s,u)du,$$

which means that, for real $s>1$,

$$\zeta(s,\alpha)\Gamma(s) = \int_0^\infty f(s,u)du.$$

Write

$$\int_0^\infty f(s,u)du = \int_0^1 f(s,u)du + \int_1^\infty f(s,u)du.$$

Now, by (1), for $0<u<2\pi$,

$$\begin{aligned} f(s,u) &= u^{s-1}e^{-\alpha u}\frac{1}{1-e^{-u}} \\ &= u^{s-2}\cdot\frac{-ue^{-\alpha u}}{e^{-u}-1} \\ &= u^{s-2}\sum_{k=0}^\infty B_k(\alpha)\frac{(-u)^k}{k!} \\ &= \sum_{k=0}^\infty (-1)^k B_k(\alpha)\frac{u^{k+s-2}}{k!}. \end{aligned}$$

For $k\geq 2$, real $s>1$, and $0<u<2\pi$, by (2),

$$\left|B_k(\alpha)\frac{u^{k+s-2}}{k!}\right| \leq \frac{2\zeta(k)k!}{(2\pi)^k}\cdot u^{k+s-2}\cdot\frac{1}{k!} = 2\zeta(k)\left(\frac{u}{2\pi}\right)^k u^{s-2},$$

which is summable, and thus by the dominated convergence theorem,

$$\begin{aligned} \int_0^1 f(s,u)du &= \int_0^1 \sum_{k=0}^\infty (-1)^k B_k(\alpha)\frac{u^{k+s-2}}{k!}du \\ &= \sum_{k=0}^\infty (-1)^k B_k(\alpha)\frac{1}{k!}\frac{1}{k+s-1}. \end{aligned}$$

Check that $s\mapsto \sum_{k=0}^\infty (-1)^k B_k(\alpha)\frac{1}{k!}\frac{1}{k+s-1}$ is meromorphic on $\mathbb{C}$, with poles of order 0 or 1 at $s=-k+1$, $k\geq 0$ (the order of the pole is 0 if $B_k(\alpha)=0$), at which the residue is $(-1)^k B_k(\alpha)\frac{1}{k!}$.[5] On the other hand, check that $s\mapsto \int_1^\infty f(s,u)du$ is entire. Therefore $\zeta(s,\alpha)\Gamma(s)$ is meromorphic on $\mathbb{C}$, with poles of order 0 or 1 at $s=-k+1$, $k\geq 0$ and the residue of $\zeta(s,\alpha)\Gamma(s)$ at $s=-k+1$ is $(-1)^k B_k(\alpha)\frac{1}{k!}$. But it is a fact that $\Gamma(s)$ has poles of order 1 at $s=-n$, $n\geq 0$, with residue $\frac{(-1)^n}{n!}$. Hence the only pole of $\zeta(s,\alpha)$ is at $s=1$, at which the residue is 1.

[5] Kazuya Kato, Nobushige Kurokawa, and Takeshi Saito, *Number Theory 1: Fermat's Dream*, p. 96.

Theorem 1. For $n \geq 1$ and for $0 < \alpha \leq 1$,

$$\zeta(1-n,\alpha) = -\frac{B_n(\alpha)}{n}.$$

Proof. For $n \geq 1$, because $\zeta(s,\alpha)$ does not have a pole at $s = 1-n$ and because $\Gamma(s)$ has a pole of order 1 at $s = 1-n$ with residue $\frac{(-1)^{n-1}}{(n-1)!}$,

$$\begin{aligned}
\lim_{s\to 1-n}(s-(1-n))\Gamma(s)\zeta(s,\alpha) &= \zeta(1-n,\alpha)\cdot \lim_{s\to 1-n}(s-(1-n))\Gamma(s)\\
&= \zeta(1-n,\alpha)\cdot \mathrm{Res}_{s=1-n}\Gamma(s)\\
&= \zeta(1-n,\alpha)\cdot \frac{(-1)^{n-1}}{(n-1)!}.
\end{aligned}$$

On the other hand, $\zeta(s,\alpha)\Gamma(s)$ has a pole of order 1 at $s = 1-n$ with residue $(-1)^n B_n(\alpha)\frac{1}{n!}$. Therefore

$$\zeta(1-n,\alpha)\cdot\frac{(-1)^{n-1}}{(n-1)!} = (-1)^n B_n(\alpha)\frac{1}{n!},$$

i.e. for $n \geq 1$ and $0 < \alpha \leq 1$,

$$\zeta(1-n,\alpha) = -\frac{B_n(\alpha)}{n}.$$

□

5 Sobolev spaces

For real $s \geq 0$, we define the **Sobolev space** $H^s(\mathbb{T})$ as the set of those $f \in L^2(\mathbb{T})$ such that

$$|\widehat{f}(0)|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{f}(n)|^2 |n|^{2s} < \infty.$$

For $f, g \in H^s(\mathbb{T})$, define

$$\langle f, g\rangle_{H^s(\mathbb{T})} = \widehat{f}(0)\overline{\widehat{g}(0)} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \widehat{f}(n)\overline{\widehat{g}(n)}|n|^{2s}.$$

This is an inner product, with which $H^s(\mathbb{T})$ is a Hilbert space.[6]

[6] See `http://www.math.umn.edu/~garrett/m/mfms/notes/09_sobolev.pdf`

For $c \in \mathbb{C}^{\mathbb{Z}}$, if $s > r + \frac{1}{2}$,

$$\begin{aligned}
&\left\| \sum_{|n| \leq N} c_n e^{2\pi i n x} \right\|_{C^r(\mathbb{T})} \\
&= \sup_{0 \leq j \leq r} \sup_{x \in \mathbb{T}} \left| \sum_{|n| \leq N} c_n (2\pi i n)^j e^{2\pi i n x} \right| \\
&\leq |c_0|^2 + \sup_{0 \leq j \leq r} \sup_{x \in \mathbb{T}} \left| \sum_{1 \leq |n| \leq N} c_n (2\pi i n)^j e^{2\pi i n x} \right| \\
&\leq |c_0|^2 + (2\pi)^r \sum_{1 \leq |n| \leq N} |c_n| |n|^r \\
&= |c_0|^2 + (2\pi)^r \sum_{1 \leq |n| \leq N} |c_n| |n|^s |n|^{-(r-s)} \\
&\leq |c_0|^2 + (2\pi)^r \left(\sum_{1 \leq |n| \leq N} |c_n|^2 |n|^{2s} \right)^{1/2} \left(\sum_{1 \leq |n| \leq N} |n|^{-(2s-2r)} \right)^{1/2} \\
&\leq |c_0|^2 + (2\pi)^r \cdot (2 \cdot \zeta(2s - 2r))^{1/2} \cdot \left(\sum_{1 \leq |n| \leq N} |c_n|^2 |n|^{2s} \right)^{1/2}.
\end{aligned}$$

For $f \in H^s(\mathbb{T})$, the partial sums $\sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}$ are a Cauchy sequence in $H^s(\mathbb{T})$ and by the above are a Cauchy sequence in the Banach space $C^r(\mathbb{T})$ and so converge to some $g \in C^r(\mathbb{T})$. Then $\widehat{g} = \widehat{f}$, which implies that $g = f$ almost everywhere.

For $k \geq 1$, $\widehat{P}_k(0) = 0$ and $\widehat{P}_k(n) = -(2\pi i n)^{-k}$ for $n \neq 0$. For $k, l > s + \frac{1}{2}$,

$$\begin{aligned}
\langle P_k, P_l \rangle_{H^s(\mathbb{T})} &= \sum_{n \in \mathbb{Z} \setminus \{0\}} -(2\pi i n)^{-k} \overline{-(2\pi i n)^{-l}} \\
&= \sum_{n \in \mathbb{Z} \setminus \{0\}} i^{-k+l} (2\pi n)^{-k-l} \\
&= i^{-k+l} (2\pi)^{-k-l} \cdot 2 \cdot \zeta(k + l).
\end{aligned}$$

Thus if $k > s + \frac{1}{2}$ then $P_k \in H^s(\mathbb{T})$, and in particular $P_k \in H^{k-1}(\mathbb{T})$ for $k \geq 1$.

For $s > r + \frac{1}{2}$, if $f \in H^s(\mathbb{T})$ then there is some $g \in C^r(\mathbb{T})$ such that $g = f$ almost everywhere. Thus if $r + \frac{1}{2} < s < k - \frac{1}{2}$, i.e. $k > r + 1$, then there is some $g \in C^r(\mathbb{T})$ such that $g = P_k$ almost everywhere. But for $k \neq 1$, P_k is continuous, so in fact $g = P_k$. In particular, $P_k \in C^{k-2}(\mathbb{T})$ for $k \geq 2$.

6 Reproducing kernel Hilbert spaces

For $x \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{C}$, define $(\tau_x f)(y) = f(y - x)$. We calculate

$$\begin{aligned}
\widehat{\tau_x f}(n) &= \int_{\mathbb{T}} f(y-x) e^{-2\pi i n y} dy \\
&= e^{-2\pi i n x} \int_{\mathbb{T}} f(y) e^{-2\pi i n y} dy \\
&= e^{-2\pi i n x} \widehat{f}(n).
\end{aligned}$$

Let $r \geq 1$. For $x \in \mathbb{T}$, define $F_x : \mathbb{T} \to \mathbb{R}$ by

$$F_x = 1 + (-1)^{r-1} (2\pi)^{2r} \tau_x P_{2r}.$$

For $n \in \mathbb{Z}$,

$$\widehat{F_x}(n) = \delta_0(n) + (-1)^{r-1} (2\pi)^{2r} \cdot e^{-2\pi i n x} \widehat{P}_{2r}(n).$$

$\widehat{F_x}(0) = 1$, and for $n \neq 0$,

$$\widehat{F_x}(n) = (-1)^{r-1} (2\pi)^{2r} \cdot e^{-2\pi i n x} \cdot -(2\pi i n)^{-2r} = |n|^{-2r} e^{-2\pi i n x}.$$

For $f \in H^r(\mathbb{T})$,

$$\begin{aligned}
\langle f, F_x \rangle_{H^r(\mathbb{T})} &= \widehat{f}(0) \overline{\widehat{F_x}(0)} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \widehat{f}(n) \overline{\widehat{F_x}(n)} |n|^{2r} \\
&= \widehat{f}(0) + \sum_{n \in \mathbb{Z} \setminus \{0\}} \widehat{f}(n) |n|^{-2r} e^{2\pi i n x} |n|^{2r} \\
&= \widehat{f}(0) + \sum_{n \in \mathbb{Z} \setminus \{0\}} \widehat{f}(n) e^{2\pi i n x} \\
&= f(x).
\end{aligned}$$

This shows that $H^r(\mathbb{T})$ is a **reproducing kernel Hilbert space**.

Define $F : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ by

$$\begin{aligned}
F(x, y) &= \langle F_x, F_y \rangle_{H^r(\mathbb{T})} \\
&= F_x(y) \\
&= 1 + (-1)^{r-1} (2\pi)^{2r} P_{2r}(y - x).
\end{aligned}$$

Thus the **reproducing kernel** of $H^r(\mathbb{T})$ is[7]

$$F(x, y) = 1 + (-1)^{r-1} (2\pi)^{2r} P_{2r}(y - x).$$

[7] cf. Alain Berlinet and Christine Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, p. 318, who use a different inner product on $H^r(\mathbb{T})$ and consequently have a different expression for the reproducing kernel.