L^0, convergence in measure, equi-integrability, the Vitali convergence theorem, and the de la Vallée-Poussin criterion

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1 Measurable spaces

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, with the order topology. We assign $\mathbb{R}$ the Borel σ-algebra. It is a fact that for $E \subset \overline{\mathbb{R}}$, $E \in \mathscr{B}_{\overline{\mathbb{R}}}$ if and only if $E \setminus \{-\infty, \infty\} \in \mathscr{B}_{\mathbb{R}}$.

Theorem 1. Let (Ω, Σ) be a measurable space. If f_j is a sequence of measurable functions $\Omega \to \overline{\mathbb{R}}$, then for each k,

$$g_k(x) = \sup_{j \geq k} f_j(x), \qquad h_k(x) = \inf_{j \geq k} f_j(x),$$

are measurable $\Omega \to \overline{\mathbb{R}}$, and

$$g(x) = \limsup_{j \to \infty} f_j(x), \qquad h(x) = \liminf_{j \to \infty} f_j(x),$$

are measurable $\Omega \to \overline{\mathbb{R}}$.

Proof. Let $a \in \mathbb{R}$. For each j, $f_j^{-1}(a, \infty] \in \Sigma$, so

$$\bigcup_{j=k}^{\infty} f_j^{-1}(a, \infty] \in \Sigma.$$

For each $j \geq k$,

$$f_j^{-1}(a, \infty] \subset \left\{ x \in \Omega : \sup_{i \geq k} f_i(x) > a \right\},$$

so

$$\bigcup_{j=k}^{\infty} f_j^{-1}(a, \infty] \subset \left\{ x \in \Omega : \sup_{i \geq k} f_i(x) > a \right\}.$$

If $y \notin \bigcup_{j=k}^{\infty} f_j^{-1}(a, \infty]$, then for each $j \geq k$, $f_j(y) \leq a$, hence $g_k(y) \leq a$, which means that $y \notin \{x \in \Omega : g_k(x) > a\}$. Therefore,

$$\bigcup_{j=k}^{\infty} f_j^{-1}(a, \infty] = \{x \in \Omega : g_k(x) > a\},$$

and thus $g_k^{-1}(a,\infty] \in \Sigma$. Because this is true for all $a \in \mathbb{R}$ and $\mathscr{B}_{\overline{\mathbb{R}}}$ is generated by the collection $\{(a,\infty] : a \in \mathbb{R}\}$, it follows that $g_k : \Omega \to \overline{\mathbb{R}}$ is measurable.

That h_k is measurable follows from the fact that if $f : \Omega \to \overline{\mathbb{R}}$ is measurable then $-f : \Omega \to \overline{\mathbb{R}}$ is measurable, and that $h_k(x) = \inf_{j \geq k} f_j(x) = -\sup_{j\geq k}(-f_j(x))$.

For $x \in \Omega$,

$$g(x) = \inf_{k\geq 1} g_k(x),$$

and because each g_k is measurable it follows that g is measurable. Likewise,

$$h(x) = \sup_{k\geq 1} h_k(x),$$

and because each h_k is measurable it follows that h is measurable. □

2 Convergence in measure

Let (Ω, Σ, μ) be a probability space. Let $L^0(\mu)$ be the collection of equivalence classes of measurable functions $\Omega \to \mathbb{C}$, where $\mathbb{C}$ has the Borel σ-algebra, and where two functions f and g are equivalent when

$$\mu\{x \in \Omega : f(x) \neq g(x)\} = 0.$$

$L^0(\mu)$ is a vector space. For $f, g \in L^0(\mu)$ we define

$$\rho(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|} d\mu.$$

This is a metric on $L^0(\mu)$, and one proves that with this metric $L^0(\mu)$ is a topological vector space. We call the topology induced by ρ the **topology of convergence in measure**.[1]

Theorem 2. Suppose that f_n is a sequence in $L^0(\mu)$. $f_n \to 0$ in the topology of convergence in measure if and only if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{x \in \Omega : |f_n(x)| \geq \epsilon\}\right) = 0.$$

Proof. Suppose that $f_n \to 0$ in the topology of convergence in measure and let $\epsilon > 0$. For each n, let

$$A_n = \{x \in \Omega : |f_n(x)| \geq \epsilon\} = \left\{x \in \Omega : \frac{|f_n(x)|}{1+|f_n(x)|} \geq \frac{\epsilon}{1+\epsilon}\right\}.$$

Because $\frac{\epsilon}{1+\epsilon}\chi_{A_n} \leq \frac{|f_n|}{1+|f_n|}$,

$$\int_\Omega \frac{\epsilon}{1+\epsilon}\chi_{A_n} d\mu \leq \int_\Omega \frac{|f_n|}{1+|f_n|} d\mu = \rho(f_n, 0),$$

[1] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 480, Lemma 13.40.

i.e. $\mu(A_n) \leq \frac{1+\epsilon}{\epsilon}\rho(f_n, 0)$, which tends to 0 as $n \to \infty$.

Let $\epsilon > 0$ and for each n, let

$$A_n = \{x \in \Omega : |f_n(x)| \geq \epsilon\} = \left\{x \in \Omega : \frac{|f_n(x)|}{1 + |f_n(x)|} \geq \frac{\epsilon}{1+\epsilon}\right\}.$$

Suppose that $\mu(A_n) \to 0$ as $n \to \infty$. There is some n_ϵ such that $n \geq n_\epsilon$ implies that $\mu(A_n) < \epsilon$. For $n \geq n_\epsilon$,

$$\begin{aligned}
\rho(f_n, 0) &= \int_{A_n} \frac{|f_n|}{1+|f_n|} d\mu + \int_{\Omega \setminus A_n} \frac{|f_n|}{1+|f_n|} d\mu \\
&\leq \int_{A_n} 1 \, d\mu + \int_{\Omega \setminus A_n} \frac{\epsilon}{1+\epsilon} d\mu \\
&= \frac{1+\epsilon}{1+\epsilon}\mu(A_n) + \frac{\epsilon}{1+\epsilon}\mu(\Omega \setminus A_n) \\
&= \frac{1}{1+\epsilon}\mu(A_n) + \frac{\epsilon}{1+\epsilon}(\mu(A_n) + \mu(\Omega \setminus A_n)) \\
&= \frac{1}{1+\epsilon}\mu(A_n) + \frac{\epsilon}{1+\epsilon} \\
&< \frac{1}{1+\epsilon} \cdot \epsilon + \frac{\epsilon}{1+\epsilon} \\
&< 2\epsilon.
\end{aligned}$$

This shows that $f_n \to 0$ in the topology of convergence in measure. □

We now prove that if a sequence in $L^0(\mu)$ converges almost everywhere to 0 then it converges in measure to 0.[2]

Theorem 3. Suppose that f_n is a sequence in $L^0(\mu)$ and that for almost all $x \in \Omega$, $f_n(x) \to 0$ as $n \to \infty$. Then $f_n \to 0$ in the topology of convergence in measure.

Proof. Let $\epsilon > 0$ and let $\eta > 0$. Egorov's theorem tells us that there is some $E \in \Sigma$ with $\mu(E) < \eta$ such that $f_n \to 0$ uniformly on $\Omega \setminus E$. So there is some n_0 such that if $n \geq n_0$ and $x \in \Omega \setminus E$ then $|f_n(x)| < \epsilon$. Thus for $n \geq n_0$,

$$\begin{aligned}
\mu(\{x \in \Omega : |f_n(x)| \geq \epsilon\}) &\leq \mu(E) + \mu(\{x \in \Omega \setminus E : |f_n(x)| \geq \epsilon\}) \\
&= \mu(E) \\
&< \eta.
\end{aligned}$$

Then $\mu(\{x \in \Omega : |f_n(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$, namely, $f_n \to 0$ in measure. □

Theorem 4. If f_n is a sequence in $L^1(\mu)$ that converges in $L^1(\mu)$ to 0, then f_n converges in measure to 0.

[2] Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 479, Theorem 13.37.

Proof. Let $\epsilon > 0$ and let $A_n = \{x \in \Omega : |f_n(x)| \geq \epsilon\}$. By Chebyshev's inequality,

$$\mu(A_n) \leq \frac{1}{\epsilon} \|f_n\|_1 ,$$

hence $\mu(A_n) \to 0$ as $n \to \infty$, namely, f_n converges to 0 in measure. □

The following theorem shows that a sequence in $L^0(\mu)$ that converges in measure to 0 then it has a subsequence that almost everywhere converges to 0.[3]

Theorem 5. Suppose that f_n is a sequence in $L^0(\mu)$ that converges in measure to 0. Then there is a subsequence $f_{a(n)}$ of f_n such that for almost all $x \in \Omega$, $f_{a(n)}(x) \to 0$.

Proof. For each n, with $\epsilon = \frac{1}{n}$, there is some $a(n)$ such that $m \geq a(n)$ implies that

$$\mu\left(\left\{x \in \Omega : |f_m(x)| \geq \frac{1}{n}\right\}\right) < \frac{1}{2^n}.$$

For each n, let

$$E_n = \left\{x \in \Omega : |f_{a(n)}(x)| \geq \frac{1}{n}\right\},$$

for which $\mu(E_n) < \frac{1}{2^n}$. Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=n} E_m,$$

and for each n,

$$\mu(E) \leq \mu\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \sum_{m=n}^{\infty} \mu(E_m) < \sum_{m=n}^{\infty} \frac{1}{2^m} = \frac{1}{2^n} \cdot 2 = 2^{1-n}.$$

Because this is true for all n, $\mu(E) = 0$. If $x \in \Omega \setminus E$, then there is some n_x such that $x \notin \bigcup_{m=n_x}^{\infty} E_m$. This means that for $m \geq n_x$ we have $x \notin E_m$, i.e. $|f_{a(m)}(x)| < \frac{1}{m}$. This implies that for $x \notin E$, $f_{a(n)}(x) \to 0$ as $n \to \infty$, showing that for almost all $x \in \Omega$, $f_{a(n)}(x) \to 0$ as $n \to \infty$. □

We now prove that ρ is a complete metric, namely that $L^0(\mu)$ with this metric is an F-**space**.[4]

Theorem 6. ρ is a complete metric on $L^0(\mu)$.

[3]Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 479, Theorem 13.38.

[4]Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., p. 481, Theorem 13.41; Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, second ed., p. 61, Theorem 2.30.

Proof. Suppose that f_n is a Cauchy sequence in $L^0(\mu)$. To prove that f_n is convergent it suffices to prove that f_n has a convergent subsequence. If (X,d) is a metric space and x_n is a Cauchy sequence in X, for any N let a_N be such that $n,m \geq a_N$ implies that $d(x_n,x_m) < \frac{1}{N}$ and a fortiori $d(x_n,x_m) < \frac{1}{a_N}$. Thus we presume that f_n itself satisfies $\rho(f_n,f_m) < \frac{1}{n}$ for $m \geq n$.

Let

$$A_{k,m}(\epsilon) = \{x \in \Omega : |f_k(x) - f_m(x)| \geq \epsilon\} = \left\{ x \in \Omega : \frac{|f_k(x) - f_m(x)|}{1 + |f_k(x) - f_m(x)|} \geq \frac{\epsilon}{1+\epsilon} \right\},$$

for which

$$\frac{\epsilon}{1+\epsilon}\chi_{A_{k,m}(\epsilon)} \leq \frac{|f_k(x) - f_m(x)|}{1 + |f_k(x) - f_m(x)|}.$$

If $m \geq k$,

$$\mu(A_{k,m}(\epsilon)) \leq \frac{1+\epsilon}{\epsilon} \int_\Omega \frac{|f_k - f_m|}{1 + |f_k - f_m|} d\mu = \frac{1+\epsilon}{\epsilon} \rho(f_k, f_m) < \frac{1+\epsilon}{\epsilon}\frac{1}{k}. \tag{1}$$

For $n = 1$ and $\epsilon_1 = \frac{1}{2^1}$, let k_1 be such that $\frac{1+\epsilon_1}{\epsilon_1}\frac{1}{k_1} \leq \frac{1}{2^1}$, i.e. $k_1 \geq 2^1 \cdot \frac{1+\epsilon_1}{\epsilon_1}$. For $n \geq 1$ and $\epsilon_n = \frac{1}{2^n}$, assume that k_n satisfies $\frac{1+\epsilon_n}{\epsilon_n}\frac{1}{k_n} \leq \frac{1}{2^n}$ and $k_n > k_{n-1}$. For $\epsilon_{n+1} = \frac{1}{2^{n+1}}$, let k_{n+1} be such that $\frac{1+\epsilon_{n+1}}{\epsilon_{n+1}}\frac{1}{k_{n+1}} \leq \frac{1}{2^{n+1}}$ and $k_{n+1} > k_n$. For any n we have, because $k_n \geq n$, $\frac{1+\epsilon_n}{\epsilon_n}\frac{1}{k_n} \leq \frac{1}{2^{k_n}}$. Then using (1) with $m \geq k_n$,

$$\mu\left(A_{k_n,m}\left(\frac{1}{2^n}\right)\right) < \frac{1+\epsilon_n}{\epsilon_n}\frac{1}{k_n} \leq \frac{1}{2^{k_n}}. \tag{2}$$

Let $g_n = f_{k_n}$ and let

$$E_n = \left\{ x \in \Omega : |g_{n+1}(x) - g_n(x)| \geq \frac{1}{2^n} \right\},$$

for which, by (2), $\mu(E_n) < \frac{1}{2^n}$. Let

$$F_n = \bigcup_{r=n}^\infty E_n,$$

which satisfies

$$\mu(F_n) \leq \sum_{r=n}^\infty \mu(E_n) < \sum_{r=n}^\infty 2^{-r} = 2^{-n+1}.$$

Hence $F = \bigcap_{n=1}^\infty F_n$ satisfies $\mu(F) = 0$.

If $x \notin F$, then there is some n for which $x \notin F_n$, i.e. for each $r \geq n$ we have $x \notin E_r$, i.e. for each $r \geq n$ we have $|g_{r+1}(x) - g_r(x)| < 2^{-r}$. This implies that for $k \geq n$ and for any positive integer p,

$$\begin{aligned}
|g_{k+p}(x) - g_k(x)| &\leq |g_{k+p}(x) - g_{k+p-1}(x)| + \cdots + |g_{k+1}(x) - g_k(x)| \\
&< 2^{-k-p+1} + \cdots + 2^{-k} \\
&< 2^{-k+1},
\end{aligned}$$

so if $j \geq k$ then

$$|g_j(x) - g_k(x)| < 2^{-k+1}. \tag{3}$$

This shows that if $x \notin F$ then $g_k(x)$ is a Cauchy sequence in $\mathbb{C}$, and hence converges. We define $g : \Omega \to \mathbb{C}$ by

$$g(x) = \chi_{\Omega \setminus F}(x) \limsup_{k \to \infty} g_k(x), \qquad x \in \Omega,$$

and by Theorem 1, $g \in L^0(\mu)$. For $x \notin F_k$ we have $x \notin F$ and so $g_l(x) \to g(x)$ as $l \to \infty$. Then for $x \notin F_k$ and $j \geq k$, using (3) we have

$$|g_j(x) - g(x)| \leq |g_j(x) - g_l(x)| + |g_l(x) - g(x)| < 2^{-k+1} + |g_l(x) - g(x)| \to 2^{-k+1}$$

as $l \to \infty$, so $|g_j(x) - g(x)| < 2^{-k+1}$. For $\epsilon > 0$, let k be such that $2^{-k+1} \leq \epsilon$. Then

$$\begin{aligned}
&\mu(\{x \in \Omega : |g_k(x) - g(x)| \geq \epsilon\}) \\
\leq &\mu(F_k) + \mu(\{x \in \Omega \setminus F_k : |g_k(x) - g(x)| \geq \epsilon\}) \\
< &2^{-k+1} + \mu(\{x \in \Omega \setminus F_k : |g_k(x) - g(x)| \geq 2^{-k+1}\}) \\
= &2^{-k+1},
\end{aligned}$$

which tends to 0 as $k \to \infty$, showing that g_k converges to g in measure, and because g_k is a subsequence of f_k this completes the proof. □

3 Equi-integrability

Let (Ω, Σ, μ) be a probability space. A subset $\mathscr{F}$ of $L^1(\mu)$ is said to be **equi-integrable** if for every $\epsilon > 0$ there is some $\delta > 0$ such that for all $E \in \Sigma$ with $\mu(E) \leq \delta$ and for all $f \in \mathscr{F}$,

$$\int_E |f| d\mu \leq \epsilon.$$

In other words, to say that $\mathscr{F}$ is equi-integrable means that

$$\lim_{\mu(E) \to 0} \sup_{f \in \mathscr{F}} \int_E |f| d\mu = 0.$$

The following theorem gives an equivalent condition for a bounded subset of $L^1(\mu)$ to be equi-integrable.[5]

Theorem 7. Suppose that $\mathscr{F}$ is a bounded subset of $L^1(\mu)$. Then the following are equivalent:

1. $\mathscr{F}$ is equi-integrable.
2. $\lim_{C \to \infty} \sup_{f \in \mathscr{F}} \int_{\{|f| > C\}} |f| d\mu = 0$.

[5] Fernando Albiac and Nigel J. Kalton, *Topics in Banach Space Theory*, p. 105, Lemma 5.2.6.

Proof. Let $K = \sup_{f \in \mathscr{F}} \|f\|_1 < \infty$ and suppose that $\mathscr{F}$ is equi-integrable. For $f \in \mathscr{F}$, Chebyshev's inequality tells us

$$\mu(\{|f| > M\}) \leq \frac{\|f\|_1}{M} \leq \frac{K}{M}.$$

Because $\mu(\{|f| > M\}) \to 0$ as $M \to \infty$ and $\mathscr{F}$ is equi-integrable,

$$\lim_{M \to \infty} \sup_{f \in \mathscr{F}} \int_{\{|f|>M\}} |f| d\mu = 0.$$

Suppose now that

$$\lim_{M \to \infty} \sup_{f \in \mathscr{F}} \int_{\{|f|>M\}} |f| d\mu = 0. \tag{4}$$

For $E \in \Sigma$ and $f \in \mathscr{F}$, if $M > 0$ then

$$\begin{aligned} \int_E |f| d\mu &= \int_{E \cap \{|f| \leq M\}} |f| d\mu + \int_{E \cap \{|f|>M\}} |f| d\mu \\ &\leq M\mu(E) + \int_{\{|f|>M\}} |f| d\mu \\ &\leq M\mu(E) + \sup_{g \in \mathscr{F}} \int_{\{|g|>M\}} |g| d\mu, \end{aligned}$$

hence

$$\sup_{f \in \mathscr{F}} \int_E |f| d\mu \leq M\mu(E) + \sup_{g \in \mathscr{F}} \int_{\{|g|>M\}} |g| d\mu. \tag{5}$$

Let $\epsilon > 0$. By (4) there is some M such that $\sup_{g \in \mathscr{F}} \int_{\{|g|>M\}} |g| d\mu \leq \frac{\epsilon}{2}$. For $\delta = \frac{\epsilon}{2M}$, if $E \in \Sigma$ and $\mu(E) \leq \delta$ then (5) yields

$$\sup_{f \in \mathscr{F}} \int_E |f| d\mu \leq M\delta + \frac{\epsilon}{2} = \epsilon,$$

showing that $\mathscr{F}$ is equi-integrable. □

Theorem 8 (Absolute continuity of Lebesgue integral)**.** Suppose that $f \in L^1(\mu)$. If $\epsilon > 0$ then there is some $\delta > 0$ such that for any $E \in \Sigma$ with $\mu(E) \leq \delta$,

$$\int_E |f| d\mu \leq \epsilon.$$

Proof. For $n \geq 1$, define

$$g_n(x) = \min\{|f(x)|, n\}, \qquad x \in \Omega.$$

Then g_n is a sequence in $L^1(\mu)$ such that for each $x \in \Omega$, $g_n(x)$ is nondecreasing and $g_n(x) \to |f(x)|$, and thus the monotone convergence theorem tells us that

$$\lim_{n\to\infty} \int_\Omega g_n d\mu = \int_\Omega |f| d\mu.$$

Then there is some N for which

$$0 \le \int_\Omega |f| d\mu - \int_\Omega g_N d\mu \le \frac{\epsilon}{2}.$$

For $\delta = \frac{\epsilon}{2N}$ and $E \in \Sigma$ with $\mu(E) \le \delta$,

$$\begin{aligned}\int_E |f| d\mu &= \int_E (|f| - g_N) d\mu + \int_E g_N d\mu \\ &\le \frac{\epsilon}{2} + \int_E g_N d\mu \\ &\le \frac{\epsilon}{2} + N\mu(E) \\ &\le \frac{\epsilon}{2} + N\delta \\ &= \epsilon.\end{aligned}$$

□

The following is the **Vitali convergence theorem**.[6]

Theorem 9 (Vitali convergence theorem)**.** Suppose that $f \in L^0(\mu)$ and f_n is a sequence in $L^1(\mu)$. Then the following are equivalent:

1. $\{f_n\}$ is equi-integrable, $\{f_n\}$ is bounded in $L^1(\mu)$, and $f_n \to f$ in measure.
2. $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$.

Proof. Suppose that $\{f_n\}$ is equi-integrable and $f_n \to f$ in measure. Because $f_n \to f$ in measure, there is a subsequence $f_{a(n)}$ of f_n that converges almost everywhere to f and so $|f_{a(n)}|$ converges almost everywhere to $|f|$. Let $K = \sup_{n\ge1} \|f_{a(n)}\|_1 < \infty$. Fatou's lemma tells us that $|f| \in L^1(\mu)$ and

$$\|f\|_1 \le \liminf_{n\to\infty} \|f_{a(n)}\|_1 \le K.$$

Because $f \in L^0(\mu)$ and $|f| \in L^1(\mu)$, $f \in L^1(\mu)$. To show that f_n converges to f in $L^1(\mu)$, it suffices to show that any subsequence of f_n itself has a subsequence that converges to f in $L^1(\mu)$. (Generally, a sequence in a topological space converges to x if and only if any subsequence itself has a subsequence that converges to x.) Thus, let g_n be a subsequence of f_n. Because f_n converges to f in measure, the subsequence g_n converges to f in measure and so there is

[6] V. I. Bogachev, *Measure Theory*, volume I, p. 268, Theorem 4.5.4.

a subsequence $g_{a(n)}$ of g_n that converges almost everywhere to f. Let $\epsilon > 0$. Because $\{f_n\}$ is equi-integrable, there is some $\delta > 0$ such that for all $E \in \Sigma$ with $\mu(E) \leq \delta$ and for all n,

$$\int_E |g_{a(n)}| d\mu \leq \epsilon.$$

If $E \in \Sigma$ with $\mu(E) \leq \delta$, then $\chi_E g_{a(n)}$ converges almost everywhere to $\chi_E f$, and $\sup_{n\geq 1} \left\|\chi_E g_{a(n)}\right\|_1 \leq \epsilon$, so by Fatou's lemma we obtain

$$\int_E |f| d\mu = \|\chi_E f\|_1 \leq \liminf_{n\to\infty} \left\|\chi_E g_{a(n)}\right\|_1 \leq \epsilon.$$

But because $g_{a(n)}$ converges almost everywhere to f, by Egorov's theorem there is some $E \in \Sigma$ with $\mu(E) \leq \delta$ such that $g_{a(n)} \to f$ uniformly on $\Omega \setminus E$, and so there is some n_0 such that if $n \geq n_0$ and $x \in \Omega \setminus E$ then $|g_{a(n)}(x) - f(x)| \leq \epsilon$. Thus for $n \geq n_0$,

$$\begin{aligned}
\int_\Omega |g_{a(n)} - f| d\mu &= \int_{\Omega\setminus E} |g_{a(n)} - f| d\mu + \int_E |g_{a(n)} - f| d\mu \\
&\leq \mu(\Omega \setminus E)\epsilon + \int_E |g_{a(n)}| d\mu + \int_E |f| d\mu \\
&\leq \epsilon + \epsilon + \epsilon,
\end{aligned}$$

which shows that $g_{a(n)} \to f$ in $L^1(\mu)$. That is, we have shown that for any subsequence g_n of f_n there is a subsequence $g_{a(n)}$ of g_n that converges to f in $L^1(\mu)$, which implies that the sequence f_n converges to f in $L^1(\mu)$.

Suppose that $f \in L^1(\mu)$ and $f_n \to f$ in $L^1(\mu)$. First, because the sequence f_n is convergent in $L^1(\mu)$ the set $\{f_n\}$ is bounded in $L^1(\mu)$. Second, $f_n \to f$ in $L^1(\mu)$ implies that $f_n \to f$ in measure. Third, for $\epsilon > 0$, let n_0 such that $n \geq n_0$ implies that $\|f_n - f\|_1 \leq \epsilon$. For each $1 \leq n < n_0$, by Theorem 8 there is some $\delta_{f_n} > 0$ such that for $E \in \Sigma$ and $\mu(E) \leq \delta_{f_n}$,

$$\int_E |f_n| d\mu \leq \epsilon,$$

and likewise there is some δ_f such that for $E \in \Sigma$ and $\mu(E) \leq f$,

$$\int_E |f| d\mu \leq \epsilon.$$

Let $\delta > 0$ be the minimum of $\delta_{f_1}, \ldots, \delta_{f_{n-1}}, \delta_f$. Thus if $E \in \Sigma$ and $\mu(E) \leq \delta$, then for $1 \leq n < n$,

$$\int_E |f_n| d\mu \leq \epsilon,$$

and for for $n \geq n_0$,

$$\int_E |f_n| d\mu \leq \int_E |f_n - f| d\mu + \int_E |f| d\mu \leq \|f_n - f\|_1 + \int_E |f| d\mu \leq \epsilon + \epsilon.$$

This shows that $\{f_n\}$ is equi-integrable, completing the proof. $\square$

The following is the **de la Vallée-Poussin criterion for equi-integrability**.[7]

Theorem 10 (de la Vallée-Poussin criterion)**.** Suppose that $\mathscr{F} \subset L^1(\mu)$. $\mathscr{F}$ is bounded and equi-integrable if and only if there is a there nonnegative nondecreasing function G on $[0, \infty)$ such that

$$\lim_{t \to \infty} \frac{G(t)}{t} = \infty \quad \text{and} \quad \sup_{f \in \mathscr{F}} \int_\Omega G(|f(x)|) d\mu(x) < \infty, \tag{6}$$

and if there is a nonnegative nondecreasing function G satisfying (6) then there is a convex nonnegative nondecreasing function G satisfying (6).

Proof. Suppose that G is a nonnegative nondecreasing function on $[0, \infty)$ satisfying (6). Let

$$\sup_{f \in \mathscr{F}} \int_\Omega G(|f(x)|) d\mu(x) \le M < \infty.$$

For $\epsilon > 0$, there is some C such that $t \ge C$ implies that $\frac{G(t)}{t} \ge \frac{M}{\epsilon}$, and hence, for $f \in \mathscr{F}$, if $x \in \Omega$ and $|f(x)| \ge C$ then $\frac{G(|f(x)|)}{|f(x)|} \ge \frac{M}{\epsilon}$, i.e. $|f(x)| \le \frac{\epsilon}{M} G(|f(x)|)$, which yields

$$\int_{\{|f| \ge C\}} |f| d\mu \le \int_{\{|f| \ge C\}} \frac{\epsilon}{M} G(|f(x)|) d\mu(x) \le \frac{\epsilon}{M} \cdot M = \epsilon.$$

Therefore by Theorem 7, $\mathscr{F}$ is bounded and equi-integrable.

Suppose that $\mathscr{F}$ is bounded and equi-integrable. For $f \in \mathscr{F}$ and $j \ge 1$, let

$$\mu_j(f) = \mu(\{x \in \Omega : |f(x)| > j\}) \in \Sigma.$$

By induction, because $\mathscr{F}$ is bounded and equi-integrable there is a strictly increasing sequence of positive integers C_n such that for each n,

$$\sup_{f \in \mathscr{F}} \int_{\{|f| > C_n\}} |f| d\mu \le 2^{-n}. \tag{7}$$

For $f \in \mathscr{F}$ and $n \ge 1$,

$$\begin{aligned}
\int_{\{|f| > C_n\}} |f| d\mu &= \sum_{j=C_n}^{\infty} \int_{\{j < |f| \le j+1\}} |f| d\mu \\
&\ge \sum_{j=C_n}^{\infty} j\mu(\{x \in \Omega : j < |f(x)| \le j+1\}) \\
&= \sum_{j=C_n}^{\infty} j(\mu_j(f) - \mu_{j+1}(f)) \\
&= \sum_{j=C_n}^{\infty} \mu_j(f).
\end{aligned}$$

[7] V. I. Bogachev, *Measure Theory*, volume I, p. 272, Theorem 4.5.9.

Using this and (7), for $f \in \mathscr{F}$,

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=C_n}^{\infty}\mu_j(f) &\leq \sum_{n=1}^{\infty}\int_{\{|f|>C_n\}}|f|d\mu\\ &\leq \sum_{n=1}^{\infty}2^{-n}\\ &= 1.\end{aligned}$$

For $n \geq 0$ we define

$$\alpha_n = \begin{cases} 0 & n < C_1 \\ \max\{k : C_k \leq n\} & n \geq C_1. \end{cases}$$

It is straightforward that $\alpha_n \to \infty$ as $n \to \infty$. We define a step function g on $[0, \infty)$ by

$$g(t) = \sum_{n=0}^{\infty}\alpha_n\chi_{(n,n+1]}(t), \qquad 0 \leq t < \infty,$$

and we define a function G on $[0, \infty)$ by

$$G(t) = \int_0^t g(s)ds, \qquad 0 \leq t < \infty.$$

It is apparent that G is nonnegative and nondecreasing. For $t_1, t_2 \in [0, \infty)$, $t_1 \leq t_2$, by the fundamental theorem of calculus,

$$G'(t_1)(t_2 - t_1) = g(t_1)(t_2 - t_1) \leq G(t_2) - G(t_1),$$

showing that G is convex. The above inequality also yields that for $t > 0$, $\frac{G(t)}{t} \geq \frac{g(t/2)}{2}$, and $g(t/2) \to \infty$ as $t \to \infty$ so we get that $\lim_{t\to\infty}\frac{G(t)}{t} = \infty$. For $f \in \mathscr{F}$, using $G(0) = 0$, $G(1) = 0$, and for $n \geq 1$,

$$G(n+1) \leq g(1) + g(2) + \cdots + g(n+1) = \alpha_0 + \alpha_1 + \cdots + \alpha_n = \alpha_1 + \cdots + \alpha_n,$$

we get

$$\begin{aligned}
\int_\Omega G(|f(x)|)d\mu(x) &= \int_{\{|f|=0\}} G(|f(x)|)d\mu(x) + \sum_{n=0}^{\infty} \int_{\{n<|f|\le n+1\}} G(|f(x)|)d\mu(x) \\
&\le \sum_{n=0}^{\infty} \int_{\{n<|f|\le n+1\}} G(n+1)d\mu(x) \\
&= \sum_{n=1}^{\infty} (\mu_n(f) - \mu_{n+1}(f))G(n+1) \\
&\le \sum_{n=1}^{\infty} (\mu_n(f) - \mu_{n+1}(f)) \sum_{j=1}^{n} \alpha_j \\
&= \sum_{n=1}^{\infty} \mu_n(f)\alpha_n \\
&= \sum_{n=1}^{\infty} \sum_{j=C_n}^{\infty} \mu_j(f) \\
&\le 1,
\end{aligned}$$

showing that $\sup_{f\in\mathscr{F}} \int_\Omega G(|f(x)|)d\mu(x) < \infty$, which completes the proof. □