C^k spaces and spaces of test functions

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April 10, 2014

1 Notation

Let $\mathbb{N}$ denote the set of nonnegative integers. For $\alpha \in \mathbb{N}^n$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We denote by $B_r(x)$ the open ball with center x and radius r.

2 Open sets

Let Ω be an open subset of $\mathbb{R}^n$ and let k be either a nonnegative integer or ∞. We define $C^k(\Omega)$ to be the set of those functions $f : \Omega \to \mathbb{C}$ such that for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, the derivative $\partial^\alpha f$ exists and is continuous. We write $C(\Omega) = C^0(\Omega)$.

One proves that there is a sequence of compact sets K_j such that each K_j is contained in the interior of K_{j+1} and $\Omega = \bigcup_{j=1}^\infty K_j$; we call this an *exhaustion of* Ω *by compact sets*. For $f \in C^k(\Omega)$, we define

$$p_{k,N}(f) = \sup_{|\alpha| \leq \min(k,N)} \sup_{x \in K_N} |(\partial^\alpha f)(x)|;$$

this definition makes sense for $k = \infty$. If f is a nonzero element of $C^k(\Omega)$, then there is some $x \in \Omega$ for which $f(x) \neq 0$ and then there is some N for which $x \in K_N$, and hence $p_{k,N}(f) \geq \sup_{y \in K_N} |f(y)| \geq |f(x)| > 0$. Thus, $p_{k,N}$ is a separating family of seminorms on $C^k(\Omega)$. Those sets of the form

$$V_{k,N} = \left\{ f \in C^k(\Omega) : p_{k,N}(f) < \frac{1}{N} \right\}$$

form a local basis at 0 for a topology on $C^k(\Omega)$, and because $p_{k,N}$ is a separating family of seminorms, with this topology $C^k(\Omega)$ is a locally convex space.[1]

[1]Walter Rudin, *Functional Analysis*, second ed., p. 27, Theorem 1.37.

Because $p_{k,N}$ is a countable separating family of seminorms, this topology is metrizable. We prove in the following theorem that $C(\Omega)$ is a Fréchet space.[2]

Theorem 1. *If Ω is an open subset of $\mathbb{R}^n$, then $C(\Omega)$ is a Fréchet space.*

Proof. Let $f_i \in C(\Omega)$ be a Cauchy sequence. That is, for every N there is some i_N such that if $i, j \geq i_N$ then

$$f_i - f_j \in V_{0,N} = \left\{ f \in C(\Omega) : \sup_{x \in K_N} |f(x)| < \frac{1}{N} \right\}.$$

For each $x \in \Omega$, eventually $x \in K_N$. If $x \in K_N$ and $i, j \geq i_N$, then

$$|f_i(x) - f_j(x)| < \frac{1}{N}.$$

Therefore, $f_i(x)$ is a Cauchy sequence in $\mathbb{C}$ and hence converges to some $f(x) \in \mathbb{C}$. We have thus defined a function $f : \Omega \to \mathbb{C}$. We shall prove that $f \in C(\Omega)$ and that $f_i \to f$ in $C(\Omega)$.

Let K be a compact subset of Ω, let $\epsilon > 0$, and let N be large enough both so that $K \subseteq K_N$ and so that $N \geq \frac{1}{\epsilon}$. For $i, j \geq i_N$,

$$\sup_{x \in K_N} |f_i(x) - f_j(x)| < \frac{1}{N} \leq \epsilon.$$

Let $i \geq i_N$ and $x \in K_N$. There is some j_x such that $j \geq j_x$ implies that $|f_j(x) - f(x)| < \epsilon$, and hence for $j \geq \max(i_N, j_x)$,

$$\begin{aligned} |f_i(x) - f(x)| &\leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| \\ &< \epsilon + \epsilon. \end{aligned}$$

This shows that for $i \geq i_N$,

$$\sup_{x \in K} |f_i(x) - f(x)| \leq \sup_{x \in K_N} |f_i(x) - f(x)| \leq 2\epsilon.$$

We have proved that for any compact subset K of Ω, we have $\sup_{x \in K} |f_i(x) - f(x)| \to 0$ as $i \to \infty$.

Let $x \in \Omega$, let $\epsilon > 0$, and let N be large enough both so that x lies in the interior of K_N and so that $N \geq \frac{1}{\epsilon}$. Because $\sup_{x \in K_N} |f_i(x) - f(x)| \to 0$ as $i \to \infty$, there is some i_0 so that $i \geq i_0$ implies

$$\sup_{x \in K_N} |f_i(x) - f(x)| < \epsilon.$$

Let $i = \max(i_0, i_N)$. Because f_i is continuous, there is some $\delta > 0$ so that $|x - y| < \delta$ implies that $|f_i(x) - f_i(y)| < \epsilon$; take δ small enough so that the open ball with center x and radius δ is contained in K_N. For $|y - x| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &\leq \sup_{z \in K_N} |f(z) - f_i(z)| + \frac{1}{N} + \sup_{z \in K_N} |f(z) - f_i(z)| \\ &< \epsilon + \epsilon + \epsilon. \end{aligned}$$

[2] Walter Rudin, *Functional Analysis*, second ed., p. 33, Example 1.44.

This shows that f is continuous at x and x was an arbitrary point in Ω, hence $f \in C(\Omega)$.

We have already established that for any compact subset K of Ω, we have $\sup_{x\in K} |f_i(x) - f(x)| \to 0$ as $i \to \infty$. Thus, for any N, there is some j_N so that if $i \geq j_N$ then $\sup_{x\in K_N} |f_i(x) - f(x)| < \frac{1}{N}$. In other words, if $i \geq j_N$, then $p_{0,N}(f_i - f) < \frac{1}{N}$, i.e. $f_i - f \in V_{0,N}$, showing that $f_i \to f$ in $C(\Omega)$. □

Theorem 2. *If Ω is an open subset of $\mathbb{R}^n$ and k is a positive integer, then $C^k(\Omega)$ is a Fréchet space.*

Proof. We have proved in Theorem 1 that $C(\Omega) = C^0(\Omega)$ is a Fréchet space. We assume that $C^{k-1}(\Omega)$ is a Fréchet space, and using this induction hypothesis we shall prove that $C^k(\Omega)$ is a Fréchet space.

Let $f_i \in C^k(\Omega)$ be a Cauchy sequence in $C^k(\Omega)$. f_i is in particular a Cauchy sequence in the Fréchet space $C(\Omega)$, hence there is some $g \in C(\Omega)$ such that $f_i \to g$ in $C(\Omega)$. We shall prove that $g \in C^k(\Omega)$ and that $f_i \to g$ in $C^k(\Omega)$.

For each $1 \leq p \leq n$ we have $\partial_p f_i \in C^{k-1}(\Omega)$, and $\partial_p f_i$ is a Cauchy sequence in $C^{k-1}(\Omega)$. Because $C^{k-1}(\Omega)$ is a Fréchet space, for each p there is some $g_p \in C^{k-1}(\Omega)$ such that $\partial_p f_i \to g_p$ in $C^{k-1}(\Omega)$. Fix p, and let $\alpha \in \mathbb{N}^n$ have pth entry 1 and all other entries 0. Then, fix $x \in \Omega$, and take N large enough so that x lies in the interior of K_N. For each i, define $F_i(t) = f(x + t\alpha)$, for which

$$F_i'(t) = (\nabla f)(x + t\alpha) \cdot \alpha = (\partial_p f_i)(x + t\alpha).$$

For nonzero τ small enough so that the line segment from x to $x+\tau\alpha$ is contained in K_N,

$$F_i(\tau) - F_i(0) = \int_0^\tau F_i'(t)dt,$$

i.e.

$$f_i(x + \tau\alpha) - f_i(x) = \int_0^\tau (\partial_p f_i)(x + t\alpha)dt.$$

Because $f_i \to g$ in $C(\Omega)$ and $\partial_p f_i \to g_p$ in $C(\Omega)$, we have $\sup_{y\in K_N} |f_i(y) - g(y)| \to 0$ and $\sup_{y\in K_N} |(\partial^p f_i)(y) - g_p(y)| \to 0$, from which it follows that

$$g(x + \tau\alpha) - g(x) = \int_0^\tau g_p(x + t\alpha)dt,$$

or

$$\frac{g(x + \tau\alpha) - g(x)}{\tau} = \frac{1}{\tau} \int_0^\tau g_p(x + t\alpha)dt.$$

As τ tends to 0, the right hand side tends to $g_\alpha(x)$, showing that $(\partial_p g)(x) = g_p(x)$. But x was an arbitrary point in Ω, so $\partial_p g = g_p \in C^{k-1}(\Omega)$. Thus, for each $1 \leq p \leq n$ we have $\partial_p g \in C^{k-1}(\Omega)$, from which it follows that $g \in C^k(\Omega)$. □

Theorem 3. *If Ω is an open subset of $\mathbb{R}^n$, then $C^\infty(\Omega)$ is a Fréchet space.*

Proof. Let $f_i \in C^\infty(\Omega)$ be a Cauchy sequence in $C^\infty(\Omega)$. Thus, for each k, f_i is a Cauchy sequence in $C^k(\Omega)$, and so by Theorem 2 there is some $g_k \in C^k(\Omega)$ for which $f_i \to g_k$ in $C^k(\Omega)$. Define $g = g_0$, and check that $g_0 = g_1 = g_2 = \cdots$, and hence that $g \in C^\infty(\Omega)$. □

3 Closed sets

Let Ω be an open subset of $\mathbb{R}^n$ such that $\overline{\Omega}$ is compact, i.e. Ω is a bounded open subset of $\mathbb{R}^n$. If k is a nonnegative integer, let $C^k(\overline{\Omega})$ be those elements f of $C^k(\Omega)$ such that for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, the function $\partial^\alpha f$ is continuous $\Omega \to \mathbb{C}$ and can be extended to a continuous function $\overline{\Omega} \to \mathbb{C}$; if there is such a continuous function $\overline{\Omega} \to \mathbb{C}$ it is unique, and it thus makes sense to talk about the value of $\partial^\alpha f$ at points in $\partial\Omega$, and thus to write $\partial^\alpha f : \overline{\Omega} \to \mathbb{C}$. We write $C(\overline{\Omega}) = C^0(\overline{\Omega})$. For $f \in C^k(\overline{\Omega})$, we define

$$\|f\|_k = \sup_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} |(\partial^\alpha f)(x)|.$$

It is straightforward to check that this is a norm on $C^k(\overline{\Omega})$.

Theorem 4. *If Ω is a bounded open subset of $\mathbb{R}^n$, then $C(\overline{\Omega})$ is a Banach space.*

Proof. Let $f_i \in C(\overline{\Omega})$ be a Cauchy sequence. Thus, $f_i : \overline{\Omega} \to \mathbb{C}$ are continuous, and for any $\epsilon > 0$ there is some i_ϵ such that if $i, j \geq i_\epsilon$ then

$$\sup_{x \in \overline{\Omega}} |f_i(x) - f_j(x)| < \epsilon.$$

Then, for each $x \in \overline{\Omega}$ we have that $f_i(x)$ is a Cauchy sequence in $\mathbb{C}$ and hence converges to some $f(x) \in \mathbb{C}$, thus defining a function $f : \overline{\Omega} \to \mathbb{C}$. For $x \in \overline{\Omega}$ and $\epsilon > 0$, because $f_i(x) \to f(x)$, there is some j_x such that $j \geq j_x$ implies that $|f_j(x) - f(x)| < \epsilon$. For $i \geq i_\epsilon$ and $j \geq \max(i_\epsilon, j_x)$,

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \epsilon + \epsilon.$$

This shows that $\sup_{x \in \overline{\Omega}} |f_i(x) - f(x)| \to 0$ as $i \to \infty$.

Fix $x \in \Omega$ and let $\epsilon > 0$. What we just proved shows that there is some i_0 for which $i \geq i_0$ implies that $\sup_{z \in \overline{\Omega}} |f_i(z) - f(z)| < \epsilon$. As $f_{i_0} : \overline{\Omega} \to \mathbb{C}$ is continuous, there is some $\delta > 0$ such that for $y \in B_\delta(x) \cap \overline{\Omega}$, we have $|f_{i_0}(x) - f_{i_0}(y)| < \epsilon$. Then, for $y \in B_\delta(x) \cap \overline{\Omega}$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{i_0}(x)| + |f_{i_0}(x) - f_{i_0}(y)| + |f_{i_0}(y) - f(y)| \\ &< \epsilon + \epsilon + \epsilon. \end{aligned}$$

This proves that f is continuous at x, and because x was an arbitrary point in $\overline{\Omega}$, we have that $f \in C(\overline{\Omega})$. □

Theorem 5. *If Ω is a bounded open subset of $\mathbb{R}^n$ and k is a positive integer, then $C^k(\overline{\Omega})$ is a Banach space.*

Proof. We proved in Theorem 4 that $C(\overline{\Omega}) = C^0(\overline{\Omega})$ is a Banach space. We assume that $C^{k-1}(\overline{\Omega})$ is a Banach space, and using this induction hypothesis we shall prove that $C^k(\overline{\Omega})$ is a Banach space.

Let $f_i \in C^k(\overline{\Omega})$ be a Cauchy sequence. In particular, f_i is a Cauchy sequence in $C(\overline{\Omega})$, and because $C(\overline{\Omega})$ is a Banach space, there is some $g \in C(\overline{\Omega})$ for which $\|f_i - g\|_0 \to 0$. For each $1 \leq p \leq n$ we have $\partial_p f_i \in C^{k-1}(\overline{\Omega})$. Because $C^{k-1}(\overline{\Omega})$ is a Banach space, for each p there is some $g_p \in C^{k-1}(\overline{\Omega})$ for which $\|\partial_p f_i - g_p\|_{k-1} \to 0$.

Let $\alpha \in \mathbb{N}^n$ have pth entry 1 and all other entries 0, and let $x \in \Omega$. For nonzero τ small enough so that the line segment from x to $x + \tau\alpha$ is contained in Ω,

$$f_i(x + \tau\alpha) - f_i(x) = \int_0^\tau (\partial_p f_i)(x + t\alpha)dt.$$

Because $\|f_i - g\|_0 \to 0$ and $\|\partial_p f_i - g_p\|_0 \to 0$ (the latter because $\|\partial_p f_i - g_p\|_{k-1} \to 0$), we obtain

$$g(x + \tau\alpha) - g(x) = \int_0^\tau g_p(x + t\alpha)dt,$$

or

$$\frac{g(x + \tau\alpha) - g(x)}{\tau} = \frac{1}{\tau}\int_0^\tau g_p(x + t\alpha)dt.$$

As τ tends to 0 the right hand side tends to $g_p(x)$, which shows that $(\partial_p g)(x) = g_p(x)$. We did this for all $x \in \Omega$, and so $\partial_p g = g_p \in C^{k-1}(\overline{\Omega})$. Because this is true for each $1 \leq p \leq n$, we obtain $g \in C^k(\overline{\Omega})$. □

If Ω is a bounded open subset of $\mathbb{R}^n$, then

$$C^\infty(\overline{\Omega}) = \bigcap_{k=0}^\infty C^k(\overline{\Omega}).$$

It can be proved that $C^\infty(\overline{\Omega})$ is the projective limit of the Banach spaces $C^k(\overline{\Omega})$, $k = 0, 1, \ldots$.[3] A projective limit of a countable projective system of Banach spaces is a Fréchet space, and thus $C^\infty(\overline{\Omega})$ is a Fréchet space.

4 Test functions

Let Ω be an open subset of $\mathbb{R}^n$. If $f : \Omega \to \mathbb{C}$ is a function, the *support of f* is the closure of the set $\{x \in \Omega : f(x) \neq 0\}$. We denote the support of f by $\operatorname{supp} f$. If $\operatorname{supp} f$ is a compact set, we say that f has *compact support*, and we denote by $C_c^\infty(\Omega)$ the set of all elements of $C^\infty(\Omega)$ with compact support. We write $\mathscr{D}(\Omega) = C_c^\infty(\Omega)$.

For $f \in \mathscr{D}(\Omega)$, we define

$$\|f\|_N = \sup_{|\alpha| \leq N} \sup_{x \in \Omega} |(\partial^\alpha f)(x)|.$$

If K is a compact subset of Ω, we define

$$\mathscr{D}(K) = \{f \in C_c^\infty(\Omega) : \operatorname{supp} f \subseteq K\}.$$

[3] See Paul Garrett, *Banach and Fréchet spaces of functions*, `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/02_spaces_fcns.pdf`

The restriction of these norms to $\mathscr{D}(K)$ are norms, in particular seminorms. Hence, with the topology for which a local basis at 0 is the collection of sets of the form $\{f \in \mathscr{D}(K) : \|f\|_N < \frac{1}{N}\}$, we have that $\mathscr{D}(K)$ is a locally convex space, and because there are countably many seminorms $\|\cdot\|_N$, the space is metrizable. One checks that the topology on $\mathscr{D}(K)$ is equal to the subspace topology it inherits from $C^\infty(\Omega)$.[4] Theorem 3 tells us that $C^\infty(\Omega)$ is a Fréchet space, and in the following theorem we show that $\mathscr{D}(K)$ is a closed subspace of this Fréchet space, and hence is a Fréchet space itself.

Theorem 6. *If Ω is an open subset of $\mathbb{R}^n$ and K is a compact subset of Ω, then $\mathscr{D}(K)$ is a closed subspace of the Fréchet space $C^\infty(\Omega)$.*

Proof. Let $f_i \in \mathscr{D}(K)$, $f \in C^\infty(\Omega)$, and suppose that $f_i \to f$ in $C^\infty(\Omega)$. If $x \in \Omega \setminus K$, then $f_i(x) = 0$. There is some K_N that contains K, and the fact that $f_i \to f$ gives us in particular that

$$|f(x)| = |0 - f(x)| = |f_i(x) - f(x)| \leq \sup_{y \in K_N} |f_i(y) - f(y)| \to 0,$$

hence $f(x) = 0$. This shows that $\operatorname{supp} f \subseteq K$, and hence that $f \in \mathscr{D}(K)$. □

Let K_j be an exhaustion of Ω by compact sets. Check that $\mathscr{D}(K_j)$ is a closed subspace of $\mathscr{D}(K_{j+1})$ and that the inclusion $\mathscr{D}(K_j) \hookrightarrow \mathscr{D}(K_{j+1})$ is a homeomorphism onto its image. We define the following topology on the set $\mathscr{D}(\Omega)$. Let $\mathscr{B}$ be the collection of all convex balanced subsets V of $\mathscr{D}(U)$ such that for all j, the set $V \cap \mathscr{D}(K_j)$ is open in $\mathscr{D}(K_j)$. (To be *balanced* means that $\alpha V \subseteq V$ if $|\alpha| \leq 1$.) We define $\mathscr{T}$ be the collection of all subsets U of $\mathscr{D}(\Omega)$ such that $x_0 \in U$ implies that there is some $V \in \mathscr{B}$ for which $x_0 + V \subseteq U$. We check that $\mathscr{T}$ is a topology on $\mathscr{D}(\Omega)$, which we call the *strict inductive limit topology*. One proves[5] that with this topology, $\mathscr{D}(\Omega)$ is a locally convex space. With the strict inductive limit topology, we call the locally convex space $\mathscr{D}(\Omega)$ the *strict inductive limit* of the Fréchet spaces $\mathscr{D}(K_1) \hookrightarrow \mathscr{D}(K_2) \hookrightarrow \cdots$, and write

$$\mathscr{D}(\Omega) = \varinjlim \mathscr{D}(K_j).$$

[4] Walter Rudin, *Functional Analysis*, second ed., p. 151.

[5] John B. Conway, *A Course in Functional Analysis*, second ed., pp. 116–123, chap. IV, §5; this is presented without using the language of inductive limits in Walter Rudin, *Functional Analysis*, second ed., p. 152, Theorem 6.4.