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Abstract. There is not much that can be said for all x and for all n about the sum

$$\sum_{k=1}^{n} \frac{1}{|\sin k\pi x|}.$$

However, for this and similar sums, series, and products, we can establish results for almost all x using the tools of continued fractions. This story includes various parts of late 19th century and early 20th century mathematics. etc.

Grattan-Guinness, p. 158

Hobson, Ch. VII, The Theory of Functions of a Real Variable and the Theory of, Volume 1, p. 730

Define (x) to be 0 if $x \in \mathbb{Z} + \frac{1}{2}$; if $x \notin \mathbb{Z} + \frac{1}{2}$ then there is an integer m_x for which $|x - m_x| < |x - n|$ for all integers $n \neq m_x$, and we define (x) to be $x - m_x$.

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Riemann [4, p. 105, §6] defines

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2};$$

for any x, the series converges absolutely because $|(-nx)| < \frac{1}{2}$. Riemann states that if p and m are relatively prime and $x = \frac{p}{2m}$, then

$$f(x^+) = \lim_{h \to 0^+} f(x+h) = f(x) - \frac{\pi^2}{16m^2}, \qquad f(x^-) = \lim_{h \to 0^-} f(x+h) = f(x) + \frac{\pi^2}{16m^2},$$

thus

$$f(x^-) - f(x^+) = \frac{\pi^2}{8m^2},$$

and hence that f is discontinuous at such points, and says that at all other points f is continuous; see Neuenschwander [2] about Riemann's work on pathological functions, and also [3, p. 37]. For any interval $[a, b]$ and any $\sigma > 0$, it is apparent from the above that there are only finitely many $x \in [a, b]$ for which $f(x^-) - f(x^+) > \sigma$, and Riemann deduces from this that f is Riemann integrable on $[a, b]$; cf. Hawkins [1, p. 18] on the history of Riemann integration. Later in the same

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paper [4, p. 129, §13], Riemann states that the function

$$x \mapsto \sum_{n=1}^{\infty} \frac{(nx)}{n},$$

is not Riemann integrable in any interval.

Hankel 1871 199-200

References

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3. Alfred Pringsheim, *Grundlagen der allgemeinen Funktionenlehre*, Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Band II, 1. Teil, 1. Hälfte (H. Burkhardt, W. Wirtinger, and R. Fricke, eds.), B. G. Teubner, Leipzig, 1899–1916, pp. 1–53.
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