Alternating series

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1684-1691, D. T. Whiteside, p. 33

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Vincent Leotaud, Examen circuli quadraturae, 1654

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1 Grégoire de Saint-Vincent

Gregory Saint-Vincent, *Opus Geometricum Quadraturae Circuli et Sectionum Coni*, 1647: 51–177.,

2 Brouncker

The squaring of the hyperbola, by an infinite series of rational numbers, together with its demonstration Philosophical Transactions No 34 (13 April 1668), 645-9.

3 Newton

I.112-5,134-42

II.166,246

III

Collins to Gregory Decembry 1760 on sine, Gregory to Collins February 1761.

Gregory *Vera Circuli* 1667

Gregory, *Exercitationes Geometriae*, "Appendicula ad Veram Circuli et Hyperbole Quadraturam" 1668

In both the Vera Quadratur aand the "Appendicuta" GREGORY was concerned more generally with sectors of central conics. See Vera Quadratura Proposition 20 for the circle and the ellipse, Proposition 25 for the hyperbola and Proposition 29 for the calculation. HUYGENS had already given the result for the circle in Proposition 5 of his De Circuli Magnitudine Inventa (Leiden, 1654). That GREGORY failed to acknowledge this was the main point of contention in the bitter controversy that ensued between him and HUYGENS. GREGORY's methods are discussed by D. T. WHITESIDE in op. cit. (see especially his pages 226-7 and 266-70).

As well as inscribed figures GREGORY considered circumscribed figures and combinations of both to produce a series of upper and lower bounds for areas of sectors in the "Appendicula" - twenty three in all. For an analysis of this work see J. E. HOFMANN, "Uber Gregorys systematische N/iherungen ffir den Sektor eines Mittelpunktkegelschnittes", Centaurus 1 (1950), 24–37.

4 Wolff

October 1674, *Schediasma de serierum summis, et seriebus quadraticibus.*

Wolff to Leibniz June 12, 1712 [4, pp. 143, Letter LXX]

Leibniz to Wolff July 13 1712 [4, pp. 147, Letter LXXI]:

> Respondissem citius, si prius vacasset elegantissimam tuam meditationem considerare attentius, qua ostendere aggrederis, ut $1-1+1-1$ etc. in infinit. est $\frac{1}{2}$, ita $1-2+4-8+16-32$ etc. esse $\frac{1}{3}$, et

> $1 - 3 + 9 - 27 + 81$ etc. esse $\frac{1}{4}$, et ita porro; in quo ego haesi, quia summationes serierum infinitarum solent postulare descrecentiam terminorum.

5 Wallis

Wallis, *Arithmetica Infinitorum*, Propositions 39–41 [7, p. 39].

6 Leibniz

Leibniz, *De vera proportione Circuli ad Quadratum circumscriptum in Numeris rationalibus expressa*, Acta Erud February 1682, [2, pp. 118–122]

De quadratura arithmetica, Proposition XLIX, p. 657 [6, p. 657].

Leibniz to Hermann, June 26, 1705 [3, pp. 272–275, Letter VII]

> Videtur mihi determinatio limitum pars esse essentialis doctrinae de seriebus infinitis plene tradendae.

Leibniz [1, p. 922]

Leibniz to Johann Bernoulli, January 10, 1714 [1, pp. 925–927, Letter CCLI],

Johann Bernoulli to Leibniz, February 28, 1714 [1, pp. 927–930, Letter CCLII].

Leibniz, *Epistola ad. V. Cl. Christianum Wolfium, Professorem Matheseos Halensem, circa Scientiam infiniti*, Acta Eruditorum Supplementa, Volume V, 1713 [2, pp. 382–386].

Leibniz, AE, February 1682, *De vera proportione Circuli ad Quadratum circumscriptum in Numeris rationalibus expressa* [2, pp. 118–122].

Grandi, *Quadratura circuli et hyperbolae per infinitas hyperbolas geometrice exhibita*, Pisa, 1703

7 Nicolaus Bernoulli

Nicolaus Bernoulli to Leibniz, October 25, 1712 and April 7, 1713.

Leibniz to Nicolaus Bernoulli, June 28, 1713 [1, p. 983]

8 Alternating series test

Theorem 1 (Alternating series test)**.** *Suppose that* $0 < a_{k+1} < a_k$, $k \geq 1$, *and that* $a_k \to 0$ *as* $k \to \infty$. *Let* $S_m = \sum_{1 \leq k \leq m} (-1)^{k-1} a_k$. *Then (i)*

$$S_2 < S_4 < \cdots < S_3 < S_1,$$

and (ii) there is some S *such that*

$$S_m \to S, \qquad m \to \infty,$$

with

$$S_2 < S_4 < \cdots < S < \cdots < S_3 < S_1,$$

and for $r \geq 1$*,*

$$S - a_{2r+1} < S_{2r} < S, \qquad S < S_{2r-1} < S + a_{2r}.$$

Proof. (i) As $a_{k+1} < a_k$,

$$S_{2r+2} - S_{2r} = -a_{2r+2} + a_{2r+1} > 0$$

and

$$S_{2r+1} - S_{2r-1} = a_{2r+1} - a_{2r} < 0.$$

Thus for $r \geq 1$,

$$S_{2r+2} > S_{2r}, \qquad S_{2r+1} < S_{2r-1}. \tag{1}$$

Furthermore, for $r \geq 1$,

$$S_{2r} = -a_{2r} + S_{2r-1} < S_{2r-1} \tag{2}$$

Fix r. For $1\rho \leq r$, by (1) we have $S_{2\rho-1} \geq S_{2r-1}$ and thus by (2) we have $S_{2r} < S_{2\rho-1}$. For $\rho \geq r$, by (1) we have $S_{2r} \leq S_{2\rho}$ and thus by (2) we have $S_{2r} < S_{2\rho-1}$. Therefore $S_{2r} < S_{2\rho-1}$ for all $\rho \geq 1$. That is, if n is even and m is odd then $S_n < S_m$. Thus

$$S_2 < S_4 < \cdots < S_3 < S_1.$$

(ii) From (i) we have

$$S_2 < S_4 < \cdots < S_3 < S_1.$$

In particular, $S_{2r} < S_1$ for all $r \geq 1$ and $S_{2r-1} > S_2$ for all $r \geq 1$. Let

$$A = \sup\{S_{2r} : r \geq 1\} \leq S_1, \qquad B = \inf\{S_{2r-1} : r \geq 1\} \geq S_2.$$

Then $S_{2r-1} - S_{2r} \to B - A$. But $S_{2r-1} - S_{2r} = -a_{2r}$ and $a_{2r+1} \to 0$. Therefore $B - A = 0$, i.e. $A = B$. Let

$$S = A = B,$$

so $S_{2r} < S$ and $S_{2r-1} > S$ for all $r \geq 1$. Then

$$S - S_{2r} < S_{2r+1} - S_{2r} = a_{2r+1},$$

giving $S_{2r} > S - a_{2r+1}$, and

$$S_{2r-1} - S < S_{2r-1} - S_{2r} = a_{2r},$$

giving $S_{2r-1} < S + a_{2r}$, completing the proof. □

Jakob Bernoulli, 1689 limits

Mengoli, Geometriae Speciosiae 1659

The Mathematical Papers of Isaac Newton:, Volume 4; Volumes 1674-1684, p. 611

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James Gregory 1668 convergent and divergent

James Gregory letter to Collins, February 15, 1671, $\arctan x$.

Knobloch [5]

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[6] Siegmund Probst and Uwe Mayer (eds.), *Gottfried Wilhelm Leibniz. Sämtliche Schriften und Briefe. Siebente Reihe: Mathematische Schriften. Sechster Band: 1673–1676. Arithmetische Kreisquadratur*, Akademie-Verlag, Berlin, 2012.

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