LAMBERT SERIES IN ANALYTIC NUMBER THEORY

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1. Lambert series

Let $d(n)$ denote the number of positive divisors of n. For $|z| < 1$,

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}.$$

The first use of the words "Lambert series" did not refer to this series, but were used by Euler to describe something to do with roots of an equation.

2. Lambert

Lambert [39, pp. 506–511, §875]

Monatsbuch, September 1764, "Singula haec in Capp. ult. Ontol. occurunt", and Anm. 5, Anm. 25, 1764, Anm. 12 1765, Anm. 19, 1765 [2].

Iohannis Henrici Lamberti Opera Mathematica. Volumen Secundum, no. 9

Youschkevitch [61]

Bullynck [7]

3. Krafft

Krafft [36, pp. 244–245]

4. Servois

Servois [52, p. 166].

5. Lacroix

Lacroix [37, pp. 465–466, §1195]

6. Klügel

Klügel [34, pp. 52–53, "Theiler einer Zahl", §12]:

> Ist $N = \alpha^m \beta^n \gamma^p \cdots$, wo α, β, γ, Primzahlen sind; so erhellet auch leicht, daßalle Theiler von N, die Einheit und die Zahl selbst mit engeschlossen, durch die Glieder des Products
>
> $$(1+\alpha+\alpha^2+\cdots+\alpha^m)(1+\beta+\beta^2+\cdots+\beta^n)(1+\gamma+\gamma^2+\cdots+\gamma^p)\cdots$$
>
> argestelle werden. Die Anzahl der Glieder dieses Products, d. i. die Anzahl aller Theiler von N, ist offenbar $= (m+1)(n+1)(p+1)\cdots$. Für das obige Beispiel $= 4 \cdot 3 \cdot 2 = 24$, wo die Einheit mit engeschlossen ist.

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In der aus der Entwickelung von

$$\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \cdots + \frac{x^n}{1-x^n} + \cdots$$

entspringenden Reihe:

$$x + 2x^2 + 2x^3 + 3x^4 + 2x^5 + 4x^6 + 2x^7 + \cdots$$

welche Lambert in seiner Architektonik S. 507. mittheilt, enthalt jeder Coefficient so viele Einheiten, als der Exponent der entsprechendenden Potenz von x Theiler hat.

7. Stern

Stern [54]

8. Clausen

Clausen [17] states that

$$\sum_{n=1}^\infty \frac{x^n}{1-x^n} = \sum_{n=1}^\infty x^{n^2} \left(\frac{1+x^n}{1-x^n} \right),$$

and that the right-hand series converges quickly for small x. Clausen does prove this expansion, and a proof is later given by Scherk [48]. Scherk's argument uses the fact

$$1+2t+2t^2+2t^3+2t^4+\cdots = (1+t+t^2+t^3+t^4+\cdots)+t(1+t+t^2+t^3+t^4+\cdots) = \frac{1+t}{1-t}.$$

We write

$$\sum_{n=1}^\infty \frac{x^n}{1-x^n} = \sum_{n=1}^\infty \sum_{m=1}^\infty x^{nm}.$$

The series is

$$\begin{array}{ccccccc}
x & +x^2 & +x^3 & +x^4 & +x^5 & +x^6 & +\text{etc.} \\
+x^2 & +x^4 & +x^6 & +x^8 & +x^{10} & +x^{12} & +\text{etc.} \\
+x^3 & +x^6 & +x^9 & x^{12} & +x^{15} & +x^{18} & +\text{etc.} \\
+x^4 & +x^8 & +x^{12} & +x^{16} & +x^{20} & +x^{24} & +\text{etc.} \\
+x^5 & +x^{10} & +x^{15} & +x^{20} & +x^{25} & +x^{30} & +\text{etc.} \\
+x^6 & +x^{12} & +x^{18} & +x^{24} & +x^{30} & +x^{36} & +\text{etc.} \\
+\text{etc.} & & & & & &
\end{array}$$

We sum the terms in the first row and column: the sum of these is

$$x + 2x^2 + 2x^3 + 2x^4 + \text{etc.} = x \left(\frac{1+x}{1-x} \right).$$

Then, from what remains we sum the terms in the second row and column: the sum of these is

$$x^4 + 2x^6 + 2x^8 + 2x^{10} + \text{etc.} = x^4 \left(\frac{1+x^2}{1-x^2} \right).$$

Then, from what remains, we sum the terms in the third row and column: the sum of these is

$$x^9 + 2x^{12} + 2x^{15} + 2x^{18} + \text{etc.} = x^9 \left(\frac{1+x^3}{1-x^3} \right),$$

etc.

9. Eisenstein

Eisenstein [23] states that for $|z| < 1$,

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nz^{n(n+1)/2}}{(1-x)\cdots(1-x^n)}.$$

For $t = \frac{1}{z}$, Eisenstein states that

$$\frac{z}{1-z} + \frac{z^2}{1-z^2} + \frac{z^3}{1-z^3} + \frac{z^4}{1-z^4} + \text{etc.}$$

is equal to

$$\cfrac{1}{t-1-\cfrac{(t-1)^2}{t^2-1-\cfrac{t(t-1)^2}{t^3-1-\cfrac{t(t^2-1)^2}{t^4-1-\cfrac{t^2(t^2-1)^2}{t^5-1-\cfrac{t^2(t^3-1)^2}{t^6-1-\cfrac{t^3(t^3-1)^2}{t^7-1-\text{etc.}}}}}}}}$$

Expressing Lambert series using continued fractions is relevant to the irrationality of the value of the series. See Borwein [3]. In fact, Euler wrote in E25 about the particular value of a Lambert series.

10. Möbius

Möbius [44]

11. Jacobi

Jacobi's *Fundamenta nova* [33, §40 and p. 185]

12. Dirichlet's paper

Dirichlet [21]

13. Cauchy

Cauchy [12] and [13] two memoirs in the same volume.

14. Burhenne

Burhenne [8] says the following about Lambert series. For

$$F(x) = \sum_{n=1}^{\infty} d(n)x^n,$$

we have

$$d(n) = \frac{F^{(n)}(0)}{n!}.$$

Define

$$F_k(x) = \frac{x^k}{1-x^k},$$

so that

$$F(x) = \sum_{k=1}^{\infty} F_k(x).$$

It is apparent that if $k > n$, then

$$F_k^{(n)}(0) = 0,$$

hence

$$F^{(n)}(0) = \sum_{k=1}^{n} F_k^{(n)}(0).$$

The above suggests finding explicit expressions for $F_k^{(n)}(0)$. Burhenne cites Sohncke [53, pp. 32–33]: for even k,

$$\begin{aligned}\frac{d^n\left(\frac{x^p}{x^k-a^k}\right)}{dx^n} &= (-1)^n \frac{n!}{ka^{k-p-1}}\left(\frac{1}{(x-a)^{n+1}} - (-1)^p \frac{1}{(x+a)^{n+1}}\right) \\ &\quad + (-1)^n \frac{n!}{\frac{1}{2}ka^{k-p-1}} \sum_{h=1}^{\frac{1}{2}k-1} \frac{\cos\left(\frac{2h(p+1)\pi}{k} + (n+1)\phi_h\right)}{\sqrt{\left(x^2 - 2xa\cos\frac{2h\pi}{n} + a^2\right)^{n+1}}}\end{aligned}$$

and for odd k,

$$\begin{aligned}\frac{d^n\left(\frac{x^p}{x^k-a^k}\right)}{dx^n} &= (-1)^n \frac{n!}{ka^{k-p-1}} \frac{1}{(x-a)^{n+1}} \\ &\quad + (-1)^n \frac{n!}{\frac{1}{2}ka^{k-p-1}} \sum_{h=1}^{\frac{k-1}{2}} \frac{\cos\left(\frac{2h(p+1)\pi}{k} + (n+1)\phi_h\right)}{\sqrt{\left(x^2 - 2xa\cos\frac{2h\pi}{n} + a^2\right)^{n+1}}},\end{aligned}$$

where

$$\cos\phi_h = \frac{x - a\cos\frac{2h\pi}{k}}{\sqrt{x^2 - 2xa\cos\frac{2h\pi}{k} + a^2}}, \quad \sin\phi_h = \frac{a\sin\frac{2h\pi}{k}}{\sqrt{x^2 - 2xa\cos\frac{2h\pi}{k} + a^2}}.$$

For $a = 1$ and $x = 0$,

$$\cos\phi_h = -\cos\frac{2h\pi}{k}, \qquad \sin\phi_h = \sin\frac{2h\pi}{k},$$

from which

$$\phi_h = \pi - \frac{2h\pi}{k},$$

and thus

$$\begin{aligned}\cos\left(\frac{2h(k+1)\pi}{k} + (n+1)\phi_h\right) &= \cos\left(\frac{2h(k+1)\pi}{k} + (n+1)\left(\pi - \frac{2h\pi}{k}\right)\right) \\ &= \cos\left(2h\pi + \frac{2h\pi}{k} + \pi - \frac{2h\pi}{k} + n\left(\pi - \frac{2h\pi}{k}\right)\right) \\ &= \cos\left((n+1)\pi - \frac{2nh\pi}{k}\right) \\ &= (-1)^{n+1}\cos\frac{2nh\pi}{k}.\end{aligned}$$

For even k, taking $p = k$ we have

$$\frac{d^n\left(\frac{x^k}{1-x^k}\right)}{dx^n} = (-1)^{n+1}\frac{n!}{k}\left(\frac{1}{(-1)^{n+1}} - 1\right) + (-1)^{n+1}\frac{n!}{\frac{1}{2}k}\sum_{h=1}^{\frac{1}{2}k-1}(-1)^{n+1}\cos\frac{2nh\pi}{k},$$

i.e.,

$$F_k^{(n)}(0) = \frac{n!}{k}(1-(-1)^{n+1}) + \frac{2\cdot n!}{k}\sum_{h=1}^{\frac{1}{2}k-1}\cos\frac{2nh\pi}{k}.$$

For odd k, taking $p = k$ we have

$$\frac{d^n\left(\frac{x^k}{1-x^k}\right)}{dx^n} = (-1)^{n+1}\frac{n!}{k}\frac{1}{(-1)^{n+1}} + (-1)^{n+1}\frac{n!}{\frac{1}{2}k}\sum_{h=1}^{\frac{k-1}{2}}(-1)^{n+1}\cos\frac{2nh\pi}{k},$$

i.e.,

$$F_k^{(n)}(0) = \frac{n!}{k} + \frac{2\cdot n!}{k}\sum_{h=1}^{\frac{k-1}{2}}\cos\frac{2nh\pi}{k}.$$

Using the identity, for $h \notin 2\pi\mathbb{Z}$,

$$\sum_{h=1}^{M}\cos h\theta = -\frac{1}{2} + \frac{\sin\left(M+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = -\frac{1}{2} + \frac{1}{2}\left(\sin M\theta\cot\frac{\theta}{2} + \cos M\theta\right),$$

we get for even k,

$$\begin{aligned} F_k^{(n)}(0) &= \begin{cases} \frac{n!}{k}\cot\frac{n\pi}{k}\sin n\pi & k \nmid n \\ \frac{n!}{k}(1-(-1)^{n+1}) + \frac{2\cdot n!}{k}\left(\frac{1}{2}k-1\right) & k|n \end{cases} \\ &= \begin{cases} 0 & k \nmid n \\ n! - \frac{n!}{k}(1+(-1)^{n+1}) & k|n. \end{cases} \end{aligned}$$

For odd k,

$$\begin{aligned} F_k^{(n)}(0) &= \begin{cases} \frac{n!}{k}\csc\frac{n\pi}{k}\sin n\pi & k \nmid n \\ \frac{n!}{k} + \frac{2\cdot n!}{k}\frac{k-1}{2} & k|n. \end{cases} \\ &= \begin{cases} 0 & k \nmid n \\ n! & k|n. \end{cases} \end{aligned}$$

15. Zehfuss

Zehfuss [62]

16. Bernoulli numbers

The *Bernoulli polynomials* are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}.$$

The *Bernoulli numbers* are defined by $B_m = B_m(0)$.

We denote by $[x]$ the greatest integer $\le x$, and we define $\{x\} = x - [x]$, namely, the fractional part of x. We define $P_m(x) = B_m(\{x\})$, the *Bernoulli functions*.

17. Euler-Maclaurin summation formula

Euler E47 and E212, §142, for the summation formula. Euler's studies the gamma function in E368. In particular, in §12 he gives Stirling's formula, and in §14 he obtains $\Gamma'(1) = -\gamma$. Euler in §142 of E212 states that

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{2n}}{2n}.$$

Bromwich [6, Chapter XII]

18. Schlömilch

Schlömilch [49] and [51, p. 238], [50]

For $m \geq 1$,

$$\int_0^\infty \frac{t^{2m-1}}{e^{2\pi t}-1} dt = (-1)^{m+1} \frac{B_{2m}}{4m}. \tag{1}$$

For $\alpha > 0$,

$$\int_0^\infty \frac{\sin \alpha t}{e^{2\pi t}-1} dt = \frac{1}{4} + \frac{1}{2}\left(\frac{1}{e^\alpha - 1} - \frac{1}{\alpha}\right) \tag{2}$$

and

$$\int_0^\infty \frac{1-\cos \alpha t}{e^{2\pi t}-1} \frac{dt}{t} = \frac{1}{4}\alpha + \frac{1}{2}\left(\log(1-e^{-\alpha}) - \log \alpha\right). \tag{3}$$

For $\xi > 0$ and $n \geq 1$, using (2) with $\alpha = \xi, 2\xi, 3\xi, \ldots, 2n\xi$ and also using

$$\sum_{k=1}^N \sin k\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos(N+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}},$$

we get

$$\begin{aligned}\sum_{m=1}^{2n}\left(\frac{1}{e^{m\xi}-1} - \frac{1}{m\xi}\right) &= \sum_{m=1}^{2n}\left(-\frac{1}{2} + 2\int_0^\infty \frac{\sin m\xi t}{e^{2\pi t}-1} dt\right)\\ &= -n + \int_0^\infty \frac{1}{e^{2\pi t}-1} \sum_{m=1}^{2n} 2\sin m\xi t dt\\ &= -n + \int_0^\infty \frac{1}{e^{2\pi t}-1}\left(\cot\frac{\xi t}{2} - \frac{\cos(2n+\frac{1}{2})\xi t}{\sin\frac{\xi t}{2}}\right) dt.\end{aligned}$$

Using $\cos(a+b) = \cos a \cos b - \sin a \sin b$, this becomes

$$\begin{aligned}\sum_{m=1}^{2n}\left(\frac{1}{e^{m\xi}-1} - \frac{1}{m\xi}\right) &= -n + \int_0^\infty \frac{1}{e^{2\pi t}-1}(1-\cos 2n\xi t)\cot\frac{\xi t}{2} dt\\ &\quad + \int_0^\infty \frac{1}{e^{2\pi t}-1}\sin 2n\xi t dt.\end{aligned} \tag{4}$$

For $\alpha = 2n\xi$, (3) tells us

$$\int_0^\infty \frac{1-\cos 2n\xi t}{e^{2\pi t}-1}\frac{dt}{t} = \frac{1}{4}\cdot 2n\xi + \frac{1}{2}\left(\log(1-e^{-2n\xi}) - \log 2n\xi\right).$$

Rearranging,

$$\frac{\log 2n}{\xi} = n + \frac{\log(1-e^{-2n\xi}) - \log\xi}{\xi} - \frac{2}{\xi}\int_0^\infty \frac{1-\cos 2n\xi t}{e^{2\pi t}-1}\frac{dt}{t} \tag{5}$$

Adding (4) and (5) gives

$$\begin{aligned}
&\sum_{m=1}^{2n} \frac{1}{e^{m\xi}-1} - \frac{1}{\xi}\left(-\log 2n + \sum_{m=1}^{2n}\frac{1}{m}\right)\\
=&\frac{\log(1-e^{-2n\xi}) - \log\xi}{\xi} - \int_0^\infty \left(\frac{2}{\xi t} - \cot\frac{\xi t}{2}\right)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\\
&+\int_0^\infty \frac{1}{e^{2\pi t}-1}\sin 2n\xi t dt.
\end{aligned}$$

Writing

$$C_n = -\log n + \sum_{m=1}^{n}\frac{1}{m}$$

and using (2) this becomes

$$\begin{aligned}
&\sum_{m=1}^{2n} \frac{1}{e^{m\xi}-1} - \frac{1}{\xi}C_{2n}\\
=&\frac{\log(1-e^{-2n\xi}) - \log\xi}{\xi} - 2\int_0^\infty \left(\frac{1}{\xi t} - \frac{1}{2}\cot\frac{\xi t}{2}\right)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\\
&+\frac{1}{4} + \frac{1}{2}\left(\frac{1}{e^{2n\xi}-1} - \frac{1}{2n\xi}\right).
\end{aligned}$$

We write

$$I_{2n}(\xi) = 2\int_0^\infty \left(\frac{1}{\xi t} - \frac{1}{2}\cot\frac{\xi t}{2}\right)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt,$$

and we shall obtain an asymptotic formula for $I_{2n}(\xi)$.

The Euler-Maclaurin summation formula [5, p. 280, Ch. VI, Eq. 35] tells us that for $f \in C^\infty([0,1])$,

$$f(0) = \int_0^1 f(t)dt + B_1(f(1)-f(0)) + \sum_{m=1}^{k}\frac{1}{(2m)!}B_{2m}(f^{(2m-1)}(1) - f^{(2m-1)}(0)) + R_{2k},$$

where

$$R_{2k} = -\int_0^1 \frac{P_{2k}(1-\eta)}{(2k)!}f^{(2k)}(\eta)d\eta.$$

Let $h > 0$. For $f(t) = \cos ht$ we have $f'(t) = -h\sin ht$, and for $m \geq 1$ we have $f^{(2m)}(t) = (-1)^m h^{2m}\cos ht$ and $f^{(2m-1)}(t) = (-1)^m h^{2m-1}\sin ht$. Thus the Euler-Maclaurin formula yields

$$1 = \int_0^1 \cos ht dt - \frac{1}{2}(\cos h - 1) + \sum_{m=1}^{k}\frac{1}{(2m)!}B_{2m}(-1)^m h^{2m-1}\sin h + R_{2k}.$$

Using the identity $\cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}$ and dividing by $\sin h$, this becomes

$$\frac{1}{2}\cot\frac{h}{2} = \frac{1}{h} + \sum_{m=1}^{k}\frac{1}{(2m)!}B_{2m}(-1)^m h^{2m-1} + \frac{1}{\sin h}R_{2k}. \tag{6}$$

Because $P_m(1-\eta)=P_m(\eta)$ for even m,

$$\begin{aligned} R_{2k} &= -\int_0^1 \frac{P_{2k}(\eta)}{(2k)!}(-1)^k h^{2k}\cos h\eta d\eta \\ &= -B_{2k}\int_0^1 \frac{1}{(2k)!}(-1)^k h^{2k}\cos h\eta d\eta - \int_0^1 \frac{(P_{2k}(\eta)-B_{2k})}{(2k)!}(-1)^k h^{2k}\cos h\eta d\eta \\ &= (-1)^{k+1}\frac{B_{2k}h^{2k}}{(2k)!}\frac{\sin h}{h} + (-1)^{k+1}\frac{h^{2k}}{(2k)!}\int_0^1 (P_{2k}(\eta)-B_{2k})\cos h\eta d\eta. \end{aligned}$$

Since $P_{2k}(\eta)-B_{2k}$ does not change sign on $(0,1)$, by the mean-value theorem for integration there is some $\theta=\theta(h,k)$, $0<\theta<1$, such that (using $\int_0^1 P_{2k}(\eta)d\eta=0$)

$$\int_0^1 (P_{2k}(\eta)-B_{2k})\cos h\eta d\eta = \cos h\theta \int_0^1 (P_{2k}(\eta)-B_{2k})d\eta = -B_{2k}\cos h\theta.$$

Therefore (6) becomes

$$\begin{aligned} \frac{1}{2}\cot\frac{h}{2}-\frac{1}{h} &= \sum_{m=1}^k \frac{1}{(2m)!}B_{2m}(-1)^m h^{2m-1} \\ &\quad + (-1)^{k+1}\frac{B_{2k}h^{2k-1}}{(2k)!} + (-1)^{k+2}\frac{h^{2k}}{(2k)!\sin h}B_{2k}\cos h\theta, \end{aligned}$$

i.e.,

$$\frac{1}{2}\cot\frac{h}{2}-\frac{1}{h} = \sum_{m=1}^{k-1}\frac{1}{(2m)!}B_{2m}(-1)^m h^{2m-1} + (-1)^k\frac{h^{2k}}{(2k)!\sin h}B_{2k}\cos h\theta.$$

Write

$$E_k(h) = (-1)^{k+1}\frac{h^{2k}}{(2k)!\sin h}B_{2k}\cos h\theta.$$

We apply the above to $I_{2n}(\xi)$, and get, for any $k\geq 1$,

$$\begin{aligned} I_{2n}(\xi) &= 2\int_0^\infty \left(E_k(\xi t) - \sum_{m=1}^{k-1}\frac{1}{(2m)!}B_{2m}(-1)^m(\xi t)^{2m-1}\right)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt \\ &= -2\sum_{m=1}^{k-1}\frac{1}{(2m)!}B_{2m}(-1)^m \xi^{2m-1}\int_0^\infty t^{2m-1}\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt \\ &\quad + 2\int_0^\infty E_k(\xi t)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt. \end{aligned}$$

Using (1),

$$\begin{aligned} \int_0^\infty t^{2m-1}\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt &= \int_0^\infty \frac{t^{2m-1}}{e^{2\pi t}-1}dt - \int_0^\infty \frac{t^{2m-1}\cos 2n\xi t}{e^{2\pi t}-1}dt \\ &= (-1)^{m+1}\frac{B_{2m}}{4m} - \int_0^\infty \frac{t^{2m-1}\cos 2n\xi t}{e^{2\pi t}-1}dt. \end{aligned}$$

Let

$$f(x) = \frac{1}{e^x-1}-\frac{1}{x}.$$

By (2),

$$f(x)+\frac{1}{2} = 2\int_0^\infty \frac{\sin xt}{e^{2\pi t}-1}dt.$$

For $m \geq 1$,

$$f^{(2m-1)}(x) = 2\int_0^\infty \frac{(-1)^{m-1}t^{2m-1}\cos xt}{e^{2\pi t}-1}dt,$$

which for $x = 2n\xi$ becomes

$$\frac{(-1)^{m-1}}{2}f^{(2m-1)}(2n\xi) = \int_0^\infty \frac{t^{2m-1}\cos 2n\xi t}{e^{2\pi t}-1}dt.$$

Therefore

$$2\int_0^\infty t^{2m-1}\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt = (-1)^{m+1}\frac{B_{2m}}{2m} + (-1)^m f^{(2m-1)}(2n\xi).$$

Thus $I_{2n}(\xi)$ is

$$\begin{aligned}
I_{2n}(\xi) = &-\sum_{m=1}^{k-1}\frac{1}{(2m)!}B_{2m}(-1)^m\xi^{2m-1}\left((-1)^{m+1}\frac{B_{2m}}{2m} + (-1)^m f^{(2m-1)}(2n\xi)\right)\\
&+2\int_0^\infty E_k(\xi t)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\\
&=\sum_{m=1}^{k-1}\frac{B_{2m}^2}{(2m)!2m}\xi^{2m-1} - \sum_{m=1}^{k-1}\frac{B_{2m}}{(2m)!}\xi^{2m-1}f^{(2m-1)}(2n\xi)\\
&+2\int_0^\infty E_k(\xi t)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt.
\end{aligned}$$

But

$$\begin{aligned}
\left|\int_0^\infty E_k(\xi t)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\right| &= \left|\int_0^\infty (-1)^{k+1}\frac{(\xi t)^{2k}}{(2k)!\sin\xi t}B_{2k}\cos\xi t\theta\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\right|\\
&\leq \frac{|B_{2k}|}{(2k)!}\int_0^\infty \frac{(\xi t)^{2k}}{|\sin\xi t|}\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt.
\end{aligned}$$

Taking as given that for all $u \in \mathbb{R}$,

$$\frac{1-\cos 2nu}{|\sin u|} \leq \frac{\pi}{2}\frac{1-\cos 2nu}{u},$$

we obtain

$$\begin{aligned}
&\left|\int_0^\infty E_k(\xi t)\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\right|\\
\leq&\frac{\pi}{2}\frac{|B_{2k}|}{(2k)!}\int_0^\infty (\xi t)^{2k-1}\frac{1-\cos 2n\xi t}{e^{2\pi t}-1}dt\\
=&\frac{\pi}{2}\frac{|B_{2k}|}{(2k)!}\xi^{2k-1}\cdot\frac{1}{2}\left((-1)^{k+1}\frac{B_{2k}}{2k} + (-1)^k f^{(2k-1)}(2n\xi)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
I_{2n}(\xi) = &\sum_{m=1}^{k-1}\frac{B_{2m}^2}{(2m)!2m}\xi^{2m-1} - \sum_{m=1}^{k-1}\frac{B_{2m}}{(2m)!}\xi^{2m-1}f^{(2m-1)}(2n\xi)\\
&+O\left(\frac{B_{2k}^2}{(2k)!2k}\xi^{2k-1}\right) + O\left(\frac{|B_{2k}|}{(2k)!}\xi^{2k-1}f^{(2k-1)}(2n\xi)\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\sum_{m=1}^{2n} \frac{1}{e^{m\xi}-1} - \frac{1}{\xi}C_{2n}\\
=&\frac{\log(1-e^{-2n\xi})-\log\xi}{\xi} + \frac{1}{4} + \frac{1}{2}\left(\frac{1}{e^{2n\xi}-1} - \frac{1}{2n\xi}\right) - I_{2n}(\xi)\\
=&\frac{\log(1-e^{-2n\xi})-\log\xi}{\xi} + \frac{1}{4} + \frac{1}{2}\left(\frac{1}{e^{2n\xi}-1} - \frac{1}{2n\xi}\right)\\
&-\sum_{m=1}^{k-1} \frac{B_{2m}^2}{(2m)!2m}\xi^{2m-1} + \sum_{m=1}^{k-1} \frac{B_{2m}}{(2m)!}\xi^{2m-1} f^{(2m-1)}(2n\xi)\\
&+O\left(\frac{B_{2k}^2}{(2k)!2k}\xi^{2k-1}\right) + O\left(\frac{|B_{2k}|}{(2k)!}\xi^{2k-1} f^{(2k-1)}(2n\xi)\right).
\end{aligned}$$

Taking $n \to \infty$,

$$\sum_{m=1}^{\infty} \frac{1}{e^{m\xi}-1} - \frac{\gamma}{\xi} = -\frac{\log\xi}{\xi} + \frac{1}{4} - \sum_{m=1}^{k-1} \frac{B_{2m}^2}{(2m)!2m}\xi^{2m-1} + O\left(\frac{B_{2k}^2}{(2k)!2k}\xi^{2k-1}\right).$$

19. Voronoi summation formula

The Voronoi summation formula [18, p. 182] states that if $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function, then

$$\begin{aligned}
\sum_{n=1}^{\infty} d(n) f(n) = &\int_0^\infty f(t)(\log t + 2\gamma) dt + \frac{f(0)}{4}\\
&+\sum_{n=1}^{\infty} d(n) \int_0^\infty f(t)(4K_0(4\pi(nt)^{1/2}) - 2\pi Y_0(4\pi(nt)^{1/2}))dt,
\end{aligned}$$

where K_0 and Y_0 are Bessel functions.

Let $0 < x < 1$. For $f(t) = e^{-tx}$, we compute using Mathematica that

$$\begin{aligned}
&\int_0^\infty f(t)(4K_0(4\pi(nt)^{1/2}) - 2\pi Y_0(4\pi(nt)^{1/2}))dt\\
=&-\frac{2}{x}\exp\left(\frac{4\pi^2 n}{x}\right)\mathrm{Ei}\left(-\frac{4\pi^2 n}{x}\right) - \frac{2}{x}\exp\left(-\frac{4\pi^2 n}{x}\right)\mathrm{Ei}\left(\frac{4\pi^2 n}{x}\right).
\end{aligned}$$

Then the Voronoi summation formula becomes

$$\begin{aligned}
&\sum_{n=1}^{\infty} d(n) e^{-nx}\\
=&\frac{\gamma}{x} - \frac{\log x}{x} + \frac{1}{4}\\
&+\sum_{n=1}^{\infty} d(n)\left(-\frac{2}{x}\exp\left(\frac{4\pi^2 n}{x}\right)\mathrm{Ei}\left(-\frac{4\pi^2 n}{x}\right) - \frac{2}{x}\exp\left(-\frac{4\pi^2 n}{x}\right)\mathrm{Ei}\left(\frac{4\pi^2 n}{x}\right)\right).
\end{aligned}$$

Egger and Steiner [22] give a proof of the Voronoi summation formula involving Lambert series.

20. Curtze

Curtze [19]

21. Laguerre

Laguerre [38]

22. V. A. Lebesgue

V. A. Lebesgue [42]

23. Bouniakowsky

Bouniakowsky [4]

24. Catalan

Catalan [9, p. 89]
Catalan [10, p. 119, §CXXIV] and [11, pp. 38–39, §CCXXVI]

25. Pincherle

, Pincherle [45]

26. Glaisher

Glaisher [26, p. 163]

27. Günther

Günther [28, p. 83] and [29, p. 178]

28. Rogel

Rogel [46] and [47]

29. Cesàro

Cesàro [14]
Cesàro [15] and [16, pp. 181–184]
Bromwich [6, p. 201, Chapter VIII, Example B, 35]

30. de la Vallée-Poussin

de la Vallée-Poussin [20]

31. Torelli

Torelli [57]

32. Fibonacci numbers

Landau [40]

33. Knopp

Knopp [35]

34. Generating functions

Hardy and Wright [31, p. 258, Theorem 307]:

Theorem 1. *For* $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ *and* $g(s) = \sum_{n=1}^\infty b_n n^{-s}$,

$$\sum_{n=1}^\infty a_n \frac{x^n}{1-x^n} = \sum_{n=1}^\infty b_n x^n, \qquad |x| < 1,$$

if and only if there is some σ *such that*

$$\zeta(s) f(s) = g(s), \qquad \Re(s) > \sigma.$$

For $f(s) = \sum_{n=1}^\infty \mu(n) n^{-s}$ and $g(s) = 1$, using [31, p. 250, Theorem 287]

$$\frac{1}{\zeta(s)} = \sum_{n=1}^\infty \mu(n) n^{-s}, \qquad \Re(s) > 1,$$

we get

$$\sum_{n=1}^\infty \frac{\mu(n) x^n}{1-x^n} = x.$$

For $f(s) = \sum_{n=1}^\infty \phi(n) n^{-s}$ and

$$g(s) = \zeta(s-1) = \sum_{n=1}^\infty n^{-s+1} = \sum_{n=1}^\infty n n^{-s},$$

using [31, p. 250, Theorem 288]

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^\infty \phi(n) n^{-s}, \qquad \Re(s) > 2,$$

we get

$$\sum_{n=1}^\infty \frac{\phi(n) x^n}{1-x^n} = \sum_{n=1}^\infty n x^n = \frac{x}{(1-x)^2}.$$

For $n = p_1^{a_1} \cdots p_r^{a_r}$, define $\Omega(n) = a_1 + \cdots + a_n$ and

$$\lambda(n) = (-1)^{\Omega(n)}.$$

For $f(s) = \sum_{n=1}^\infty \lambda(n) n^{-s}$ and

$$g(s) = \zeta(2s) = \sum_{n=1}^\infty n^{-2s} = \sum_{n=1}^\infty (n^2)^{-s},$$

using [31, p. 255, Theorem 300]

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^\infty \lambda(n) n^{-s}, \qquad \Re(s) > 1,$$

we get

$$\sum_{n=1}^\infty \frac{\lambda(n) x^n}{1-x^n} = \sum_{n=1}^\infty x^{n^2}.$$

We define the *von Mangoldt function* $\Lambda : \mathbb{N} \to \mathbb{R}$ by $\Lambda(n) = \log p$ if n is some positive integer power of a prime p, and $\Lambda(n) = 0$ otherwise. For example, $\Lambda(1) = 0$,

$\Lambda(12) = 0$, $\Lambda(125) = \log 5$. It is a fact [31, p. 254, Theorem 296] that for any n, the von Mangoldt function satisfies

$$\sum_{m|n} \Lambda(m) = \log n. \tag{7}$$

For $f(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}$ and

$$g(s) = -\zeta'(s) = \sum_{n=1}^{\infty} \log n n^{-s},$$

using [31, p. 253, Theorem 294]

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s},$$

we obtain

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) x^n}{1 - x^n} = \sum_{n=1}^{\infty} \log n x^n.$$

35. Preliminaries on prime numbers

We define

$$\vartheta(x) = \sum_{p \le x} \log p = \log \prod_{p \le x} p$$

and

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n).$$

One sees that

$$\psi(x) = \sum_{p \le x} [\log_p x] \log p = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p.$$

As well,

$$\psi(x) = \sum_{m=1}^{\infty} \sum_{p \le x^{1/m}} \log p = \sum_{m=1}^{\infty} \vartheta(x^{1/m}); \tag{8}$$

there are only finitely many terms on the right-hand side, as $\vartheta(x^{1/m}) = 0$ if $x < 2^m$.

Theorem 2.

$$\psi(x) = \vartheta(x) + O(x^{1/2} (\log x)^2).$$

Proof. For $x \ge 2$, $\vartheta(x) < x \log x$, giving

$$\begin{aligned}
\sum_{2 \le m \le \frac{\log x}{\log 2}} \vartheta(x^{1/m}) &< \sum_{2 \le m \le \frac{\log x}{\log 2}} x^{1/m} \frac{1}{m} \log x \\
&\le x^{1/2} \log x \sum_{2 \le m \le \frac{\log x}{\log 2}} \frac{1}{m} \\
&= O(x^{1/2} (\log x)^2).
\end{aligned}$$

Thus, using (8) we have

$$\psi(x) = \vartheta(x) + \sum_{2 \le m \le \frac{\log x}{\log 2}} \vartheta(x^{1/m}) = \vartheta(x) + O(x^{1/2}(\log x)^2).$$

□

We prove that if $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$ then $\frac{\pi(x)}{x/\log x} = 1$.

Theorem 3.

$$\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} = \liminf_{x\to\infty} \frac{\vartheta(x)}{x}$$

and

$$\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} = \limsup_{x\to\infty} \frac{\vartheta(x)}{x}.$$

Proof. From (8), $\vartheta(x) \le \psi(x)$. And,

$$\psi(x) = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p \le \sum_{p \le x} \frac{\log x}{\log p} \log p = \log x \sum_{p \le x}.$$

Hence

$$\frac{\vartheta(x)}{x} \le \frac{\pi(x)\log x}{x},$$

whence

$$\liminf_{x\to\infty} \frac{\vartheta(x)}{x} \le \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x}$$

and

$$\limsup_{x\to\infty} \frac{\vartheta(x)}{x} \le \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}.$$

Let $0 < \alpha < 1$. For $x > 1$,

$$\vartheta(x) = \sum_{p \le x} \log p \ge \sum_{x^\alpha < p \le x} \log p > \sum_{x^\alpha < p \le x} \log x^\alpha = \alpha \log x(\pi(x) - \pi(x^\alpha)).$$

As $\pi(x^\alpha) < x^\alpha$,

$$\vartheta(x) > \alpha\pi(x)\log x - \alpha x^\alpha \log x,$$

i.e.,

$$\frac{\vartheta(x)}{x} > \alpha\frac{\pi(x)\log x}{x} - \alpha\frac{\log x}{x^{1-\alpha}}.$$

This yields

$$\liminf_{x\to\infty} \frac{\vartheta(x)}{x} \ge \alpha \liminf_{x\to\infty} \frac{\pi(x)\log x}{x} - \alpha \liminf_{x\to\infty} \frac{\log x}{x^{1-\alpha}} = \alpha \liminf_{x\to\infty} \frac{\pi(x)\log x}{x}$$

and

$$\limsup_{x\to\infty} \frac{\vartheta(x)}{x} \ge \alpha \limsup_{x\to\infty} \frac{\pi(x)\log x}{x} - \alpha \limsup_{x\to\infty} \frac{\log x}{x^{1-\alpha}} = \alpha \limsup_{x\to\infty} \frac{\pi(x)\log x}{x}.$$

Since these are true for all $0 < \alpha < 1$, we obtain respectively

$$\liminf_{x\to\infty} \frac{\vartheta(x)}{x} \ge \liminf_{x\to\infty} \frac{\pi(x)\log x}{x}$$

and

$$\limsup_{x\to\infty} \frac{\vartheta(x)}{x} \ge \limsup_{x\to\infty} \frac{\pi(x)\log x}{x}.$$

□

36. Wiener's tauberian theorem

Wiener [59, Chapter III].

We say that a function $s : (0, \infty) \to \mathbb{R}$ is *slowly decreasing* if

$$\liminf(s(\rho v) - s(v)) \geq 0, \qquad v \to \infty, \quad \rho \to 1^+.$$

Widder [58, p. 211, Theorem 10b]: Wiener's tauberian theorem tells us that if $a \in L^\infty(0, \infty)$ and is slowly decreasing and if $g \in L^1(0, \infty)$ satisfies

$$\int_0^\infty t^{ix} g(t) dt \neq 0, \qquad t \in \mathbb{R},$$

then

$$\lim_{x \to \infty} \frac{1}{x} \int_0^\infty g\left(\frac{t}{x}\right) a(t) dt = A \int_0^\infty g(t) dt$$

implies that

$$\lim_{v \to \infty} a(v) = A.$$

It is straightforward to check the following by rearranging summation.

Lemma 4. *If $\sum_{n=1}^\infty a_n z^n$ has radius of convergence ≥ 1, then for $|z| < 1$,*

$$\sum_{n=1}^\infty a_n \frac{z^n}{1 - z^n} = \sum_{n=1}^\infty \left(\sum_{m|n} a_m\right) z^n.$$

Using Lemma 4 with $a_n = \Lambda(n)$ and $z = e^{-x}$ and using (7), we get

$$\sum_{n=1}^\infty \Lambda(n) \frac{z^n}{1 - z^n} = \sum_{n=1}^\infty \log(n) z^n. \tag{9}$$

Using (9) we have

$$\sum_{n=1}^\infty (\Lambda(n) - 1) \frac{e^{-nx}}{1 - e^{-nx}} = \sum_{n=1}^\infty (\log n - d(n)) e^{-nx}.$$

We follow Widder [58, p. 231, Theorem 16.6].

Theorem 5. *As $x \to 0^+$,*

$$\sum_{n=1}^\infty (\log n - d(n)) e^{-nx} = -\frac{2\gamma}{x} + O(x^{-1/2}).$$

Proof. Generally,

$$\begin{aligned}(1 - z) \sum_{n=1}^\infty z^n \sum_{m=1}^n a_m &= (1 - z) \sum_{m=1}^\infty a_m \sum_{n=m}^\infty z^n \\ &= (1 - z) \sum_{m=1}^\infty a_m \frac{z^m}{1 - z} \\ &= \sum_{m=1}^\infty a_m z^m.\end{aligned}$$

Using this with $a_m = \log m - d(m)$ and $z = e^{-x}$ gives

$$\begin{aligned}\sum_{n=1}^\infty (\log n - d(n))e^{-nx} &= (1-e^{-x})\sum_{n=1}^\infty e^{-nx}\left(\sum_{m=1}^n \log m - \sum_{m=1}^n d(m)\right)\\ &= (1-e^{-x})\sum_{n=1}^\infty e^{-nx}\left(\log(n!) - \sum_{m=1}^n d(m)\right).\end{aligned}$$

Using

$$\log(n!) = n\log n - n + O(\log n)$$

and

$$\sum_{m=1}^n d(m) = n\log n + (2\gamma-1)n + O(n^{1/2}),$$

we get

$$\log(n!) - \sum_{m=1}^n d(m) = -2\gamma n + O(n^{1/2}).$$

Therefore,

$$\sum_{n=1}^\infty (\log n - d(n))e^{-nx} = (1-e^{-x})\sum_{n=1}^\infty e^{-nx}(-2\gamma n + O(n^{1/2})).$$

One proves that there is some K such that for all $0 \le y < 1$,

$$(1-y)\left(\log\frac{1}{y}\right)^{1/2}\sum_{n=1}^\infty n^{1/2}y^n \le K,$$

whence, with $y = e^{-x}$,

$$\sum_{n=1}^\infty n^{1/2}e^{-nx} \le K\frac{x^{-1/2}}{1-e^{-x}}.$$

Also,

$$\sum_{n=1}^\infty ne^{-nx} = \frac{e^{-x}}{(1-e^{-x})^2},$$

and thus we have

$$\begin{aligned}\sum_{n=1}^\infty (\log n - d(n))e^{-nx} &= -2\gamma\frac{e^{-x}}{1-e^{-x}} + O(x^{-1/2})\\ &= -2\gamma\frac{1}{e^x-1} + O(x^{-1/2}).\end{aligned}$$

But

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + O(x),$$

so

$$\sum_{n=1}^\infty (\log n - d(n))e^{-nx} = -\frac{2\gamma}{x} + O(x^{-1/2}).$$

□

Define

$$f(x) = \sum_{n=1}^{\infty} (\Lambda(n) - 1) \frac{e^{-nx}}{1 - e^{-nx}},$$

and

$$h(x) = \sum_{n \le x} \frac{\Lambda(n) - 1}{n}.$$

and

$$g(t) = \frac{d}{dt} \left(\frac{te^{-t}}{1 - e^{-t}} \right).$$

First we show that h is slowly decreasing.

Lemma 6. *$h(x)$ is slowly decreasing.*

Proof. Using

$$\sum_{1 \le n \le x} \frac{1}{n} = \log x + \gamma + O(n^{-1}), \qquad x \to \infty,$$

we have, for $\rho > 1$,

$$\begin{aligned}
h(\rho x) - h(x) &= \sum_{x < n \le \rho x} \frac{\Lambda(n) - 1}{n} \\
&\ge - \sum_{x < n \le \rho x} \frac{1}{n} \\
&= - \sum_{1 \le n \le \rho x} \frac{1}{n} + \sum_{1 \le n \le x} \frac{1}{n} \\
&= - \log(\rho x) + \log x + O((\rho x)^{-1}) + O(x^{-1}) \\
&= - \log \rho + O((\rho x)^{-1}) + O(x^{-1}).
\end{aligned}$$

Hence as $x \to \infty$ and $\rho \to 1^+$,

$$h(\rho x) - h(x) \to 0,$$

which shows that h is slowly decreasing. □

The following is from Widder [58, pp. 231–232].

Lemma 7. *As $x \to \infty$,*

$$\frac{1}{x} \int_0^\infty g\left(\frac{t}{x}\right) h(t) dt = 2\gamma + O(x^{-1/2}).$$

Proof. Let $I(t) = 0$ for $t < 0$ and $I(t) = 1$ for $t \ge 0$. Writing

$$h(x) = \sum_{n=1}^{\infty} I(x - n) \frac{\Lambda(n) - 1}{n},$$

we check that for $x > 0$,

$$\begin{aligned}\int_0^\infty \frac{te^{-xt}}{1-e^{-xt}} dh(t) &= \sum_{n=1}^\infty \int_0^\infty \frac{te^{-xt}}{1-e^{-xt}} \frac{\Lambda(n)-1}{n} d(I(t-n))\\ &= \sum_{n=1}^\infty \int_0^\infty \frac{te^{-xt}}{1-e^{-xt}} \frac{\Lambda(n)-1}{n} d\delta_n(t)\\ &= \sum_{n=1}^\infty \frac{ne^{-nx}}{1-e^{-nx}} \frac{\Lambda(n)-1}{n}\\ &= f(x).\end{aligned}$$

On the other hand, integrating by parts,

$$\begin{aligned}f(x) &= \int_0^\infty \frac{te^{-xt}}{1-e^{-xt}} dh(t)\\ &= \int_0^\infty \frac{1}{x} \frac{xte^{-xt}}{1-e^{xt}} dh(t)\\ &= \int_0^\infty \frac{1}{x} \frac{xte^{-xt}}{1-e^{-xt}} dh(t)\\ &= \int_0^\infty \frac{1}{x} \frac{te^{-t}}{1-e^{-t}} dh\left(\frac{t}{x}\right)\\ &= \frac{1}{x} \frac{te^{-t}}{1-e^{-t}} h\left(\frac{t}{x}\right)\Big|_0^\infty - \int_0^\infty \frac{1}{x} g(t) h\left(\frac{t}{x}\right) dt\\ &= -\int_0^\infty \frac{1}{x} g(t) h\left(\frac{t}{x}\right) dt\\ &= -\int_0^\infty g(xt)h(t)dt.\end{aligned}$$

By Theorem 5, as $x \to 0^+$,

$$f(x) = -\frac{2\gamma}{x} + O(x^{-1/2}),$$

i.e., as $x \to 0^+$,

$$\int_0^\infty g(xt)h(t)dt = \frac{2\gamma}{x} + O(x^{-1/2}).$$

Thus, as $x \to \infty$,

$$\int_0^\infty g\left(\frac{t}{x}\right) h(t)dt = 2\gamma x + O(x^{1/2}).$$

□

The following is Widder [58, p. 232].

Lemma 8.

$$\int_0^\infty t^{-ix} g(t)dt = \begin{cases} -1 & x = 0\\ ix\zeta(1-ix)\Gamma(1-ix) & x \neq 0.\end{cases}$$

Proof.

$$\begin{aligned}\int_0^\infty t^{-ix} g(t)dt &= \int_0^\infty t^{-ix}\frac{d}{dt}\left(\frac{te^{-t}}{1-e^{-t}}\right)dt\\ &= \lim_{\delta\to 0}\int_0^\infty t^{-ix+\delta}\frac{d}{dt}\left(\frac{te^{-t}}{1-e^{-t}}\right)dt\\ &= \lim_{\delta\to 0}\left(t^{-ix+\delta}\frac{te^{-t}}{1-e^{-t}}\bigg|_0^\infty + (ix-\delta)\int_0^\infty t^{-ix+\delta-1}\frac{te^{-t}}{1-e^{-t}}dt\right)\\ &= \lim_{\delta\to 0}(ix-\delta)\int_0^\infty t^{-ix+\delta-1}\frac{te^{-t}}{1-e^{-t}}dt\\ &= \lim_{\delta\to 0}(ix-\delta)\int_0^\infty \frac{t^{(-ix+\delta+1)-1}e^{-t}}{1-e^{-t}}dt.\end{aligned}$$

Using

$$\int_0^\infty \frac{t^{s-1}}{e^t-1}dt = \zeta(s)\Gamma(s), \qquad \Re(s) > 1,$$

this becomes

$$\int_0^\infty t^{-ix}g(t)dt = \lim_{\delta\to 0^+}(ix-\delta)\zeta(1+\delta-ix)\Gamma(1+\delta-ix).$$

If $x = 0$, then using

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|), \qquad s \to 1,$$

we get

$$\lim_{\delta\to 0^+}(-\delta)\zeta(1+\delta)\Gamma(1+\delta) = -1.$$

If $x > 0$, then

$$\lim_{\delta\to 0^+}(ix-\delta)\zeta(1+\delta-ix)\Gamma(1+\delta-ix) = ix\zeta(1-ix)\Gamma(1-ix).$$

□

By Weiner's tauberian theorem, it follows that

$$\sum_{n=1}^\infty \frac{\Lambda(n)-1}{n} = -2\gamma.$$

Lemma 9.

$$h(x) = \int_{\frac{1}{2}}^x \frac{d(\psi(t)-[t])}{t}.$$

Proof. Let $I(t) = 0$ for $t < 0$ and $I(t) = 1$ for $t \geq 0$. Writing

$$\psi(x) = \sum_{n=1}^\infty I(x-n)\Lambda(n), \qquad [x] = \sum_{n=1}^\infty I(x-n),$$

we have

$$\begin{aligned}\int_{\frac{1}{2}}^{x} \frac{d(\psi(t)-[t])}{t} &= \int_{\frac{1}{2}}^{x} \frac{1}{t} d\left(\sum_{n=1}^{\infty} I(t-n)(\Lambda(n)-1)\right)\\ &= \int_{\frac{1}{2}}^{x} \frac{1}{t} \sum_{n=1}^{\infty} (\Lambda(n)-1) d\delta_n(t)\\ &= \sum_{1\le n\le x} \frac{\Lambda(n)-1}{n}\\ &= h(x).\end{aligned}$$

□

Thus, we have established that

$$\int_{\frac{1}{2}}^{\infty} \frac{d(\psi(t)-[t])}{t} = -2\gamma.$$

37. Hermite

Hermite [32]:

38. Levi-Civita

Levi-Civita [43]

39. Franel

Franel [25] and [24]

The next theorem shows that the set of points on the unit circle that are singularities of $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}$ is dense in the unit circle. Titchmarsh [56, pp. 160–161, §4.71].

Theorem 10. *For $|z| < 1$, define*

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n}.$$

Suppose that $p > 0, q > 1$ are relatively prime integers. As $r \to 1^-$,

$$(1-r)f(re^{2\pi i/q}) \to \infty.$$

Proof. Set $z = re^{2\pi ip/q}$ and write

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n\equiv 0 \pmod q} \frac{z^n}{1-z^n} + \sum_{n\not\equiv 0 \pmod q} \frac{z^n}{1-z^n}.$$

On the one hand,

$$
\begin{aligned}
(1-r) \sum_{n \equiv 0 \pmod q} \frac{z^n}{1-z^n} &= (1-r) \sum_{m=1}^\infty \frac{z^{mq}}{1-z^{mq}} \\
&= (1-r) \sum_{m=1}^\infty \frac{(re^{2\pi i p/q})^{mq}}{1-(re^{2\pi i p/q})^{mq}} \\
&= (1-r) \sum_{m=1}^\infty \frac{r^{mq}}{1-r^{mq}} \\
&= \frac{1-r}{1-r^q} \sum_{m=1}^\infty \frac{r^{mq}}{1+r^q+\cdots+r^{(m-1)q}} \\
&= \frac{1}{1+r+\cdots+r^{q-1}} \sum_{m=1}^\infty \frac{r^{mq}}{1+r^q+\cdots+r^{(m-1)q}} \\
&\geq \frac{1}{q} \sum_{m=1}^\infty \frac{r^{mq}}{m} \\
&= -\frac{1}{q} \log(1-r^q) \\
&\to \infty
\end{aligned}
$$

as $r \to 1$.

On the other hand, for $n \not\equiv 0 \pmod q$ we have

$$
\begin{aligned}
|1-z^n|^2 &= |1-r^n e^{2\pi i p n/q}|^2 \\
&= (1-r^n e^{2\pi i p n/q})(1-r^n e^{-2\pi i p n/q}) \\
&= 1 - r^n (e^{2\pi i p n/q} + e^{-2\pi i p n/q}) + r^{2n} \\
&= 1 - 2r^n \cos 2\pi p n/q + r^{2n} \\
&= 1 - 2r^n + 4r^n \sin^2 \frac{\pi p n}{q} + r^{2n} \\
&= (1-r^n)^2 + 4r^n \sin^2 \frac{\pi p n}{q}.
\end{aligned}
$$

So far we have not used the hypothesis that $n \equiv 0 \pmod q$. We use it to obtain

$$
\sin \frac{\pi p n}{q} \geq \sin \frac{\pi}{q}.
$$

With this we have

$$
|1-z^n|^2 \geq 4r^n \sin^2 \frac{\pi}{q},
$$

and therefore, as $r < 1$,

$$\begin{aligned}(1-r)\left|\sum_{n\not\equiv 0 \pmod q}\frac{z^n}{1-z^n}\right| &\le (1-r)\sum_{n\not\equiv 0 \pmod q}\frac{|z|^n}{|1-z^n|}\\ &\le (1-r)\sum_{n\not\equiv 0 \pmod q}\frac{r^n}{2r^{n/2}\sin\frac{\pi}{q}}\\ &\le \frac{1-r}{2\sin\frac{\pi}{q}}\sum_{n=0}^\infty r^{n/2}\\ &= \frac{1-r}{2\sin\frac{\pi}{q}}\cdot\frac{1}{1-\sqrt{r}}\\ &= \frac{1+\sqrt{r}}{2\sin\frac{\pi}{q}}\\ &< \frac{1}{\sin\frac{\pi}{q}}.\end{aligned}$$

□

40. Wigert

The following result is proved by Wigert [60]. Our proof follows Titchmarsh [55, p. 163, Theorem 7.15]. Cf. Landau [41].

Theorem 11. *For $\lambda < \frac{1}{2}\pi$ and $N \ge 1$,*

$$\sum_{n=1}^\infty d(n)e^{-nz} = \frac{\gamma}{z} - \frac{\log z}{z} + \frac{1}{4} - \sum_{n=0}^{N-1}\frac{B_{2n+2}^2}{(2n+2)!(2n+2)}z^{2n+1} + O(|z|^{2N})$$

as $z \to 0$ in any angle $|\arg z| \le \lambda$.

41. Unsorted

In 1892, in volume VII, no. 23, p. 296 of the weekly *Naturwissenschaftliche Rundschau*, it is stated that for the year 1893, one of the six prize questions for the Belgian Academy of Sciences in Brussels is to determine the sum of the Lambert series

$$\frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \cdots,$$

or if one cannot do this, to find a differential equation that determines the function.

Gram [27] on distribution of prime numbers.

Hardy [30]

Bohr and Cramer [1, p. 820]

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