PROOF OF THE PENTAGONAL NUMBER THEOREM

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1752 Bibliothèque impartiale (juillet et août), tome VI, première partie, article IX, "Demonstration de la Loi d'une suite de termes de la Quantité composée qui est faite par la multiplication des Binomes $1-x$, $1-x^2$, $1-x^3$, & c.", pp. 111–126.

$$y - 2y1' + y2' = 0$$

That is, $y_p - 2y_{p+1} + y_{p+2} = 0$, $p \geq 0$.

Let

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 + zx + z'x^2 + z''x^3 + z'''x^4 + z^{\mathrm{IV}}z^5 + \cdots$$

In other words,

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 + \sum_{p\geq 0} z_p x^{p+1}.$$

Let

$$S = (1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$$

Let $i = 1+x$, $i' = 1+x^2$, $i'' = 1+x^3$, $i''' = 1+x^4$, etc. In other words,

$$i_p = 1 + x^{p+1}, \qquad p \geq 0.$$

Let $k = 1$, $k' = 5$, $k'' = 12$, $k''' = 22$, $k^{\mathrm{IV}} = 35$, etc. In other words,

$$k_p = \frac{(p+1)(3p+2)}{2}, \qquad p \geq 0.$$

We shall prove that

$$S = 1 - ix^{k'} - i''x^{k''} + i'''x^{k'''} - i^{\mathrm{IV}}x^{k^{\mathrm{IV}}} + \cdots$$

Let $m = 1-x$, $m' = 1-x^2$, $m'' = 1-x^3$, $m''' = 1-x^4$, $m^{\mathrm{IV}} = 1-x^5$, etc. In other words,

$$m_p = 1 - x^{p+1}, \qquad p \geq 0.$$

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