# THE HARMONIC TRIANGLE IN MENGOLI'S AND LEIBNIZ'S WORKS

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## 1.- Introduction[1].

Mathematics in the seventeenth century has specific features. Firstly, the significant development of geometrical researches into the calculation of quadratures and volumes, due above all to the heritage of classic mathematics as exemplified by Euclid's and Archimedes' works. Secondly, at the same time, the appearance and diffusion of work *In Artem Analyticen Isagoge* (1591) by François Viète (1540-1603) promoted the use of symbols in the mathematics connecting the algebra with the geometry. Viète introduced the specious logistic, that is, calculations with "species", in contrast to the "numerous logistic"; that is, calculations with numbers, which was already used in the Renaissance algebras. The "species" used in Viète's analytic art (or algebra) consisted of all kinds of magnitudes, numerical magnitudes, such as natural and rational numbers, and also geometric magnitudes such as lengths, areas, volumes or angles. As Viète's work came to prominence at the beginning of the seventeenth century, other authors, like René Descartes (1596-1650) with his work *La Géométrie* (1637), and later Pietro Mengoli (1627-1686) and Gottfried Wilhelm Leibniz (1646-1716), among others, began to consider the utility of symbolic language and of algebraic procedures for solving all kinds of problems[2]. A further feature of mathematics in the seventeenth century

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2 There are many useful studies on Descartes' *La Géometrie*, including BOS (2001), 225–412; GIUSTI (1987), 409–432, and MANCOSU (1996), 62–84. On the use of algebraic procedures in Mengoli's works, see MASSA-ESTEVE (1998, 2006b) and MASSA-ESTEVE (2012). On Leibniz's mathematics there are many useful studies, including KNOBLOCH (1973, 2002), HOFMANN (1974), RABOUIN (2018), and PROBST (2015). It is also necessary to mention Pierre de Fermat (1601–1665), who was one of the mathematicians who used algebraic analysis to solve geometric problems. He did not publish any of his work during his lifetime, although it circulated in the form of letters and manuscripts and was referred to in other publications. On Fermat, see FERMAT (1891–1922), 65–71 and 286–292 and MAHONEY (1973), 229–232.

was the beginning of the use of the infinite in mathematics, that is, the introduction of infinity procedures, especially in Mengoli's and Leibniz's works. In fact, these three features of the seventeenth century mathematics are found in the mathematical object involved in this research: the harmonic triangle.

The harmonic triangle is deduced from the arithmetical triangle, which is the most famous set of numbers in mathematics arranged in a triangular table; it was useful in many fields and had been studied since ancient times and by many civilizations[3]. The rule for forming the arithmetical triangle is simple: every row begins and ends with 1, and the other numbers are obtained by the addition of two numbers more near the row immediately above (see Figure 1).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Figure 1. The arithmetical triangle

The numbers that form the arithmetical triangle arranged diagonally are well known and date back at least as far the ancient Greeks, if not earlier, and are known as the figurate numbers (triangulars, tetrahedrals or pentagonals)[4]. The numbers in the rows of the triangle were subsequently recognized as the terms of a binomial development (now called binomial coe-

3 The fascination exerted by the many properties and applications of the arithmetical triangle is reflected in the many references made by mathematicians who have studied it. References to the history of the arithmetical triangle are numerous. CASSINA (1923), for instance, who quotes Stifel, Tartaglia, Oughtred, Brasser, Scheubelius, Briggs, Harriot, Faulhaber and Van Schooten; EDWARDS (1987, 2002), who quotes Faulhaber, Oughtred, Briggs and Pascal and MASSA-ESTEVE (1998) who quotes Mengoli.

4 In the first diagonal after the unities, the numbers are the succession of positive integers: 1, 2, 3, 4 and so on. In the second diagonal, the numbers are called the triangular numbers: 1, 3, 6, 10, etc., which are sums of successive positive integers (1=1; 3=1+2; 6=1+2+3; 10=1+2+3+4). In the third diagonal, the numbers are called the tetrahedral or pyramidal numbers: 1, 4, 10, 20, etc., which are sums of successive triangular numbers (1=1; 4=1+3; 10=1+3+6; 20=1+3+6+10). In the fourth diagonal, they are called the pentagonal or triangulo-triangular numbers: 1, 5, 15, 35, etc., which are sums of successive pyramidal numbers, and so on. For references on the figurate numbers, see BEERY (2009) and BEERY; STEDALL (2009).

fficients), and later on, as may already be seen in Pascal's arithmetical triangle, the numbers apply to solving combinatorial problems. Thus, depending on the context in which they appear, we may refer to three interpretations of the numbers of the arithmetical triangle as follows: the figurate numbers, the binomial numbers and the combinatory numbers[5]. Indeed, in the work, written in 1654 and published in 1665, *Traité du Triangle arithmétique, avec quelques autres petits traités sur la même matière. Usage du Triangle Arithmétique pour les ordres numériques, pour les combinaisons, pour trouver les puissances des binômes et des apotomes...*, Pascal, after defining the arithmetical triangle, wrote and subsequently published three further treatises in which he put forward and explained in a very clear style these three interpretations, their properties and uses[6]. For this reason, this arithmetical triangle has been known throughout history under the name of Pascal's arithmetical triangle.

However, in this article it is not our aim to study the arithmetical triangle, but to analyze the mathematical treatment of the harmonic triangle in Mengoli's and Leibniz's works in order to show the relevance of these triangular tables for the development of some mathematical ideas in the seventeenth century.

The harmonic triangle (also known as Leibniz's triangle) that arises from the arithmetical triangle enables one to work with infinite series and can even be used to calculate areas[7]. The harmonic triangle is formed by the reciprocal of the elements of the arithmetical triangle times their own numbers. Its construction is as follows: take the arithmetical triangle and, considering the vertex apart, multiply each row by the number of terms it contains; that is, the first row by two, the second by three; that is, the row ($m$) by ($m$ +1). Next, the reciprocals of these numbers are found and placed in a triangular table, thus obtaining the harmonic triangle. Indeed, the sides of this triangle form the harmonic series (see Figure 2):

5 EDWARDS (1987, 2002), xiii-xiv.

6 In fact, Pascal regarded the arithmetical triangle as a mathematical object useful for obtaining indefinitely new results, such as combinations or powers of binomial. References to Pascal's arithmetical triangle are numerous; see especially CASSINA (1923), BOSMANS (1924), KNOBLOCH (1973), EDWARDS (1987, 2002) and MASSA; ROMERO (2009).

7 It should also be noted that the texts on the arithmetical triangle and the harmonic triangle are suitable for use in the teaching-learning of mathematics since they allow working with the sums of powers of the integers and with the infinite sums, see MASSA; ROMERO (2009).

1

1/2 1/2

1/3 1/ 6 1/3

1/4 1/12 1/12 1/4

1/5 1/20 1/30 1/20 1/5

1/6 1/30 1/60 1/60 1/30 1/6

Figure 2. The Harmonic Triangle

The harmonic triangle was defined by Leibniz in 1673 and its definition is related to the successive differences of the harmonic series[8]. Leibniz studied it in many different texts throughout his life (see section third below). Mengoli, rather at the same time, used the harmonic triangle as a triangular table for obtaining some results in quadratures. Few references to harmonic triangle exist, and there is no research about the relationship between Mengoli's and Leibniz's approaches to harmonic triangle[9]. Thus, in this article we analyze and compare the independent treatment of harmonic triangle by Mengoli and Leibniz in their works, referring to their sources, their aims and their uses. We show that, on the one hand, Mengoli uses triangular tables as a tool of calculus, and uses the harmonic triangle, but as a triangular table more, to perform quadratures through one procedure called by him "homology". On the other hand, at the same time Leibniz, defines the harmonic triangle from the study on harmonic series, analyses its properties and uses it to perform the summations of infinite series through one procedure called by him their "sums of all the differences".

8 See EDWARDS (2002), 106.

9 The researches on Leibniz's harmonic triangle include an article by HENRY (1881), expressed in modern notation; more recently, a paper by DE MORA (1990) on Leibniz's harmonic triangle; an article by SERFATI (2013) about Leibniz's harmony, focused in the relationship between Descartes and Leibniz, and further article by SERFATI (2014) about the "Sum of all Differences". On Mengoli's harmonic triangle, MASSA-ESTEVE (1998) and MASSA-ESTEVE; DELSHAMS (2009) showed its use in Mengoli's *Circolo* to perform quadratures, specially the quadrature of circle.

## 2.- Mengoli's triangular tables and harmonic triangle.

Mengoli's original treatment of the arithmetical triangle and the triangular tables derived from it, as the extensive use in his works, constitutes one of the foundations of Mengoli's mathematics. Indeed, throughout Mengoli's *Geometriae Speciosae Elementa* (1659, hereafter *Geometria*) and *Circolo* (1672), he introduced triangular tables as useful algebraic tools for calculations. In *Geometria*, Mengoli was probably not familiar with the treatise by Pascal, since it was published in 1665, but he may have read Herigone's *Cursus* (1634), which was Pascal's source[10].

In the *Elementum primum* of Mengoli's *Geometria,* the terms of the triangular tables are numbers and serve to obtain the development of any binomial power, including the arithmetical triangle[11] . In the *Elementum secundum,* the terms of the triangular tables are summations and serve to obtain the values of the summations[12]. Finally, in the *Elementum sextum* [13] of the *Geometria* and in the *Circolo* (1672), the terms of the triangular tables are geometric figures or forms and are employed to obtain the values of the quadratures of these geometric figures (expressed in the harmonic triangle). Mengoli`s originality resided not only in the presentation of these triangular tables, but also in his treatment of them. On the one hand, he used the arithmetical triangle and the symbolic notation to create other triangular tables with algebraic expressions, clearly stating their laws of formation, while on the other hand he employed the relations between these algebraic expressions and the terms of the arithmetical triangle in order to find rules that applied indefinitely to any exponent and also to prove them. In addi-

10 On Mengoli's triangular tables see MASSA-ESTEVE (1998), 27-62.

11 Indeed, as Pascal explained in his work: "I do not give the demonstration of all this, because others, such as Hérigone, have already treated it. Besides, it is self-evident." "Je ne donne point la démonstration de tout cela, parce que d'autres en ont deja traité, comme Hérigone, outre que la chose est évidente d'elle-même" PASCAL (1954).

12 In fact, Mengoli placed these summations in triangular tables and obtained new relations between the terms of these tables, which allowed him to calculate the values of the summations. In this way, while not arriving at a common rule, he was nevertheless able to calculate the summations of powers and products of powers for any positive integer indefinitely. Unlike other contemporaries, Mengoli did not calculate the summations of powers for some values in order to obtain a general rule, but rather gave and proved the rule and applied it by assigning values for 36 cases. On Mengoli's summations of powers of integers see MASSA-ESTEVE (2006a), 89-92.

13 In the *Elementum quartum* and *quintum,* there are not triangular tables.

tion, Mengoli accepted without question that if a result was true for one row of the table, it was also true for all rows and there was no need to prove it in the remaining rows.

In *Geometria*, Mengoli introduced his method based on the construction of triangular tables and on the use of the theory of quasi-proportions[14] to compute the quadratures of these geometric figures $FO.a^m r^n$. Indeed, he proved that all quadratures of geometric figures with the appropriate coefficients have value 1[15].

Later, in *Circolo* (1672), he displayed in an infinite triangular table the numerical values of their quadratures, which is nothing other than the harmonic triangle. Indeed, Mengoli calls his harmonic triangle simply "terza tavola triangulare". Mengoli constructed the harmonic triangle from the arithmetical triangle, and recognized as a source for this the books of specious algebra and of angular sections by Viète (see Figure 3).

> *"Here, we can see that by adding one unity to the sides and at the top, as in the first element of my Geometria Speciosa, I represent it, define it and explain its properties; while in the second, the third and the sixth, I also use it. And Viète was the author, in Algebra Speciosa, in his book on Angular Sections[16]."*

14 On Mengoli's theory of quasi –proportions see MASSA-ESTEVE (1997), 268-277.

15 On Mengoli's quadratures see MASSA-ESTEVE (2006a), 92-109.

16 "Anzi vedasi con l'aggiunta delle unità ne i lati, e in cima, come nel primo de gli Elementi Della mia Geometria Speciosa io la rappresento, e la definisco, e spiego ivi le sue proprietà, en el secondo, terzo, e sesto, l'uso ancora, e il Vietta, che ne fu l'autore, nell' Algebra Speciosa, en el suo Libro delle Settioni angolari.", MENGOLI (1672), 3. See similar triangular tables in VIETE (1983), 297-299.

7 Anzi vedasi con l'aggiunta delle vnità ne i lati, e in cima, come nel primo de gli Elementi della mia Geometria Speciosa io la rappresento, e la definisco, e spiego iui le sue proprietà, e nel secondo, terzo, e sesto, l'vso ancora, e il Vietta, che ne fù l'autore, nell'Algebra Speciosa, e nel suo Libro delle Settioni angolari.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

8 E moltiplicando i termini di ciascuna base, per la loro moltitudine, i due della prima base, per 2, i tre della seconda, per 3, e quattro della terza, per 4, e così i termini delle altre basi, per gli altri numeri, facciasi vn'altra tavola triangolare, che segue.

4 *Circolo del Mengoli.*

1
2 2
3 6 3
4 12 12 4
5 20 30 20 5
6 30 60 60 30 6
7 42 105 140 105 42 7
8 56 168 280 280 168 56 8

9 E denominando l'vnità, per ciascuno di questi termini, facciasi la terza tauola triangolare delle parti, che segue.

1
1(2) 1(2)
1(3) 1(6) 1(3)
1(4) 1(12) 1(12) 1(4)
1(5) 1(20) 1(30) 1(20) 1(5)
1(6) 1(30) 1(60) 1(60) 1(30) 1(6)
1(7) 1(42) 1(105) 1(140) 1(105) 1(42) 1(7)
1(8) 1(56) 1(168) 1(280) 1(280) 1(168) 1(56) 1(8)

Figure 3. Mengoli's construction of the harmonic triangle from the arithmetical triangle (MENGOLI, 1672: 3-4)

Mengoli then identified these numbers, the inverse of the coefficients of algebraic expressions in geometric figures, with the values of their quadratures of the *Tabula Formosa* by homology. Homologous terms for Mengoli are terms situated in the same place in each of the tables, which also preserve their proportions. Thus, the square of the vertex is homologous to the unity, and the geometric figures in the first row are homologous to ½ and so on (see Figure 4).

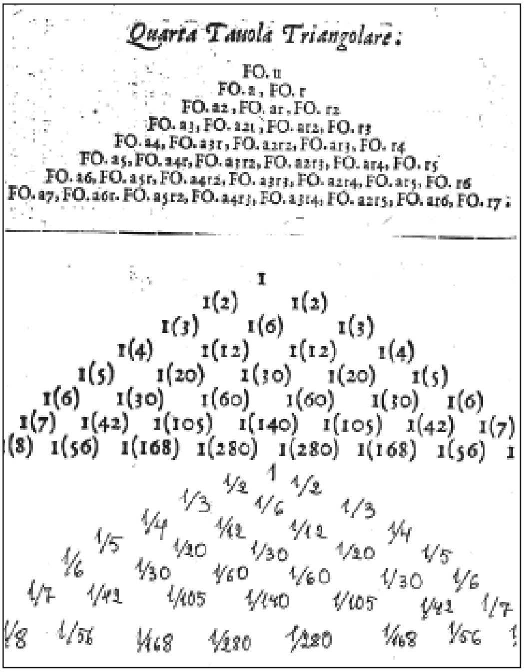


Figure 4. Mengoli's *Tabula Formosa,* Mengoli's harmonic triangle (MENGOLI, 1672 :4 and 7), and the author's harmonic triangle with fractions.

Mengoli explained this relation of the homologous terms rhetorically:

*"15. Which [figures] I have proved to be proportional to the quantities arranged in the third triangular table; and the square of the Rational FO.u, is homologous to the unity; and the triangles FO.a, and FO.r, are homologous to the half; and the [parabolas] FO.a2, FO.ar, FO.r2, are homologous to a third, a sixth and a third of the unity; and FO.a3, FO.a2.r, FO.a.r2, FO.r3, are homo-*

*logous to a fourth, a twelfth, a twelfth and a fourth of the unity, and of the square itself; and thus all the other forms in order, as also in the sixth element, one may deduce by corollary from Proposition 10*[17]*."*

Indeed, Mengoli identified the values of the quadratures with the areas of the corresponding figures, calling the homologous terms, in modern notation (compare Figure 4 and Figure 5),

$$\int_0^1 1\,dx = 1$$

$$\int_0^1 x\,dx = 1/2 \qquad \int_0^1 (1-x)\,dx = 1/2$$

$$\int_0^1 x^2\,dx = 1/3 \qquad \int_0^1 x\,(1-x)\,dx = 1/6 \qquad \int_0^1 (1-x)^2\,dx = 1/6$$

$$\int_0^1 x^3\,dx = 1/4 \quad \int_0^1 x^2\,(1-x)\,dx = 1/12 \quad \int_0^1 x\,(1-x)^2\,dx = 1/12 \quad \int_0^1 (1-x)^3\,dx = 1/4$$

Figure 5. Identification of the harmonic triangle and the areas of the figures

In fact, Mengoli's aim was the computation of the quadrature of the semicircle of diameter 1, which corresponds to the integral $\int_0^1 \sqrt{x\,(1-x)}\,dx$. Mengoli subsequently applied his method to computing infinitely many values of geometric figures through the interpolation of both tables for half-integer values of the exponents, where the notation with halves in the symbols is introduced. First, he displayed the interpolated geometric figures $\int_0^1 x^{m/2}\,(1-x)^{n/2}\,dx$ for natural numbers *m* and *n*, in an infinite interpolated triangular table, which I will call the interpolated *Tabula Formosa* (see Figure 6).

17 "15. Le quali tutte hò dimostrato, che sono proportionali, come le quantità disposte nella terza tavola triangulare; ed è il quadrato Della Rationale FO. u, homologa all'unità; e i triangoli FO. a, e FO. r, homologhi allà metà; e le FO. a2, FO. ar, FO. r2, homologhe alle parti dell'unità terza, sesta, e terza; e le FO. a3, FO. a2r, FO. ar2, FO. r3, homologue alle parti quarta, duodecima, duodecima, e quarta, dell'unità, e dello stesso quadrato; e cosí tutte le altre forme per ordine: como ivi en el sesto elemento si può dedurrè per corollario dalla prop. 10." MENGOLI (1672), 6.

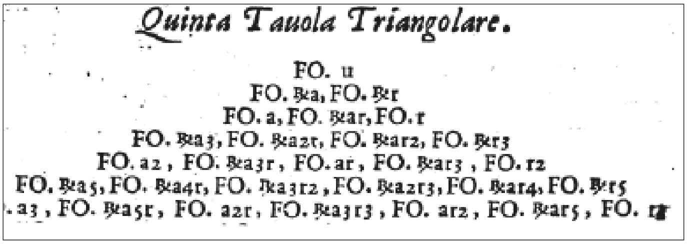
Quinta Tauola Triangolare.
FO. u
FO. ℞a, FO. ℞r
FO. a, FO. ℞ar, FO. r
FO. ℞a3, FO. ℞a2r, FO. ℞ar2, FO. ℞r3
FO. a2 , FO. ℞a3r , FO. ar , FO. ℞ar3 , FO. r2
FO. ℞a5, FO. ℞a4r, FO. ℞a3r2 , FO. ℞a2r3, FO. ℞ar4, FO. ℞r5
. a3 , FO. ℞a5r , FO. a2r , FO. ℞a3r3 , FO. ar2 , FO. ℞ar5 , FO. r3

Figure 6. Mengoli's interpolated *Tabula Formosa* (MENGOLI, 1672: 5)

He then obtained an infinite interpolated triangular table of values of their quadratures. With the help of the properties of combinatorial triangle, Mengoli was then able to fill the interpolated combinatorial triangle and thereby constructed the interpolated harmonic triangle (see Figure 7), except for an unknown number "*a*"[18] closely related to the quadrature of the circle (1/2 $a = \pi/8$). In fact, he obtained successive approximations of the number "*a*" in order to approximate the number $\pi$ up to eleven decimal places[19].

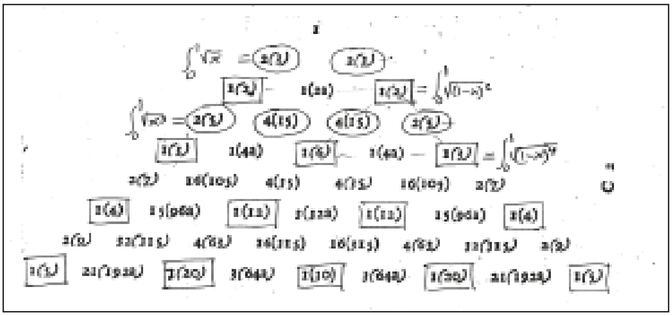

Figure 7. Mengoli's interpolated harmonic triangle with our annotations (MENGOLI, 1672: 19)

Once again the proportion between the geometric figures (interpolated *Tabula Formosa*) and the homologous values of their areas (interpolated harmonic triangle) is preserved thanks to their construction. Mengoli makes it clear that this relation, referred to by him as "the co-ordination of the two

18 MASSA-ESTEVE; DELSHAMS (2009), 337-345.

19 Explanations and proofs for theses calculations can be found in MASSA-ESTEVE (1998), 193-234.

tables", is maintained, since the interpolated tables (geometric figures and values of areas) are reproduced inside the first tables.

Mengoli used the harmonic triangle to perform quadratures and also interpolated his third triangular table (harmonic triangle) to perform the quadratures of the figures with half-integer in the exponent. However, Mengoli's aim was not to define this third triangular table (harmonic triangle) as a singular table, or to show its properties, but rather to calculate the quadrature of the circle using the interpolated harmonic triangle. On the other hand, Leibniz defined the harmonic triangle and showed its properties from the harmonic series, as we analyze in the following section.

## 3.- Leibniz's harmonic triangle.

From the beginning of Leibniz's works on mathematics, he stressed his interest in the infinite and for the difficulties that it involved, as may be seen in his treatise on the arithmetic quadrature of the circle, the ellipse and the hyperbola on which he worked during his stay in Paris and in his writings on series[20]. From 1672, and on the recommendation of Christian Huygens (1629-1695)[21], Leibniz was engaged in the demonstration of the divergence of the harmonic series (1, 1/2, 1/3, 1/4, ...), and in that of the sum of the infinite series that have for denominator figurate numbers (1, 1/3, 1/6, 1/10, ...)[22]. His work on the addition of these series eventually led Leibniz to the construction the harmonic triangle from Pascal's arithmetic triangle.

Leibniz's main texts on the harmonic triangle are as follows[23]: 1) *Differentiae*

20 KNOBLOCH (2002), 59-73, and HOFMANN (1974), 14-15.

21 Huygens had a considerable influence on Leibniz's mathematics during his stay in Paris (1672-1676), HOFMANN (1974). Oldenburg drew Huygen's attention to Leibniz ANTOGNAZZA (2009), 143-144.

22 In Oldenburg's correspondence we find a first letter dated February 26th, 1673, in which Leibniz expressed an interest in the work on series by Mengoli. Oldenburg replied (March 6th, 1673) by forwarding a letter to him by John Collins, in which this author explained that Mengoli had found the sum of infinite series with the reciprocal figurate numbers and the proof of the impossibility of the sum of the harmonic series in 1650. On Leibniz's and Mengoli's divergence of harmonic series, see in this volume KNOBLOCH (2018) and on Mengoli's sum of infinite series with the reciprocal figurate numbers see GIUSTI (1991).

23 I wish to acknowledge Siegmund Probst for these references on Leibniz's harmonic triangle. In Leibniz's references, the series is denoted by Roman numerals, the volume by Arabic numerals. II, 3 means: Series II, volume 3.

*Numerorum Harmonicorum et Reciprocorum Triangularium* AVII, 3, N2, 10-16 (September/October 1672); 2) *Leibniz für Jean Gallois* AII, 1, N109, 342-356 (End 1672); 3) *Leibniz für Die Royal Society* AIII, 1, N4, 22-29 (3/13 February 1673)[24]; 4) *De Triangulo Harmonico* AVII, 3, N30, 336-341 (End 1673-Mid 1674); 5) *De Triangulo Harmonico.* Under this title we find three manuscripts: a) *De Progressione Harmonica,* b) *Triangulum harmonicorum et Triangulum Pascalium* and c) *Origo inventionis trianguli harmonici* AVII, 3, N53, 704-714 (December 1675-February 1676) and a letter to L'Hospital, *Leibniz an Guillaume François de l'Hospital* AIII, 6, N84, 249-257 (December 1694)[25].

Thus, one may see that Leibniz began work on the harmonic series in a text of 1672 entitled *Differentiae Numerorum Harmonicorum et Reciprocorum Triangularium,* in which he presented the differences of terms of the harmonic series, but did not deal with the harmonic triangle.

In his visit to Paris (1672-1676), Huygens recommended Leibniz to read *Arithmetica Infinitorum* (Oxford, 1656) by John Wallis (1616-1703) and *Opus geometricum* (Antwerp, 1647) by Grégoire de Saint-Vincent (1584-1667), in which summations of infinite terms of series are treated. Thus, the second paper that Leibniz wrote towards the end of 1672 (*Leibniz für Jean Gallois*) was entitled *Accessio ad Arithmeticam Infinitorum* (referred to Wallis' work) and contained his results on the sums of the reciprocal figurate numbers. Leibniz intended to send it to the secretary of the Académie Royale des Sciences and editor of the *Journal des Sçavans*, Jean Gallois, for publication in the journal, but the paper remained unpublished because no new numbers of the journal were issued in the following year[26]. In this work, Leibniz first explained the arithmetical triangle, with quotes from Pascal's work and references to the figurate numbers with the names: *Triangularium*, *Pyramidalium*, *Triangulo-Triangularium*, *Triangulo-Pyramidalium*, *Pyramido-Pyramidalium* (see Figure 8).

24 There are two more texts in which Leibniz treated the harmonic series, and also his discussions with Tschirnhaus, but not the harmonic triangle: *Summa Progressionis Harmonicae II,* LEIBNIZ (End 1673-Mid 1674), AVII, 3, N28, 2003: 320-326 and *De Progressione Harmonica et De differentiis Differentiarum,* LEIBNIZ (End 1673-Mid 1674), AVII, 3, N29, 2003: 327-336.

25 In 1714, Leibniz once again explained his researches on the harmonic triangle in a text entitled *Historia et Origo Calculi Differentialis* (after September 1714).

26 ANTOGNAZZA (2009), 143

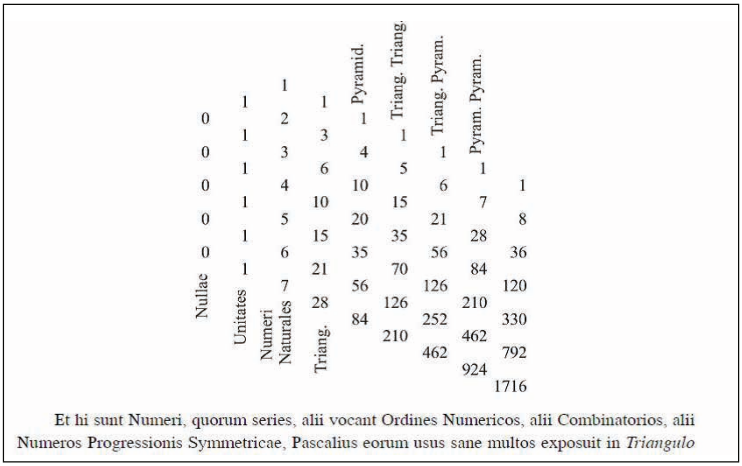


Figure 8. Pascal's arithmetical triangle in Leibniz's text of 1672 (LEIBNIZ, 1672, AII, 1, N109, 2006: 345)

Leibniz went on to analyze the fractions with numerator the unity and denominator the figurate numbers as well as providing a rule of relation between the "exponentes" for the sums of these fractions, always using the preceding summation. Leibniz placed these results in a rectangular table (see Figure 9).

Res ex hac Tabula clarius patet:

Series Fractionum Progressionis Arithmeticae Replicatae

| | semel | bis | ter | quater | quinquies | sexies | septies |
|---|---|---|---|---|---|---|---|
| Exponentes | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| | Unitatum | Naturalium | Triangularium | Pyramidalium | Triangulo-Triangularium | Triangulo-pyramidalium | Pyramido-pyramidalium |
| | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ |
| | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ |
| | $\frac{1}{1}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{10}$ | $\frac{1}{15}$ | $\frac{1}{21}$ | $\frac{1}{28}$ |
| | $\frac{1}{1}$ | $\frac{1}{4}$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{35}$ | $\frac{1}{56}$ | $\frac{1}{84}$ |
| | $\frac{1}{1}$ | $\frac{1}{5}$ | $\frac{1}{15}$ | $\frac{1}{35}$ | $\frac{1}{70}$ | $\frac{1}{126}$ | $\frac{1}{210}$ |
| | $\frac{1}{1}$ | $\frac{1}{6}$ | $\frac{1}{21}$ | $\frac{1}{56}$ | $\frac{1}{126}$ | $\frac{1}{252}$ | $\frac{1}{462}$ |
| | $\frac{1}{1}$ | $\frac{1}{7}$ | $\frac{1}{28}$ | $\frac{1}{84}$ | $\frac{1}{210}$ | $\frac{1}{462}$ | $\frac{1}{924}$ |
| Summae | $\frac{0}{0}$ | $\frac{1}{0}$ | $\frac{2}{1}$ | $\frac{3}{2}$ | $\frac{4}{3}$ | $\frac{5}{4}$ | $\frac{6}{5}$ etc. |

Figure 9. Rectangular table with the sums of the reciprocal figurate numbers (LEIBNIZ, 1672, AII, 1, N109, 2006: 347)

It is worth drawing attention to the infinite sum of unities equal to 0/0 and the infinite sum of harmonic series equal to 1/0, which he will present again later these results and he will call a conjecture. Leibniz did not present the proof of the results of other summations on this occasion, but subsequently went on to prove them in two ways, as explained below.

In the text *Leibniz für Die Royal Society* of 1673, Leibniz once again analyzed the arithmetical triangle and the sum of the reciprocal figurate numbers. However, in the paper entitled *De triangulo harmonico* (1673-1674) we find an illustration of the harmonic triangle accompanied by the following sentence "The harmonic triangle. Admirable properties of the numbers of harmonic series"[27] (see Figure 10).

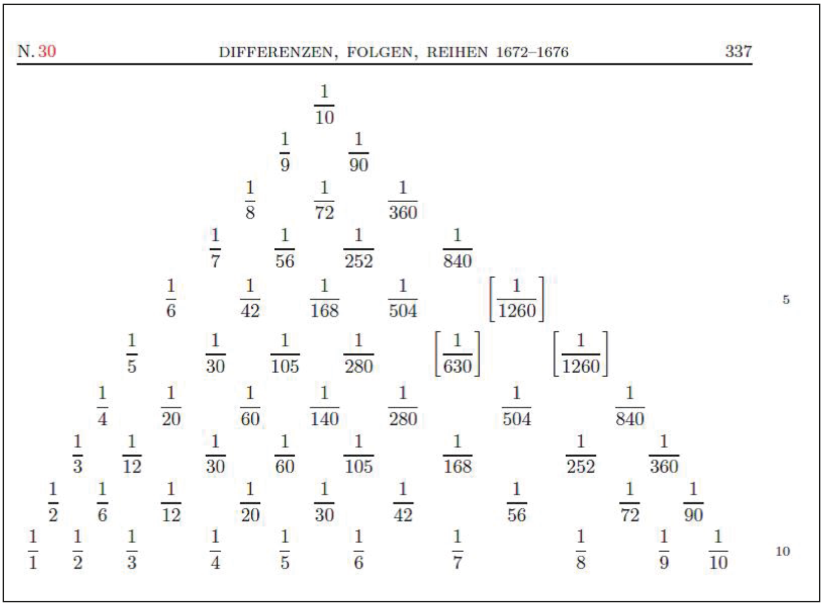


Figure 10. Representation of the harmonic triangle in Leibniz's *De Triangulo Harmonico* (LEIBNIZ, 1673-1674, AVII, 3, N30, 2003: 337)

In December, 1675, Leibniz presented a text also entitled "De Triangulo Harmonico" consisting of three sections, the first of which is called "De progressione harmonica" and contains the definition of harmonic series and discusses the progression on the harmonic triangle.

27 "De triangulo harmonico. Proprietas admirabilis numerorum progressionis harmonicae", LEIBNIZ (End 1673-Mid 1674), AVII, 3, N30, 2003: 336.

The harmonic triangle devised by Leibniz was constructed from the successive differences of the harmonic series. Given the harmonic series (which is the side of the triangle or first column): 1, 1/2, 1/3, 1/4, 1/5 … , he calculates their differences:

1–1/2 = 1/2; 1/2–1/3 = 1/6; 1/3–1/4 = 1/12; 1/4–1/5 = 1/20... And consequently he obtains the first diagonal (or second column): 1/2, 1/6, 1/12, 1/20,... of the harmonic triangle.

And again performing the differences of these terms, he obtains the second diagonal (or third column): 1/3, 1/12, 1/30, 1/60,... thereby arriving at this harmonic triangle (see Figure 11)[28],

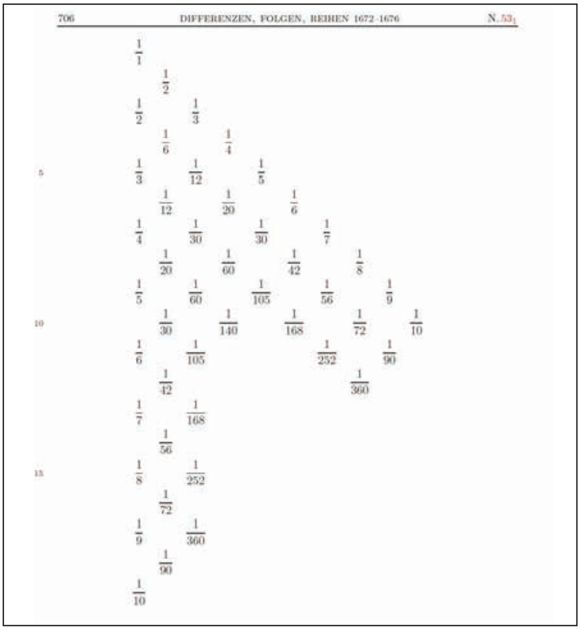


Figure 11. Harmonic triangle constructed by Leibniz (LEIBNIZ, 1675, VII, 3C, N531, 2003: 706)

Later, in February 1676, in the second section "Triangulum Harmonicum et Triangulum Pascalii", Leibniz compared his harmonic triangle with Pascal's arithmetic triangle and described its uses, both for the sums of the finite series in the arithmetic triangle and infinite series in the harmonic triangle, and for the quadratures of parabolas and hyperbolas, stating as follows (see Figure 12),

28 See COSTABEL(1978), 81-101.

*"Pascal's arithmetic triangle serves for the discovery of the finite sums of integer numbers and for square parabolas, in all degrees, in which the figures lack asymptotes[29]."*

We show by example the use of the arithmetical triangle for finding the summations and their powers from the properties of formation of the triangle and of the figurate numbers[30]. We consider the properties of triangular numbers: 1, 3, 6, 10, 15,.., obtained as semi-products of two integer numbers, 1·2, 2·3, 3·4, 4·5, ... , and that in addition they are the summation of the preceding integer numbers, expressed in the following way:

| | |
|---|---|
| 1 = 1 | 1 = 1/2 (1·2); |
| 3 = 1 + 2 | 3 = 1/2 (2·3); |
| 6= 1 + 2 + 3 | 6 = 1/2 (3·4); |
| 10 = 1 + 2 + 3 + 4 | 10 = 1/2 (4·5)… |

From these equalities the sum of the *n* first integers can be obtained, which expressed in modern notation are ½ $n\cdot(n+1)$.

In the same way, the tetrahedral (which Leibniz called pyramidal) numbers: 1, 4, 10, 20,..., are obtained like the sixth part of the products of three integers, 1·2·3, 2·3·4, 3·4·5, 4·5·6,..., and which at the same time are the summation of the preceding triangular numbers: 1=1; 1 + 3 = 4; 1 + 3 + 6 = 10; 1 + 3 + 6 + 10 = 20;... From these two properties, and from the previous result, it can be deduced that the summation of the square of the *n* first integers and the summation of the *n* first integers, expressed in modern notation, are,

$$\sum_{a=1}^{n} \frac{(a^2+a)}{2} = \frac{n.(n+1).(n+2)}{6}$$

From this equality and substituting the value of the summation of the first integers found before, we obtain the sum of the square of the *n* first integers,

29 The use of Pascal's triangle can be found in Leibniz's manuscripts on the harmonic triangle. "Triang. Arithmet Pascalii servit ad inveniendas summas integrorum finitorum figuratorum et quadraturas parabolarum rationalium, omnium graduum, quae sunt figurar carentes asymptotis". LEIBNIZ (1676), VII, 3C, N532, 2003: 709.

30 For the sums of powers of integers, see BOYER (1943); BURROWS; TALBOT(1984); EDWARDS (1982, 1986) and MASSA-ESTEVE (1997, 2006a).

which expressed in modern notation are,

$$\sum_{a=1}^{n} a^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Thus, with the properties of the pentagonal (which Leibniz called triangulo-triangular) numbers, we deduce the sum of the cubes of the first integers and thereby deduce the other sums of powers of integers.

Pascal, Faulhaber and Mengoli deduced similar formulas of sums of powers of integers from the arithmetical triangle. Leibniz explained that he had known this use of the arithmetical triangle in his comparison with his own harmonic triangle. Leibniz explicitly claimed that the harmonic triangle was his: "The harmonic triangle is mine"[31] (see Figure 12):

> *"My harmonic triangle serves for the discovery of the finite sums of the fractions with figurate numbers, and infinite sums that have finite sum and for the quadratures of hyperbolas, in all the degrees, in which the figures have a surface of finite magnitude, and asymptotes of infinite length[32]."*

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Triang. arithmet.
Pascalii

servit ad inveniendas
summas integrorum
finitorum figuratorum,
et quadraturas parabolarum
rationalium, omnium
graduum, quae sunt
figurae carentes asymptotis.

Triang. harmon.
meum

servit ad inveniendas
summas fractorum
figuratorum, finitorum
et infinitorum summam
finitam habentium et
quadraturas hyperbolarum
rationalium, omnium
graduum, quae spatia
habent finitae magnitudinis,
at ob asymptotos,
infinitae longitudinis.

Figure 12. Leibniz's comparison between Pascal's arithmetic and harmonic triangle (LEIBNIZ 1676, VII, 3C, N532, 2003: 709)

31 "Triang. Harmon. Meum", LEIBNIZ (1676), VII, 3C, N532, 2003: 709.

32 "Triang. Harmon. meum servit ad inveniendas summas fractorum figuratorum, finitorum et infinitorum summam finitam habentium et cuadraturas hyperbolarum rationalium, omnium graduum, quae spatia habent finitae magnitudinis, at ob asymptotos, infinitat longitudinis" LEIBNIZ (1676), VII, 3C, N532, 2003: 709.

Leibniz concluded the paper with an illustration of two triangles: the arithmetic triangle and the triangular table of fractions with figurate numbers in the denominator (see Figure 13):

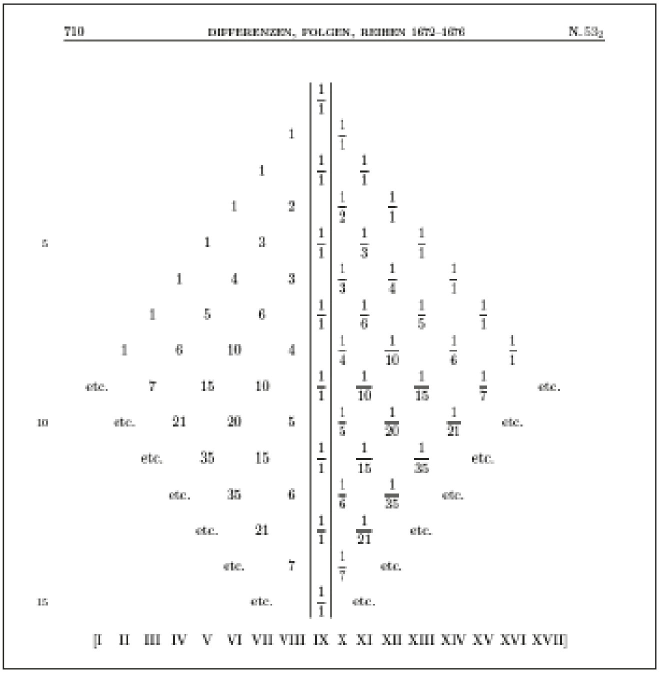



Figure 13. Pascal's arithmetical triangle and the triangular table of reciprocal figurate numbers (LEIBNIZ, 1676, VII, 3C, $N53_2$, 2003: 710)

In the third section entitled "Origo inventiones trianguli harmonici", Leibniz explained that in the year 1673 Huygens suggested that he find the summation of reciprocal figurate numbers, to which he responded with the reasoning given below[33]. In addition, Leibniz claimed "The summation of arithmetic triangle reciprocal series with the harmonic triangle"[34]. In fact, Leibniz used this series, B/2= ½ + 1/6 +1/12+ 1/20 + 1/30…., which is the first diagonal of the harmonic triangle, in the proof of the summation of reciprocal figurate numbers.

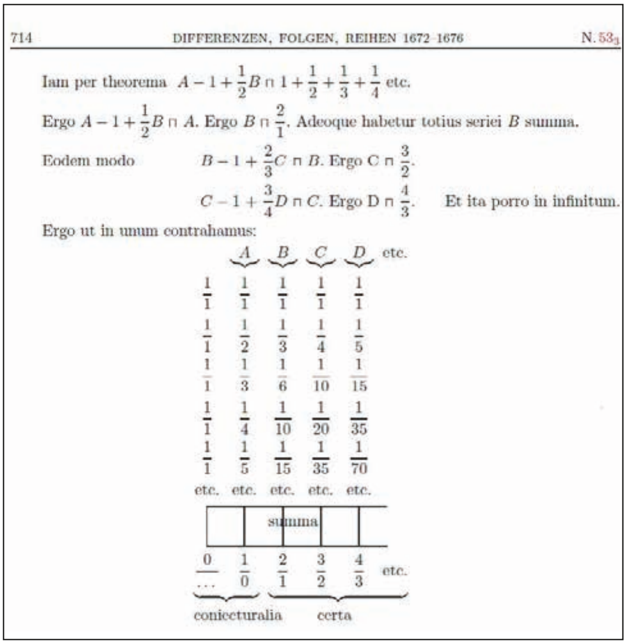
33 In fact, when Leibniz again explained his researches on the harmonic triangle in the text entitled *Historia et Origo Calculi Differentialis* (after September 1714), he referred to the year 1672, DE MORA (2014), 401.

34 "Summae serierum trianguli arithmetici reciproci seu trianguli harmonici", LEIBNIZ (1676), AVII, 3, N533, 2003: 713.

Assuming that A= 1 +1/2+1/3+1/4+1/5+… and B=1+1/3+1/6+1/10+1/15+…, we divide B by half and add to A without 1, which yields:

A-1+B/2= (1/2+1/2) + (1/6+1/3) + (1/12+1/4) + (1/20+1/5) +… =
1 + 1/2 + 1/3 + 1/4 +…= A.

Then, the equation A – 1 + B/2 = A is stated and the solution is B= 2[35]. Although the result is true, this demonstration using equations by Leibniz is no longer acceptable, because in this equation the term A is the harmonic series, one divergent series with infinite terms. However, Leibniz had arrived at this result by the sum of all the differences, as explained below. Leibniz concluded the paper with a table of the relations between these results. It is interesting to point out that the sum of unities and the summation of the harmonic series is "coniecturalia" (conjecture), while the other summations are claimed "certa" (true) (see Figure 14).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Iam per theorema $A - 1 + \frac{1}{2}B \sqcap 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ etc.

Ergo $A - 1 + \frac{1}{2}B \sqcap A$. Ergo $B \sqcap \frac{2}{1}$. Adeoque habetur totius seriei $B$ summa.

Eodem modo $B - 1 + \frac{2}{3}C \sqcap B$. Ergo $C \sqcap \frac{3}{2}$.

$C - 1 + \frac{3}{4}D \sqcap C$. Ergo $D \sqcap \frac{4}{3}$. Et ita porro in infinitum.

Ergo ut in unum contrahamus:

| | A | B | C | D | etc. |
|---|---|---|---|---|---|
| $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | |
| $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | |
| $\frac{1}{1}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{10}$ | $\frac{1}{15}$ | |
| $\frac{1}{1}$ | $\frac{1}{4}$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{35}$ | |
| $\frac{1}{1}$ | $\frac{1}{5}$ | $\frac{1}{15}$ | $\frac{1}{35}$ | $\frac{1}{70}$ | |
| etc. | etc. | etc. | etc. | etc. | |
| summa | | | | | |
| $\frac{0}{\dots}$ | $\frac{1}{0}$ | $\frac{2}{1}$ | $\frac{3}{2}$ | $\frac{4}{3}$ | etc. |
| coniecturalia | | certa | | | |

Figure 14. Leibniz's table of results of summations of reciprocal figurate numbers (LEIBNIZ, 1676, AVII, 3, N533, 2003: 714)

35 Similar proof of this sum of series in *Theorema arithmeticae infinitorum* (LEIBNIZ, August-September, 1674, AVII, 3, N35, 2003: 361-364,), and *Summa fractionum a figuratis, per aequationes* (LEIBNIZ, September, 1674, AVII, 3, N36, 2003: 365-369).

Many years later, in a letter to Guillaume de l'Hospital (1661-1704) dated December 27th, 1694, Leibniz returned again to this question of the properties of the sums of the figurate numbers in the arithmetical triangle,

> *"… the Arithmetic Triangle of M. Pascal, in which M. Pascal had shown how the sums of the figurate numbers can be given, that they arise from the search for the sums and sums of the sums, etc. of the natural arithmetic progression*[36]*."*

Leibniz also explained that the harmonic triangle served to find the infinite sum of the fractions with triangular numbers in the denominator: 1 + 1/3 + 1/6 + 1/10 + ... with the procedure of sum of all the differences. Without referring to the harmonic triangle, Leibniz had already pointed out how to find this sum in a paper of 1673 entitled *De summis serierum fractionum* (see figure 15),

De summis serierum fractionum, quarum numeratores unitas nominatores sunt numeri figurati.

Data aliqua serie fractionum, invenire aliam seriem ad cuius differentias series data aliquam habeat rationem constantem, si id possibile est, aut ostendere impossibilitatem.

Ut series 1. $\frac{1}{3}$. $\frac{1}{6}$. $\frac{1}{10}$. $\frac{1}{15}$. etc. est duplum $\frac{1}{2}$. $\frac{1}{6}$. $\frac{1}{12}$. $\frac{1}{20}$. $\frac{1}{30}$. etc. differentiarum seriei 1. $\frac{1}{2}$. $\frac{1}{3}$. $\frac{1}{4}$. $\frac{1}{5}$. $\frac{1}{6}$. etc.

Figure 15. Leibniz's explanation of the procedure for the sum
(LEIBNIZ, 1673, AVII, 3, N12, 2003: 149)

Leibniz used in the diagonals of the harmonic triangle his principle of the sums of all the differences for the proof; that is, given a series (in this case a diagonal) expressed through the sum of the differences of the terms two by two of other series, its sum (of the first series), expressed by the consecutive differences, is equal to the first term minus the last.

Thus, Leibniz calculated the infinite sum of the first diagonal
S = 1/2 + 1/6 + 1/12 + 1/20 + ..., since when doubled it is the sum of the fractions

36 "... Triangle Arithmetique de M. Pascal, car M. Pascal avoit monstré comment on peut donner les sommes des nombres figurés, qui proviennent en cherchant les sommes et les sommes des sommes etc. de la progression arithmetique naturelle" LEIBNIZ (1694), AIII, 6, N84, 2004: 256.

with triangular numbers in the denominator, 1 + 1/3 + 1/6 + 1/10 + ...

To this end, the finite sum S*n* of the first *n* terms is expressed with the sum of all the differences of the harmonic series and its value is equal to 1 minus 1/(*n*+1),

S*n* = 1/2 + 1/6 + 1/12 + 1/20 + ... =

(1/1 - 1/2) + (1 / 2 - 1/3) + 1/3 -1 / 4) + (1 / 4 - 1 / 5) + ... =

1 / 1 - 1 / (*n* + 1).

Leibniz explained that this sum in infinity is worth 1, and that therefore the infinite sum of the fractions with triangular numbers in the denominator that is its double: 1 + 1/3 + 1/6 + 1/10 + ... is worth 2.

He also calculated the infinite sum of the second diagonal T = 1/3 + 1/12 + 1/30 + 1/60 + ..., since when triplicate it is the sum of the fractions with tetrahedral ("pyramidal", in Leibniz's words) numbers in the denominator, 1 + 1/4 + 1/ 10 + 1/20 + ...

To this end, the finite sum T*m* of the first *m* terms is expressed with the sum of all the differences of the first diagonal and its value is equal to 1/2 minus 1/(*m*+1) · (*m*+2),

T*m* = 1/3 + 1/12 + 1/30 + 1/60 + ... =

(1/2 - 1 / 6) + (1 / 6 - 1 / 12) + (1 / 12 - 1 / 20) + (1 / 20 - 1 / 30) =

1/2 - 1 / (*m* + 1) · (*m* + 2).

In infinity it is worth ½, and therefore the infinite sum of the fractions with tetrahedral numbers in the denominator that is its triple 1 + 1/4 + 1/ 10 + 1/20 + ... is worth 3/2.

Leibniz does not mention the limit in his explanation, but it can be understood in this letter to l'Hospital of 1694, in which he says that in the series decreasing to infinity the sum is worth the first term:

> *"I have for a long time enjoyed looking for sums of the series of numbers, and for this purpose I used the differences in a well-known theorem that* ***in series decreasing to infinity its first term is equal to the sum of all the differences***[37] *. This gave me what I called the* Harmonic Triangle, *contrasting with the Arithmetic Triangle of M. Pascal, ..., and I found that the fractions of the figurate numbers are the differences and the differences of the differences, etc. of the natural harmonic progression ( that is, fractions 1 / 1 ,1 / 2 ,1 / 3 ,1 / 4 etc.) and that the sum of the series of the figurate fractions can be given as 1/1 + 1/3 + 1/6 + 1/10 etc and 1/1 + 1/4 + 1/10 + 1/20 etc.* [38]*".*

37 The emphasis is mine.

38 "J'avois pris plaisir long temps auparavant de chercher les sommes des series des nombres

## 4.- Differences and similarities in Mengoli's and Leibniz's treatment.

The treatment of the harmonic triangle by Mengoli and Leibniz is very different. Leibniz defined and studied the harmonic triangle independently of Mengoli's operations on quadratures, and based his researches on the harmonic series and the sums of all the differences. It was not Mengoli's intention to define the harmonic triangle ("Terza Tavola Triangolare"), and he did not wish to calculate the sums of the terms of the harmonic triangle; he simply used it like a triangular table more for his quadratures. In fact, Mengoli's aim was to calculate the quadrature of the circle, and in this way he also constructed the interpolated harmonic triangle[39].

Leibniz knew Mengoli's algorithms on quadratures later and published his excerpts from Mengoli's *Circolo* in 1676. In these excerpts Leibniz constructed the triangular table of ordinates of figures that he wants to square, whereas Mengoli had constructed the triangular tables of geometric figures through algebraic expressions (Mengoli's *Tabula Formosa*), and in this way Leibniz made his own interpretation (compare Figure 4 and Figure 16)[40].

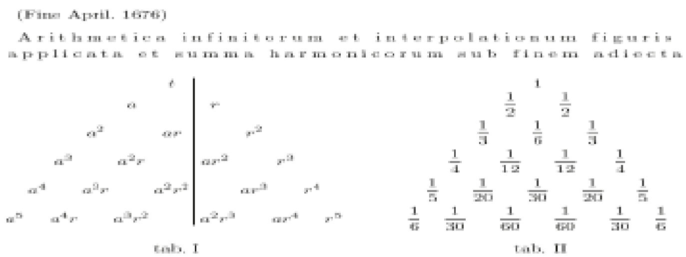


Figure 16. Leibniz's table of ordinates of geometric figures (tab. I) and Leibniz's harmonic triangle (tab. II) (LEIBNIZ, 1676, AVII, 3, N572, 2003: 736)

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et je m'estois servi pour cela des differences sur un theoreme assez Cornu qu'une series decroissant à l'infini son premier terme est egal à la somme de toutes les differences. Cela m'avoit donné ce que j'appellois le Triangle Harmonique, opposé au Triangle Arithmetique de M´Pascal, … ,et moi je trouvay que les fractions des nombres figurés sont les differences et les differences des differences, etc de la progresión harmonique naturelle (c'est à dire des fractions 1/1, ½, 1/3, ¼ etc) et qu'ainsi on peut donner les sommes des series des fractions figurées, comme 1/1+ 1/3 + 1/6 +1/10 etc et 1/1+ ¼ + 1/10 +1/20 etc" LEIBNIZ (1694), AIII, 6, N 84, 2004: 256.

39 See MASSA-ESTEVE; DELSHAMS (2009).

40 See the analysis of Leibniz's excerpts in MASSA-ESTEVE (2017).

In his letter to l'Hospital in 1694, Leibniz refers again to the use of the harmonic triangle in quadratures, pointing out the relationship of the sums of the ordinates with the quadratures, quoting Descartes. He states as follows:

> *"Recognizing then this great utility of the differences, and seeing that by the calculation of M. Descartes the ordinate of the curve can be expressed, I saw that finding the quadratures or the sums of the ordinates is nothing else than finding an ordinate (from the quadratrix) in such a way that the difference is proportional to the given ordinate[41]."*

On the contrary, Mengoli did not quote Descartes, did not use the sums of the ordinates, nor Descartes' ideas in his works on quadratures.

A further difference can be found in the sources. Leibniz and Mengoli both constructed the harmonic triangle from the arithmetical triangle, but while Leibniz recognized Pascal's arithmetical triangle as a source, Mengoli acknowledged and quoted Hérigone's and Viète's works.

Nevertheless, some similarities also exist; it is significant that both Leibniz and Mengoli used the symmetry of triangular tables and the regularity of their rows in order to generalize the proofs and to state the rules. They took it for granted that if a result was correct for one row of the table, it could be applied to all rows and there was no need for further proof. They used the harmonious construction of these infinite triangular tables to arrive at all the terms that they wished to know in triangular tables. Thus, there is no doubt about the relevance of the triangular tables as a generalizing tool in the development of mathematical ideas from the seventeenth century.

Regarding Mengoli's and Leibniz's mathematics, this brief presentation of the harmonic triangle, with the demonstrations of finite and infinite sums, reveals a harmonic mathematics arising from the symmetry of the triangles and the regularity of the rows. Triangles also have an open visual structure in which the number of terms arranged in this way can be made infinite. The infinite therefore becomes one more element in the mathematical calculations of these authors, which in seventeenth century mathematics opened up a world of possibilities in the series and in their relations with infinitesimal calculus.

41 "Reconnaissant donc cette grande utilité des différences et voyant que par le càlcul de M. des Cartes l'ordonnée de la courbe peut être exprimée, je vis que trouver les quadratures ou les sommes des ordonnées n'est autre chose que trouver une ordonnée de la quadratrice dont la difference est proportionnelle à l'ordonnée donnée." LEIBNIZ (1694), AIII, 6, N 84, 2004: 256.

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