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JACQUELINE A. STEDALL

THE ARITHMETIC OF INFINITESIMALS

John Wallis 1656

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A.W. Grootendorst
Jan de Witt's *Elementa Curvarum Linearum, Liber Primus*

Continued after Index

The Arithmetic of Infinitesimals

John Wallis
1656

Translated from Latin to English with an Introduction
by
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Springer

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For June Barrow-Green and Jeremy Gray,
and in memory of John Fauvel
who did so much to make this and many other things possible

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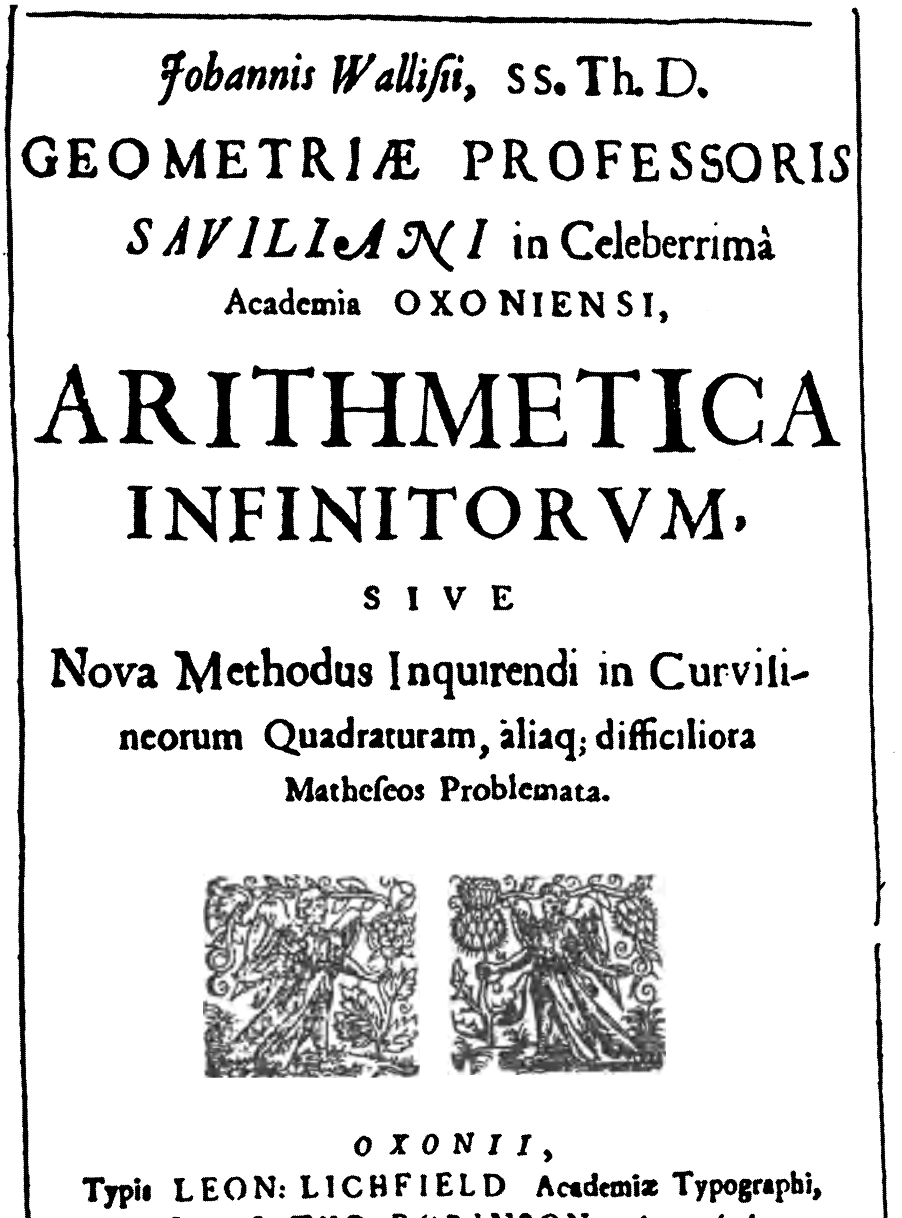
Contents



Johannis Walliſii, SS. Th. D.

GEOMETRIÆ PROFESSORIS

SAVILIANI in Celeberrimâ

Academia OXONIENSI,

ARITHMETICA

INFINITORUM,

SIVE

Nova Methodus Inquirendi in Curvilineorum Quadraturam, àliaq; difficiliora

Matheſeos Problemata.

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Title page of the *Arithmetica infinitorum* 1656

The Arithmetic of Infinitesimals

John Wallis (1616–1703) was appointed Savilian Professor of Geometry at Oxford in 1649. He was then a relative newcomer to mathematics, and largely self-taught, but in his first few years at Oxford he produced his two most significant works: *De sectionibus conicis* and *Arithmetica infinitorum*. Both were printed in 1655, and published in 1656 in the second volume of Wallis's first set of collected works, *Operum mathematicorum.*[1] In *De sectionibus conicis*, Wallis found algebraic formulae for the parabola, ellipse and hyperbola, thus liberating them, as he so aptly expressed it, from 'the embranglings of the cone'.[2] His purpose in doing so was ultimately to find a general method of quadrature (or cubature) for curved spaces, a promise held out in *De sectionibus conicis* and taken up at length in the *Arithmetica infinitorum.*[3] In both books Wallis drew on ideas originally developed in France, Italy, and the Netherlands: algebraic geometry and the method of indivisibles, but he handled them in his own way, and the resulting method of quadrature, based on the summation of indivisible or infinitesimal quantities,[4] was a crucial step towards the development of a fully fledged integral calculus some ten years later.

To the modern reader the *Arithmetica infinitorum* reveals much that is of historical and mathematical interest, not least the mid seventeenth-century tension between classical geometry on the one hand, and arithmetic and

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[1] For the first editions of *De conicis sectionibus and Arithmetica infinitorum*, see *Operum mathematicorum*, 1656–57, II, 49–108 and 1–199 (separate pagination). Both works were reprinted in Wallis's second set of collected works, *Opera mathematica*, Wallis 1693–99, I, 291–354 and 355–478.

[2] Wallis 1685, 291–292.

[3] See *De sectionibus conicis*, Proposition 48; *Arithmetica infinitorum*, Proposition 45.

[4] Strictly speaking an 'indivisible' has at least one of its dimensions zero, (for example, a point, line or plane), whereas an 'infinitesimal' has arbitrarily small but non-zero width or thickness. Wallis blurred the distinction between the two and generally spoke only of 'infinitely small quantities'. For him 'indivisible' and 'infinitesimal' were more usefully seen as geometric and arithmetic categories, respectively.

algebra on the other. Newton was to take up Wallis's work and transform it into mathematics that has become part of the mainstream, but in Wallis's text we see what we think of as modern mathematics still struggling to emerge. It is this sense of watching new and significant ideas force their way slowly and sometimes painfully into existence that makes the *Arithmetica infinitorum* such a relevant text even now for students and historians of mathematics alike.

Wallis's mathematical background

Wallis was educated from the age of nine by a private tutor then, at the age of fourteen, for a year at Felsted School in Essex, and then at Emmanuel College, Cambridge.[5] He later claimed that he had learned little or no mathematics at Cambridge (though he did study some astronomy). Instead he taught himself elementary arithmetic from the textbooks of a younger brother who was preparing to go into trade. After the brief tenure of a Fellowship at Queens' College, Cambridge, Wallis was employed as a private chaplain, but his mathematical bent came to the fore during the years of civil war in England (1642–1648) when he regularly decoded letters for Parliament.[6] His eventual appointment as Savilian Professor at Oxford was no doubt at least in part a reward for his loyalty and political service to the winning side.

By the time he took up his post, indeed possibly in preparation for it, Wallis had begun to extend his mathematical knowledge by reading William Oughtred's *Clavis mathematicae*, the first edition of which had been published in 1631. (Second editions appeared in English and Latin in 1647 and 1648, respectively, but the copies owned and annotated by Wallis were first editions.[7]) The *Clavis* provided Wallis with his first taste of algebraic notation and, as for other English readers, an elementary introduction to the new subject of algebraic geometry first developed by Viète during the 1590s. When Wallis took up his professorship he knew little more than he had learned from the *Clavis*, but once established at Oxford, he had a wealth of books available to him, especially in the Savile Library, created by Henry Savile for the use of the Savilian professors when he endowed the chairs of Geometry and Astronomy in 1619. The two books that were to influence Wallis most were Descartes' *La Géométrie*, originally published in French in 1637

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[5] The most important source of biographical material on Wallis is the autobiography he wrote when he was eighty years old; see Scriba 1970.

[6] Wallis made two identical copies of letters deciphered up to 1653, both now in the Bodleian Library, MS e. Musaeo 203 and MS Eng. misc. e. 475, ff. vii-243.

[7] Wallis's copies of the *Clavis* are now in the Bodleian Library, Savile Z.16, Z.19 and Z.24; see Stedall 2002, 77–82.

but republished in Latin as the *Geometria* by Frans van Schooten in 1649,[8] and Torricelli's *Opera geometrica* of 1644.[9]

Descartes' *Geometria* would have taken Wallis far beyond anything he had learned from the *Clavis.* Oughtred, like Viète, had seen how to express and solve certain geometric problems algebraically, but it was Descartes who introduced the coordinate systems that made it possible to describe loci or curves by means of equations between two variables.[10] Curves defined in this way Descartes called *geometric*,[11] and the simplest class of such curves, according to him, consisted of conic sections, the circle, ellipse, parabola, and hyperbola. There can be little doubt that it was Descartes' work that inspired Wallis to define conics by means of algebraic equations. He was not the first to do so, for Fermat had completed a similar exercise, though still using traditional geometric notation, by 1635.[12] Fermat's work circulated in manuscript amongst mathematicians on the continent, and Charles Cavendish read it in Paris in 1646 and wrote about it to the English mathematician John Pell, then in Amsterdam,[13] but Wallis in 1650 did not belong to such circles and was unlikely to have known of it.

In *De sectionibus conicis* Wallis based his treatment on the traditional definitions of the parabola, ellipse, and hyperbola as sections of a (right or inclined) cone, and derived from those definitions equations that related ordinates labelled p, e or h to diameters d. Thus the equation of the parabola is $p^2 = ld$, of the ellipse $e^2 = ld - \frac{l}{t}d^2$, and of the hyperbola $h^2 = ld + \frac{l}{t}d^2$, where l and t are constants associated with each curve (the *latus rectum* and the transverse diameter). In modern notation d becomes x; and p, e, and h become y. Wallis was satisfied that such equations alone were sufficient to define the curves:[14]

> It is no more necessary that a parabola is the section of a cone by a plane parallel to a side than that a circle is a section of a cone by a plane parallel to the base, or that a triangle is a section through the vertex.

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[8] For a hint that Wallis read the *Geometria* and corresponded with Van Schooten as early as 1649, see Beeley and Scriba 2003, 13.

[9] The copy of Torricelli's *Opera geometrica* read and annotated by Wallis is in the Bodleian Library, Savile Y.1.

[10] Descartes' coordinates were not necessarily the orthogonal coordinates that later came to be called Cartesian.

[11] *'ie ne sçache rien de meilleur que de dire que tous les poins, de celles qu'on peut nommer Geometriques, c'est a dire qui tombent sous quelque mesure precise & exacte, ont necessairement quelque rapport a tous les poins d'une ligne droite, qui peut estre exprimé par quelque equation, en tous par un mesme';* Descartes 1637, II, 319.

[12] *Ad locos planos et solidos isagoge*, Fermat 1679, 1–8; see also Mahoney 1973, 76–92.

[13] Cavendish's copy survives in British Library MS Harleian 6083, ff. 113–114.

[14] *'Non enim est Parabolae magis essentiale, ut fiat Sectione Coni Plano lateri parallelo; quam Circulo, ut fiat Sectione Coni plano basi parallelo; aut Triangulo, ut fiat Sectione Coni per Verticem';* *De sectionibus conicis*, Proposition 21.

Wallis went on in the second half of the book to find equations of tangents, and other properties of conic sections. His main interest, however, was in quadrature. His stated purpose at the beginning of *De sectionibus conicis* was to find the areas enclosed by the curves (or rather, the ratios of those areas to inscribed or circumscribed rectangles), and the *Arithmetica infinitorum* took up the same challenge. One might think, therefore, that in the *Arithmetica infinitorum* Wallis would use the algebraic tools he had so carefully developed in *De sectionibus conicis*, but he rarely did. Only in Proposition 163 when struggling with the quadrature of the hyperbola, did he specifically make use of the algebraic formula he had previously found: $h^2 = ld + \frac{l}{t}d^2$. For the simplest curve, the parabola, he used the geometric relationship between ordinates and diameters rather than the algebraic equation, while for the circle and ellipse he expressed the ordinates simply as mean proportionals between the two corresponding segments of the diameter. In other words, he used the geometrical concepts on which his equations were based rather than the equations themselves. Thus although Wallis was aware of Descartes' work and was almost certainly inspired by it, in the end he used his knowledge of the algebra of curves very little, falling back instead on more traditional geometric definitions.

It was in Torricelli's *Opera geometrica* of 1644 that Wallis first encountered the idea of indivisibles. The methods had been developed independently by Gilles Persone de Roberval, Pierre de Fermat, and Bonaventura Cavalieri from about 1629 onwards, but Roberval and Fermat had not published their results, and Wallis remained unaware of their work until many years later.[15] Cavalieri, however, gave a full exposition of the method in his *Geometria indivisibilibus continuorum nova quadam ratione promota* of 1635. Cavalieri's treatise was divided into seven books,[16] and his fundamental definitions and theorems were presented in Book II. His ideas were based on the notion of a plane moving through a given figure and intersecting it in 'All the lines of the figure':[17]

> If through any opposite tangents to any given plane figure there are drawn two planes parallel to each other, either perpendicular or inclined to the plane of the given figure, and produced indefinitely, and if one of them is

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[15] For Roberval's *Traité des Indivisibles*, see Roberval 1693; see also Walker 1932 and Auger 1962, 14–38. For Fermat's early work on quadrature and his 1636 correspondence with Roberval, see Mahoney 1973, 218–222.

[16] In the 1635 edition each of the seven books has its own pagination, but in the second edition, in 1653, the pages are numbered consecutively. References to both editions are given in the following notes.

[17] *'Si per oppositas tangentes cuiuscunque datae planae figurae ducantur duo plana invicem parallela, recta, sive inclinata ad planum datae figurae, hinc inde indefinite producta; quorum alterum moveatur versus reliquem eidem semper aequidistans donec illi congruerit: singulae rectae lineae, quae in toto motu fiunt communes sectiones plani moti, et datae figurae, simul collectae vocentur: Omnes lineae talis figurae, sumptae regula una earundem; et hoc cum plana sunt recta ad datam figuram: Cum vero ad illam sunt inclinata vocentur: Omnes lineae eiusdem obliqui transitus datae figurae, regula pariter earundem una';* Definition 1, Cavalieri 1635, II, 1–2 or 1653, 99.

> moved towards the other always remaining parallel until it coincides with it, then the single lines which in the motion as a whole are the intersections of the moving plane and the given figure, collected together, are called: All the lines of the figure, taken with one of them as *regula*; this when the planes are at right angles to the given figure. But if they are inclined to it, they are called: All the lines of an oblique passage of the same given figure, likewise with one of them as *regula*.

Cavalieri's fundamental theorem was that two figures could then be said to be in the same ratio as 'all their lines':[18]

> Plane figures have the same ratio to each other as all their lines taken to whatever *regula*; and solid figures as all their planes taken to whatever *regula*.

After developing this theory in the remainder of Book II, Cavalieri went on in Books III–VI to apply his methods to circles, ellipses, parabolas, hyperbolas, and spirals, and to solids created from them (a range of figures similar to those handled later by Wallis). In Book VII he returned to the theory of indivisibles, now hoping to avoid the problems of treating collections of 'All the lines' by instead comparing individual pairs of lines. Thus both here and later, in his *Exercitationes geometricae sex* of 1647, he made repeated efforts to put his theory on a sound footing, carefully trying to avoid the paradoxes that could arise, as he and others recognized, from handling an infinite number of dimensionless quantities.[19] For a full discussion the reader is referred to the work of Enrico Giusti and Kirsti Andersen.[20] The details are not repeated here because in one way they are irrelevant to the present story; Wallis never read Cavalieri's books, which were almost impossible to obtain, but instead learned of his work at second hand from the more easily available *Opera* of Torricelli.

Torricelli's *Opera geometrica* of 1644 contained three separate treatises: *De solidis sphaeralibus* on the mensuration of cylindrical, conical and spherical solids; *De motu proiectorum* on the motion of projectiles; and, the book that interested Wallis, *De dimensione parabola solidique hyperbolici problematis duo*, on the quadrature of the parabola and cubature of a hyperbolic solid.[21] On the title page of this third treatise, Torricelli explained that he had handled two problems: one ancient, one new. The ancient problem was the quadrature of the parabola, which he had solved by no fewer than twenty different methods, some geometric, some mechanical, and some based on the concept of

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[18] *'Figurae planae habent inter se eandem rationem, quam eorum omnes lineae iuxta quamvis regulam assumptae; Et figurae solidae, quam eorum plana iuxta quamvis regulam assumptae'*; Theorem III, Cavalieri 1635, II, 20 or 1653, 113.

[19] See also Cavalieri 1647.

[20] Giusti 1980; Andersen 1985.

[21] The *Opera geometrica* is paginated as follows: *De solidis sphaeralibus*, 1–94; *De motu proiectorum*, 95–243; *De dimensione parabola ... problematis duo*, 1–150. The pagination in *De dimensione* occasionally goes awry, especially towards the end, with some page numbers repeated and others left out.

indivisibles.[22] The new problem concerned a 'wonderful solid' generated by the revolution of a hyperbola,[23] which Torricelli had found to be infinite in extent but finite in volume. The book also contained appendices on properties of the cycloid and cochlea. In the text itself, in connection with both the parabola and the hyperbolic solid, Torricelli sang the praises of Cavalieri, and Wallis carefully noted his references on the flyleaf of the Savile Library copy of the *Opera geometrica*.[24]

Torricelli paid little heed to the precautions taken by Cavalieri, but offered an altogether simpler version of the theory, in which a plane figure was supposed equal to a collection of lines and a solid to a collection of planes or surfaces.[25] Torricelli found the cubature of his 'acute hyperbolic solid' by treating it as a collection of concentric cylinders whose surfaces could be added to give the volume of the solid:[26]

> Therefore all the surfaces of the cylinders taken together, that is the acute solid EBD itself, is the same as the cylinder of base FEDC, which will be equal to all its circles taken together, that is to cylinder ACGH.

Torricelli's version of the theory was both simple and intuitive, and it inspired Wallis to try his hand at similar quadratures and cubatures. Wallis's advance on Torricelli was to see that the necessary summations could be carried out arithmetically rather than geometrically. For the area of a triangle, for example, one simply needed to sum a sequence of regularly increasing terms, that is, an arithmetic progression; the area of a parallelogram could be regarded as the sum of a sequence of equal terms; while the area of a parabola was a sum of squares or square roots (depending on orientation). Wallis therefore shifted the focus of his own enquiry to the problem of finding sums of sequences of powers, or at least the ratio of such sums to certain known quantities.

Wallis called his sequences of powers 'infinite' and so they are, but not in the sense now generally understood, where the terms increase or decrease indefinitely. Wallis's sequences, beginning from 0, have a finite greatest term, reached initially by a finite number of steps. If, keeping the same end point, the steps are made smaller their number must be made larger, and eventually, according to Wallis, infinite. Thus, keeping his end point fixed and finite, Wallis moved from a finite number of steps to an infinite number of infinitely small, or infinitesimal, steps. Where Cavalieri and Torricelli had summed

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[22] *'Antiquum alterum. In quo parabola XX modis absolvitur, partim geometricis, mecanisque; partim ex indivisibilium geometria deducto rationibus'*; Torricelli 1644, title page.

[23] *'Novum alterum. In quo mirabilis cuiusdam solidi ab hyperbola geniti accidentia nonnulla demonstratur'*; Torricelli 1644, title page.

[24] In Wallis's handwriting are the notes: *'Geometria indivisibilium Cavalerij pag. 56. 57. de Dimensione parabola ... Geom. indivisib. Cavallerij. pag. 94. de Append de mens. parab.'*; Bodleian Library, Savile Y.1, flyleaf.

[25] Andersen 1985, 355–358.

[26] *'Propterea omnes simul superficies cylindricae, hoc est ipsum solidum acutum* ebd, *una cum cylindro basis* fedc, *aequale erit omnibus circulis simul, hoc est cylindro* acgh. *Quod erat etc.'*; Torricelli 1644, 116.

geometric indivisibles, Wallis now needed to sum infinite sequences of arithmetic infinitesimals, or infinitely small parts.[27] Wallis saw the two processes, geometric and arithmetic, as exactly analogous. Just as Cavalieri's method could be described as *geometria indivisibilium*, or the geometry of indivisibles, his own, he claimed, could be described as *arithmetica infinitorum* or the arithmetic of infinitely small parts. Wallis himself translated the title of his book as 'The Arithmetick of Infinites', but the 'infinites' in question were in fact infinitely small quantities, and the single modern word 'infinitesimals' thus conveys Wallis's meaning rather better than 'infinites'.[28]

The writing of the Arithmetica infinitorum

As Wallis explained in the Dedication to the *Arithmetica infinitorum*, he wrote *De sectionibus conicis* in 1652, and most of the *Arithmetica infinitorum* in the same year. Printing of both books began in 1655 and they finally appeared in *Operum mathematicorum* in 1656. Wallis claimed that the three years between the completion and publication of the texts were due to delays at the printers. This may have been true, but it was also the case that in 1652 Wallis had reached an impasse, and the final part of the *Arithmetica infinitorum* was not in fact written until 1655, just before the book went to press.

When Wallis began writing in late 1651 and early 1652, the first part of the *Arithmetica infinitorum* proceeded easily. Gradually extending the scope of his method, by analogy and by what he called 'induction', Wallis was able to produce a steady flow of results, and in particular quadratures of curves of the form (in modern notation) $y = kx^n$, not only for the cases where n was a positive integer, but also for n a fraction or a negative integer. Not all these results were new; apart from those already published by Cavalieri and Torricelli, others had been discovered by Fermat, Roberval, Descartes, and Torricelli, but had been discussed only in private correspondence that Wallis had never seen.[29] It was Wallis who therefore provided the first systematic

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[27] Wallis generally used the description 'infinitely small', but occasionally also 'infinitesimal', as in '*pars infinitesima, seu infinite parva*', 'an infinitesimal, or infinitely small, part', *Arithmetica infinitorum*, Proposition 5.

[28] See notes 4 and 27.

[29] As an example of how individual results on quadrature were circulated without proof, consider the quadrature of (in modern notation) $y^4 = x^3$ published by Mersenne in 1644 in *Cogitata physico-mathematica*, 'Tractatus mechanicus', sig.a2r. Charles Cavendish wrote of this to John Pell: 'Mersennus tells me it is Monsr: De Cartes his proposition but that he sent him not the demonstration of it; Mr: Robervall tells me it is Monsr: Fermats proposition but that he never sawe the demonstration of it; but saies he thinkes he could doe it but that it would be a verie longe demonstration. I thinke Mersennus [would have] sent me the demonstration of it into england but has forgotten it'; British Library Add MS 4278, f. 238.

Wallis, preoccupied with domestic or political affairs, knew nothing of these intellectual exchanges through Mersenne in Paris during the 1630s and 1640s. Evelyn Walker in 1932, 25, wrote that it was 'inconceivable that by 1651 [Wallis] should not have had some knowledge of Roberval's approach', but in 1651 Wallis was still new to mathematics and had read only what was easily and publicly available.

and general exposition. From his starting point of sequences of simple powers, he could easily handle sums (or differences) of sequences, and hence eventually quadratures of any curves of the form $y = (1 - x^{1/p})^q$ provided p and q were integers. But because his ultimate aim was the quadrature of the circle, the curve he was really interested in was $y = (1 - x^2)^{1/2}$, that is, where $p = q = \frac{1}{2}$.

At this point Wallis's method appears to have been driven by the skills that had served him so well as a code-breaker; working on an unspoken but intuited assumption of continuity, he proceeded to carry out a series of increasingly sophisticated interpolations. By now geometry was receding into the background, and his work became almost entirely arithmetical. Wallis made considerable progress, but at Proposition 190 he came to a halt. The last step, the final interpolation that would give him the ratio of a square to an inscribed circle, eluded him. At this point, in the spring of 1652, he put the problem to other mathematicians of his acquaintance (he named Seth Ward, Laurence Rook, Richard Rawlinson, Robert Wood, and Christopher Wren, all then resident in Oxford) but none could help, and indeed without a detailed knowledge of Wallis's techniques probably failed to understand his question. Wallis was asking for interpolated means in the sequence 1, 6, 30, 140, 630, ... but since the means were neither arithmetic or geometric, their required properties cannot have been very clear to anyone but Wallis.

In February 1655 he addressed his problem to Oughtred, once again asking for means between 1, 6, 30, 140, 630, ... which he now wrote also in the alternative forms 1, $1 \times \frac{6}{1}$, $6 \times \frac{10}{2}$, $30 \times \frac{14}{3}$, $140 \times \frac{18}{4}$, ... or 1, $1 \times 4\frac{2}{1}$, $1 \times 4\frac{2}{1} \times 4\frac{2}{2}$, $1 \times 4\frac{2}{1} \times 4\frac{2}{2} \times 4\frac{2}{3}$, $1 \times 4\frac{2}{1} \times 4\frac{2}{2} \times 4\frac{2}{3} \times 4\frac{2}{4}$, ... As Wallis described it to Oughtred:[30]

> These terms *in locis paribus* [in even places] (supposing the second to be 1) are made up by continued multiplication of these numbers $1 \times \frac{6\times10\times14\times18, etc.}{1\times2\times3\times4, etc.}$ or $1 \times \frac{12\times20\times28\times36, etc.}{2\times4\times6\times8, etc.}$. And (if I mistake not in my conjecture), supposing the first to be Q, the rest *in locis imparibus* [in odd places] will be made up by continued multiplication of these numbers $Q \times \frac{8\times16\times24\times32, etc.}{1\times3\times5\times7, etc.}$ which I thought it requisite to give you notice of, that you might see how far I had proceeded towards the solution of what I seek ... wherein if you can do me the favour to help me out, it will be a very great satisfaction to me, and (if I do not delude myself) of more use than at the first view it may seem to be.

It is clear that early in 1655 Wallis was still grappling with this final problem, now to be found in Proposition 190. Only a short time later, however, the problem was solved. In what was perhaps the one real stroke of genius in Wallis's long mathematical career, he saw how to complete his interpolations by a method now set out in Proposition 191, and so arrived at his infinite

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[30] Wallis to Oughtred, 28 February 1655, Rigaud 1841, I, 85–86.

fraction for $4/\pi$ (denoted by $\square$) in the form:

$$\square = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \textit{etc.}}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \textit{etc.}}$$

How Wallis was inspired to his breakthrough he did not say, but we do know that he discussed his work in some detail with William Brouncker. In particular, he seems to have put to Brouncker a similar problem to the one he put to Oughtred, and probably at about the same time (Brouncker's name does not appear in the list of colleagues whom Wallis consulted in 1652). How, asked Wallis, was he to interweave, for example, the two sequences $A, 2A, \frac{8A}{3}, \frac{48A}{15}, \ldots$ (or $A \times \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \ldots$) and $1, \frac{3}{2}, \frac{15}{8}, \frac{105}{48}, \ldots$ (or $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \frac{9}{8} \times \ldots$) into a single sequence beginning $A, 1, 2A, \frac{3}{2}, \frac{8A}{3}, \frac{15}{8}, \ldots$, in which the multipliers would follow some regular order? Brouncker came up with a brilliant answer, by producing a sequence of what are now called continued fractions, which served Wallis's purpose exactly. Brouncker's work enabled Wallis to answer the question he had left open in Proposition 190, and the solution was fully set out in a piece entitled *Idem aliter* following Wallis's discovery of his own infinite fraction in Proposition 191.

Unfortunately, Wallis complained, Brouncker could not be persuaded to write this piece himself, nor to explain how he had discovered his fractions, and Wallis was unable to do so either. In fact at the beginning of the *Idem aliter* (and again later in *A treatise of algebra*)[31] Wallis seriously misled his readers and posterity by quoting just the first of Brouncker's fractions,

$$1\cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{2+}}}}$$

as an alternative to his own $\frac{3}{2} \times \frac{3}{4} \times \frac{5}{4} \times \frac{5}{6} \times \ldots$. This led Euler and several other later mathematicians to look for ways of deriving Brouncker's fraction from Wallis's, an impossible task as it stands because Brouncker's fraction can only be related to Wallis's by taking the entire infinite sequence of which it is the first. I would suggest that Brouncker's fractions were derived not as alternatives to Wallis's, but in response to the problem that Wallis set but failed to solve in Proposition 190, and that only in that context does the relationship between Brouncker's fractions and Wallis's fall into place. Wallis's Proposition 191 and the mathematics discovered by Brouncker and expounded in the *Idem aliter* comprise some of the best mathematical writing of the mid seventeenth century. The material amply rewarded both Wallis and his readers for the long and, it has to be said, often tedious approach through

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[31] Wallis 1685, 317–318.

scores of uninspiring Propositions and Corollaries, and Wallis could justifiably feel proud of his achievement.

Wallis was by no means the only seventeenth-century mathematician seeking the quadrature of the circle. Grégoire de Saint-Vincent had claimed to solve the problem in his massive (1226 page) *Opus geometricum quadraturae circuli et sectione coni* of 1647. The *Opus geometricum* consists of ten books, the first six of which deal with properties of lines, circles, ellipses, parabolas and hyperbolas. In the seventh book de Saint-Vincent introduced the idea of 'drawing a plane into a plane' to produce a solid, and in the ninth book he handled cylinders, cones, spheres, and conoids. Finally in the tenth book he addressed the quadrature of the circle, parabola, and hyperbola. Wallis searched his work carefully but came to the conclusion that de Saint-Vincent had not come any closer to the quadrature of the circle than he himself had done in Proposition 136 of the *Arithmetica infinitorum* (where he had related the quadrature of the circle to a solid formed by 'multiplying' two parabolas).[32] In de Saint-Vincent's huge volume he found many propositions similar to his own, including the idea of 'drawing' a plane into a plane (Wallis described it rather more carefully as drawing the lines of one plane respectively into the lines of another). Wallis may have been led to some of his theorems by what he found in de Saint-Vincent but it is equally likely that he arrived at them independently by multiplying (or dividing) his infinite series term by term, and then looking, as he always did, for geometrical examples to illustrate his findings.

By 1655 when Wallis was finally ready to go to press, a more serious threat to his priority appeared to be looming closer to home. Thomas Hobbes, provoked by Seth Ward, Savilian Professor of Astronomy at Oxford and Wallis's close colleague, had promised, or threatened, to reveal his own quadrature of the circle along with solutions to other geometrical problems.[33] Wallis therefore made sure that he laid claim to his own success in a leaflet printed at Easter (April) 1655 advertising the key results of the *Arithmetica infinitorum*, even though the book itself was not yet printed. And in the 'Dedication' to Oughtred, written in the Spring of 1655 he was careful to emphasize, with supporting evidence, that he had been working on the problem since 1651. The Dedication ends on a note of relief, for by the time Wallis completed it in July, he had seen the first impression of Hobbes's *De corpore*,[34] and knew that he could demolish Hobbes's arguments with ease, as he went on to do in *Elenchus geometria Hobbianae*. Hobbes's attempts at quadrature were easily

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[32] *Arithmetica infinitorum*, Proposition 136; De Saint-Vincent 1647, 794, Proposition 143.

[33] Ward's challenge to Hobbes was put out in the appendix to his *Vindicae academiarum* of 1654, written in reply to Hobbes's attack on the English Universities; see Ward 1654, 57.

[34] The first impression of Hobbes's *Elementorum philosophiae sectio prima de corpore* appeared in April 1655. His three (unsuccessful) attempts at the quadrature of the circle were in Chapter 20.

dealt with but his philosophical objections to the *Arithmetica infinitorum* were not, as will be discussed further below.

The mathematics of the Arithmetica infinitorum

'Thus a geometric problem is reduced purely to arithmetic.'[35] Wallis's major contribution to the development of seventeenth-century mathematics was perhaps, as he himself recognized, the transformation of geometric problems to the summation of arithmetic sequences. Many of the results demonstrated by Wallis were already well known but, as he repeatedly pointed out, his aim was to establish a method by which those results, and others, could be systematically obtained. To prove the soundness and applicability of his method he therefore returned over and over again to justifications and applications in geometry. Every new result in summing sequences was followed by corollaries that showed how it could be interpreted geometrically, so the book describes quadratures and cubatures of all kinds of likely and unlikely plane or solid figures. At one point (Proposition 38) there is a glimpse of how the method might also work for rectification (straightening, or finding the length, of a curve), but Wallis attempted this only for the parabola and failed to complete his argument.

To the modern reader, unused to thinking in the language of Apollonius, the continual references to classical geometry are probably the most difficult parts of Wallis's book to follow. As pointed out above, Wallis had found algebraic formulae for conics and could easily have done so for the other curves he described (usually only higher parabolas) but instead he reverted to traditional Apollonian concepts of 'applied ordinates', 'intercepted diameters', and ratios of lines or spaces to each other. It is not easy to translate such geometrical language in a way that retains the essence of Wallis's thought yet renders it comprehensible to a modern reader.

In his attempt to relate arithmetic to geometry Wallis even used two distinct but parallel vocabularies: for example, *first power*, *second power*, and *third power* in arithmetic, but *side, square,* and *cube* in geometry; the Latin verbs *multiplicare* and *dividere* in arithmetic, but *ducere* and *applicare* in geometry. He sometimes slid haphazardly, however, from one usage to the other; thus on Proposition 75 he speaks of 'multiplying' lines (*rectas sic multiplicatas*) while in Proposition 140 he uses *ducatur* for term by term multiplication of series (*series ... ducatur in seriem*). To the modern reader the two sets of terms are sometimes more or less interchangeable (as 'square roots' and 'second roots', for example) but often one form, usually the geometric, has now fallen completely out of use. Thus when, for example, Wallis describes the product of a quantity multiplied by itself as a 'square' one can interpret his meaning either geometrically or arithmetically, but it is not so easy to do so when he describes the product of two unequal quantities as a 'rectangle'. The

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[35] *Arithmetica infinitorum*, Preface, sig Aa3^{v}.

general term 'product' will sometimes serve for 'rectangle' but is inadequate where Wallis goes on to compare his 'rectangles' with plane figures, in which case only a purely geometric interpretation will do.

This parallel language is seen in the use of the word '*infinitorum*' in Wallis's title as an arithmetic analogy to the geometric '*indivisibilium*', though in the text itself Wallis never used either term, referring instead to 'infinitely small parts'. This brings us to the most fundamental difficulty in Wallis's work. What is the precise nature of these infinitely small quantities? Do Wallis's lines have breadth or not? For the most part Wallis regarded a plane figure as the sum of its lines (see, for example, Proposition 3 and many others) but at other times (most notably in the *Comment* to Propositions 13 and 182) as a sum of arbitrarily thin parallelograms. In the *Arithmetica infinitorum* itself Wallis did not discuss the distinction between lines and parallelograms, nor the difficulties to which the alternative definitions could give rise, but they were already inherent and unresolved in the first page of *De sectionibus conicis.* There in Proposition 1, Wallis wrote:[36]

> I suppose, as a starting point (according to Bonaventura Cavalieri's geometry of indivisibles) that any plane is constituted, as it were, from an infinite number of parallel lines. Or rather (which I prefer) from an infinite number of parallelograms of equal altitude, the altitude of each of which indeed may be $\frac{1}{\infty}$ of the whole altitude, or an infinitely small part (for let ∞ denote an infinite number), and therefore the altitude of all taken together is equal to the altitude of the figure.

Wallis argued that a parallelogram of infinitely small altitude was no more than a line,[37] but at the same time such a line could be considered 'dilatable', or of some thickness, so that when infinitely multiplied it attained a definite height or width.[38] Therefore, said Wallis, he would call these infinitely small parts 'lines' rather than 'parallelograms', but with the understanding that they are in some definite ratio to the altitude of the whole figure, so that when infinitely multiplied they make up the total altitude of the figure.[39]

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[36] '*Suppono in limine (juxta Bonaventura Cavalerii* Geometriam Indivisbilium) *Planum quodlibet ex infinitis lineis parallelis conflari: Vel potius (quod ego mallem) ex infinitis Parallelogrammis aeque altis; quorum quidem singulorum altitudo sit totius altitudinis* $\frac{1}{\infty}$, *sive aliquota pars infinite parva; (esto enim* ∞ *nota numeri infiniti;) adeoque omnium simul altitudo aequalis altitudini figurae'; Wallis, De sectionibus conicis,* Proposition 1.

[37] '*Nam Parallelogrammum cujus altitudo supponitur infinite parva, hoc est, nulla, (nam quantitas infinite parva perinde est atque non-quanta,) vix aliud est quam linea'; Wallis, De sectionibus conicis,* Proposition 1.

[38] '*... quod linea haec supponitur dilatabilis esse, sive tantillam saltem spissitudinem habere ut infinita multiplicatione certam tandem altitudinem sive latitudinem possit acquirere';* Wallis, *De sectionibus conicis*, Proposition 1.

[39] '*... exiguae illius altitudinis eousque ratio habenda erit, ut ea infinities multiplicata totam figurae altitudinem supponatur adaequare';* Wallis, *De sectionibus conicis*, Proposition 1.

A similar argument could obviously be applied to planes with a thickness of $\frac{1}{\infty}$ of the total height of a solid.

This single proposition at the beginning of *De sectionibus conicis* contains the only serious discussion Wallis entered into on the nature of his infinitely small quantities. His method did at times lead to some paradoxical comments, as in Proposition 108 where he claimed that a finite altitude A was equal to some number that he had just claimed could be taken to be infinite, but Wallis merely ignored such problems. Hobbes, however, saw the difficulties immediately:[40]

> 'The triangle consists as it were' ('as it were' is no phrase of a geometrician) 'of an infinite number of straight lines.' Does it so? Then by your own doctrine, which is, that 'lines have no breadth', the altitude of your triangle consisteth of an infinite number of 'no altitudes', that is of an infinite number of nothings, and consequently the area of your triangle has no quantity. If you say that by the parallels you mean infinitely little parallelograms, you are never the better; for if infinitely little, either they are nothing, or if somewhat, yet seeing that no two sides of a triangle are parallel, those parallels cannot be parallelograms.

In a long *Scholium* following Proposition 182 Wallis set out some of his rules for handling an infinite number of small parts. Adding 1 to an infinite number, for example, left it unchanged, since according to Wallis, $\infty + 1 = \infty$ and $\infty - 1 = \infty$. As for multiplication and division, the reciprocal of zero is infinite and *vice versa*, so Wallis could write, for example, $\frac{1}{\infty} = 0$ or $\frac{1}{\infty} \times \infty = 1$ without qualms. Such rules can in the right circumstances be given a rigorous and correct interpretation, so Wallis was not as far adrift as he might have been, and his mathematical instincts enabled him for the most part to handle his infinite sums successfully. He was not always safe, however; his original assertion in Proposition 5, that the Archimedean spiral was equal in length to half the circumscribed circle, was wrong, and he was forced to add a caveat explaining that his result applied not to the true spiral but to a series of inscribed arcs. And in the *Scholium* after Proposition 182 he attempted to explain why an infinite sum of infinitely small parts might not always give the expected answer: a sum of parallelograms, for example, could be used to find the area of a triangle, but the sum of their sides would not, except in special circumstances, give the length of a side of the same triangle.

For Wallis, as for any other mathematician of the time, acceptable standards of rigour and proof were those of the Greeks, and Wallis was to argue thirty years later that the method of indivisibles was grounded in the classical method of exhaustions, by which a figure was approximated by a series of inscribed or circumscribed polygons:[41]

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[40] Hobbes 1656, 46.

[41] Wallis 1685, 280; see 280–290 for three consecutive chapters entitled 'The Method of Exhaustions', Of Cavallerius his Method of Indivisibles', and 'Of the Arithmetick of Infinites'.

> ... it will be necessary to premise somewhat concerning (what is wont to be called) the *Method of Exhaustions*, ... and the *Method of Indivisibles*, introduced by Cavalierius, (which is but a shorter way of expressing that Method of Exhaustions;) and of the *Arithmetick of Infinites*, (which is a further improvement of that Method of Indivisibles.)

Wallis had no concept in the modern sense of allowing a quantity to decrease continuously to zero or increase continuously to infinity. He did, however, use something very similar to a limit argument when he stated that a quantity that can always be made smaller than any assigned quantity can be taken to be zero. The idea seems modern, but again Wallis later argued that he found his justification in classical sources, in Euclid from Book X onwards, and in Archimedes:[42]

> And when in those Books following, [Euclid] had occasion to compare Quantities, wherein it was not easy by direct Demonstration, to prove their Equality; he takes this for a Foundation of his Process in such Cases: that *those Magnitudes* (or quantities,) *whose Difference may be proved to be Less than any Assignable are equal.* For if unequal, their Difference, how small soever, may be so Multiplied, as to become Greater than either of them: And if not so, then it is nothing.
> ... it is manifest in the opinion of *Archimedes*, (and as he tells us of Mathematicians before him,) that no Unequal Magnitudes can differ by so little, but that the difference may be so Multiplied as to exceed either or any other that bears any Proportion to either of them.

Basing his argument on such principles, Wallis was able to argue correctly, in the first published proof of its kind, that the difference between $1\frac{1}{z}$ and $1\frac{1}{z+1}$ tends to zero, and that both quantities tend to 1 as z becomes infinitely large.

The strangest of Wallis's concepts concerning infinity is that the ratio of a positive number to a negative number might be somehow 'greater than infinite'. He was led to this conclusion by the fact that $1/a$ grows infinitely large as a moves towards zero. If, therefore, a decreases *through* zero, the quantity $1/a$ must become both negative and 'greater than infinite'. At other times, however, Wallis used the usual rules of division for negative numbers, thus $\frac{1}{-2} = -\frac{1}{2} = \frac{-1}{2}$, so had no reason to consider the reciprocal of a negative quantity as 'greater than infinite', and his assertion has to be read in the specific geometric context to which it pertains, the quadrature of curves whose equations contain negative indices.

Two other fundamental mathematical concepts run through the whole of the *Arithmetica infinitorum.* From the first page to the end, Wallis relied on *induction*, and throughout the second half of the book, on *interpolation.* By induction, Wallis meant that a pattern established for a few cases could reasonably be assumed to continue indefinitely. Again his mathematical intuition rarely led him astray on this point, but some of his critics argued that

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[42] Wallis 1685, 282, see also 285.

it was hardly a satisfactory method of proof. Wallis replied that there were strong precedents, most recently in the work of Viète, who employed a similar kind of reasoning in *Ad angularium sectionum analyticen theoremata*,[43] and in Briggs, who made use of Viète's results on angular section to interpolate his tables of trigonometric logarithms. A much earlier precedent, according to Wallis, was to be found in Euclid, who allowed one triangle, for example, to stand for an infinite number of others. This was indeed a form of induction, albeit a rather loose one, but the reasoning used by Viète and Wallis began to have elements of modern mathematical induction, insofar as they supposed that an argument from one case to the next could be continued indefinitely. Wallis failed to make a distinction between Euclid's inductive arguments and his own, and was probably not even interested in doing so; for him induction was an obvious and natural process that needed no further justification.

Interpolation was the second cornerstone of Wallis's method, and all his later results depended upon it, but again he relied on intuition and made no attempt to justify the process beyond the fact that it worked. Perhaps the most remarkable example in the *Arithmetica infinitorum* was Wallis's willingness to interpolate between the triangular numbers 1, 3, 6, 10 ... or the pyramidals 1, 4, 10, 20, ... etc. Such figurate numbers had always been thought of as, by definition, integers, arrangements of pebbles or points, and it made no geometric sense to look for, say, a triangular number between 3 and 6. In fact, without explicitly saying so, and indeed without even being aware of it to begin with, Wallis began to treat the numbers 1, 3, 6, 10,... as equally spaced points on a continuous curve, so that all intermediate values existed and could in principle be calculated or described. Wallis was correct, of course, in that the figurate numbers are the integer values of continuous polynomial functions (whose equations he went to some lengths to find in Propositions 171 to 182), but formal definitions of functions or continuity still lay far into the future. In the final three short propositions of the book, however, Wallis did attempt to describe the underlying continuity on which his entire method depended, using the image of a smooth curve, which could be constructed from a few known (integer) points.

The final aspect of Wallis's mathematics to which we must draw attention here is his sense that the number we now call π, the ratio of the circumference of a circle to its diameter, could not be expressed in any numbers so far known, either rationals or surds. Nevertheless, he pointed out, the number could be calculated to any degree of accuracy and clearly satisfied all the usual rules of arithmetic, and therefore must be considered as valid as any other commonly accepted number. The irrationality of π would not be proved for another

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[43] '*Atque eo in infinitum progressu, dabitur laterum ratio in ratione anguli ad angulum multipla, ut praescriptum est*'; 'By this process to infinity, there will be given the ratio of the sides for the ratio of any angle to multiple angles, as prescribed'. Viète 1646, 290; Viète 1983, 424

hundred years, and its transcendence only a century after that,[44] but there can be no doubt at all that Wallis in 1655 was aware of the special and elusive nature of the number he was dealing with.[45]

The reception of the Arithmetica infinitorum

The first critical reaction to the *Arithmetica infinitorum* came from Christiaan Huygens, who was sent a copy of the book by Frans van Schooten in July 1656.[46] Huygens thanked Wallis politely for mentioning his own *Exetasis* in the Dedication, and promised to pass Wallis's result on to Aynscom, a disciple of de Saint-Vincent, who had recently refuted various methods of quadrature of the circle, including one by Huygens.[47] As to the text itself, Huygens expressed some reservations: he missed the crucial point of Wallis's argument in Proposition 191 and so could not understand Wallis's final proof; he felt that Wallis should proceed further with concrete numerical examples since he felt that induction was not a clear or certain enough method to resolve his doubts; and he argued that the curves produced by Wallis at the end of his book were not *geometric* in Descartes' sense, for there was no known formula for finding a general point. It was not enough, said Huygens, to say that the curves were smooth, for there could be many smooth curves that would pass through the few fixed points.

Wallis replied that Brouncker had now calculated a value for the ratio of the circumference to the diameter using Wallis's fraction (or one of Brouncker's own) and found it in perfect agreement with the values known from other methods.[48] To support his use of induction he pointed to Viète's *Ad angularium sectionum*, Briggs' *Arithmetica logarithmica*, Clavius' edition of Euclid V. 1–34, Euclid himself in propositions I.21, VI.20 and XII.1 (all of which are general propositions about triangles or polygons), and Archimedes almost everywhere. To Huygens' final objection, that Wallis's curves were not geometric in Descartes' sense, Wallis repeated what he had said in the *Arithmetica infinitorum* itself: that half of his curves were certainly geometric, and that the rest were equally well defined even if there was no known formula.[49]

Fermat in Toulouse received the *Arithmetica infinitorum* a year later, in the summer of 1657, through Kenelm Digby, and like Huygens raised some objections. His first complaint was that he himself had already found many of the same results; his second was that he could not understand why Wallis had

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[44] The irrationality of π was first proved by Johann Heinrich Lambert in 1761 and its transcendence by Ferdinand von Lindemann in 1882.

[45] See Panza 1995.

[46] Huygens to Wallis, [11]/21 July 1656, Beeley and Scriba 2003, 189–192.

[47] Huygens 1654; Aynscom 1656.

[48] Wallis's fraction for $4/\pi$ converges too slowly to be of practical use, and it seems much more likely that Brouncker used one of his own related continued fractions to calculate upper and lower bounds for π.

[49] Wallis to Huygens, 12/[22] August 1656, Beeley and Scriba 2003, 193–197.

chosen to work in symbols rather than by traditional Archimedean methods (even though Wallis's book, as pointed out above, uses algebraic notation only sparingly, and far less than might have been expected).[50] In a supplementary letter headed *Remarques sur l'Arithmetique des Infinis du S. J. Wallis*,[51] he also put forward four specific criticisms, none of them concerned with anything beyond Proposition 2, suggesting that he had not in fact read very far. First, he argued, Wallis's hope of finding the ratio of a sphere to a cylinder was impossible without first finding the quadrature of the circle itself; second, that it made little sense to ask for intermediate numbers between 1, 6, 30, 140, 630, ..., and indeed if one regarded 6 (as Wallis did) as $1 \times \frac{6}{1}$, then the number between 1 and 6 had to be found using a multiplier greater than 6 itself, which was absurd; third, the sum of an arithmetic progression could be found without resorting to induction; and fourth, that such a sum did not require the second term of the progression to be 1. Wallis's replies were part of a very long letter he wrote to Fermat on a number of subjects in November 1657.[52] He pointed out that his intention was not simply to obtain results but, unlike the Ancients, to demonstrate the methods by which they could be found, and he was unapologetic about his use either of induction or algebraic notation. He did not disagree with any of Fermat's specific criticisms, but considered them adequately answered in the *Arithmetica infinitorum* itself, and ended by saying that if Fermat were to look at the book again and ponder it a little more carefully, he would find his objections long since answered.[53]

Fermat's peevishness arose in part, no doubt, from Wallis and Brouncker's somewhat dismissive treatment of the number problems he had sent them earlier in 1657,[54] but perhaps also from the fact that he had indeed obtained some of Wallis's results many years before. Wallis could not have known it, for Fermat had never published his findings, but he had found quadratures for the higher parabolas as early as 1636, and for the higher hyperbolas by 1646.[55] Wallis had gone beyond this, and by different methods; nevertheless it was probably the appearance of the *Arithmetica infinitorum* in 1656 that prompted Fermat at last to write down some of his own results in 1658 or 1659.[56]

The most forthright criticism of the *Arithmetica infinitorum* undoubtedly came from Thomas Hobbes whose first (but not last) attack appeared in his

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[50] Fermat to Digby, [5]/15 August 1657, *ibid.* 294–297.

[51] Enclosed in Brouncker to Wallis, 6/[16] October 1657, *ibid.* 311–316.

[52] Wallis to Digby for Fermat, 21 Nov/[1 Dec] 1657, *ibid.* 334–337.

[53] *'Si enim exinde otii quid nactus sit Fermatius eadem secundo inspiciendi, & paulo accuratius pensitandi, non dubito quin jam ipse sibi pridem satisfecerit'*; *ibid.* 337. Wallis repeated Fermat's objections and his own refutations of them many years later (long after Fermat himself had died) in *A treatise of algebra*, Wallis 1685, 305–309.

[54] See Stedall 2002, 196–207.

[55] Mahoney 1973, 214–238; 244–267.

[56] *De aequationum localium transmutatione ... cui annectitur proportionis geometricae in quadrandis infinitis parabolis et hyperbolis usus*, published in Fermat 1679, 44–57.

Six lessons to the professors of the mathematiques of 1656. He did not mince his words: 'I verily believe that since the beginning of the world there has not been nor ever shall be so much absurdity written in geometry as is to be found in those books of [Wallis's]'.[57] Hobbes's objections were many, and some were absurd, but as Augustus De Morgan pointed out a century ago, Hobbes 'was not the ignoramus in geometry that he is sometimes supposed. His writings, erroneous as they are in many things, contain acute remarks on points of principle.'[58] Hobbes's criticisms pinpointed three main areas: Wallis's use of algebraic symbols; of induction; and of indivisibles.

Like Fermat a year later, Hobbes objected strongly to Wallis's use of algebraic symbolism:[59]

> [Wallis] mistook the study of *Symboles* for the study of *Geometry*, and thought *Symbollical* writing to be new kind of Method, and other men's Demonstrations set down in *Symboles* new Demonstrations. ... I never saw anything added thereby to the Science of Geometry, as being a way wherein men go round from the Equality of rectangled Plains to the Equality of Proportion, and thence to the Equality of rectangled Plains, wherein the *Symboles* serve only to make men go faster about, as greater Winde to a Winde-mill.

As we have noted already, Wallis used only a limited amount of algebraic notation in the *Arithmetica infinitorum*, so Hobbes was perhaps tilting at windmills in a different sense. His struggle against algebraic symbolism in this and other contexts now seems like a futile attack on the wrong enemy, but it arose from Hobbes's belief that mathematics should be based on the material and sense-perceptible, that is on space and movement. Thus for Hobbes geometry was the true foundation of mathematics, and the introduction of symbols served merely to confuse the reader and obscure the truth:[60]

> Had Pappus no analytiques? Or wanted he the wit to shorten his reckoning by signes? Or has he not proceeded analytically in an 100 problems and never used symbols? Symboles are poor unhandsome (though necessary) scaffolds of demonstration; and ought no more to appear in publique, then the most deformed necessary business which you do in your chambers.

Hobbes's second objection was to induction, and he railed against 'egregious logicians and geometers that think an *Induction* without a *numeration* of all the particulars sufficient to infer a Conclusion universall'. Wallis merely replied

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[57] *Six lessons to the professors of mathematiques, one of geometry, the other of astronomy: in the chaires set up by Sir Henry Savile in the University of Oxford*, Hobbes 1656, Introduction [dated 10 June 1656].

[58] De Morgan 1915, 110; for a modern analysis of Hobbes's mathematics see Jesseph 1993 and 1999.

[59] Hobbes 1656, Introduction.

[60] Hobbes 1656, 23. For more on Hobbes's philosophy of mathematics and his objections to algebraic geometry see Jesseph 1993, 167–181 and Jesseph 1999, 240–246.

that induction was justified 'if after the enumeration of some particulars comes the general clause: "and the like in other cases" ', and again sought justification in Euclid: 'If not, no proposition of Euclid is demonstrated.'[61]

Hobbes's most accurate and damaging criticism was aimed at Wallis's use of indivisibles. Part of his argument has already been quoted above; indivisibles must be either 'something or nothing', and in either case, according to Hobbes, contradictions followed:[62]

> The least altitude, is somewhat or nothing. If somewhat, then the first character of your arithmetic progression must not be a cipher, and consequently the first eighteen propositions of this your *Arithmetica infinitorum* are all nought. If nothing, then your whole figure is without altitude, and consequently your understanding nought.

Hobbes was not quite right here; the first term of an arithmetic progression did not need to be zero, as Wallis had explained elsewhere. The real problem with a quantity that was 'somewhat' was that it could not be multiplied infinitely many times to produce a finite result. In *Due correction for Mr. Hobbes*, published in 1656,[63] Wallis tried to explain more clearly what he meant by an indivisible, now shifting slightly from lines to parallelograms, but still unable to escape the fundamental problem:[64]

> I do not mean precisely a line but a parallelogram whose breadth is very small, viz an aliquot part [divisor] of the whole figures altitude, denominated by the number of parallelograms (which is a determination geometrically precise).

This did not answer Hobbes's argument, and indeed contradicted Wallis's own claim elsewhere that the number of such very small parallelograms could be considered infinite. Wallis ended his chapter entitled '*Arithmetica infinitorum* vindicated' with the words: 'Well, *Arithmetica infinitorum* is come off clear',[65] but it had not, for Hobbes had made valid objections.[66] The truth was, however, that Wallis did not greatly care about the philosophical foundations of his method provided that it worked, and clearly it did. The argument with Hobbes raged backwards and forwards through further pamphlets. Hobbes in his *ΣΤΙΓΜΑΙ* of 1657 protested:[67]

> You do shift and wriggle and throw out ink, that I cannot perceive which way you go, nor need I, especially in your vindication of your *Arithmetica*

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[61] Hobbes 1656, 46; Wallis 1656c, 41.

[62] Hobbes 1656, 46.

[63] *Due correction for Mr Hobbes, or school discipline, for not saying his lessons right*, Wallis 1656c, 41–50.

[64] Wallis 1656c, 47.

[65] Wallis 1656c, 50.

[66] See also Jesseph 1993, 187–189 and Jesseph 1999, 177–185.

[67] *ΣΤΙΓΜΑΙ or markes of the absurd geometry, rural language, Scottish church-politicks and barbarisms of John Wallis*, Hobbes 1657, 12.

> *infinitorum* ... your book of *Arithmetica infinitorum* is all nought from the beginning to the end.

Wallis retaliated in *The undoing of Mr Hobs's points*, also published in 1657,[68] but by now the quarrel between them had taken on its own momentum. It was to end only with Hobbes's death in 1679, and the details have been fully described by others.[69]

All the early readers of the *Arithmetica infinitorum*, Huygens, Hobbes, and Fermat, homed in on those parts of the Wallis's argument that were indeed less than soundly based: his use of indivisibles and induction, and his assumption of a range of a continuous and definable values between the numbers of a sequence. Nevertheless, methods based on indivisible or infinitely small quantities came increasingly into use amongst his contemporaries. In 1657, William Neile, a young student at Wadham College, Oxford, found the rectification of the semicubical parabola (in modern notation $9y^2 = 4kx^3$) by a method that was geometric but involved a comparison of sums of infinitely small quantities. Wallis was easily able to make Neile's proof algebraic using the notation defined in *De sectionibus conicis*, while William Brouncker went further and came up with a formula for the length of a portion of the curve in terms of its coordinates.[70] At about the same time, Hendrick van Heuraet in the Netherlands arrived independently at a general method of rectification, and it applied it to the semicubical parabola,[71] and in 1659 Fermat rectified both the semicubical parabola and the cycloid.[72] Wallis later claimed that all these attempts were based on the hints he had given in the *Arithmetica infinitorum*:[73]

> And I do not at all doubt that this notion there hinted, gave the occasion (not to Mr Neil only, but) to all those others (mediately or immediately,) who have since attempted such Rectification of Curves (nothing in that way having been attempted before;)

It was true that Wallis had outlined a method of rectification in the *Arithmetica infinitorum*, and Neile may have been inspired by it, but Neile's method was expressed in traditional geometric terms and he handled a curve that Wallis had not thought about at all. Meanwhile Huygens had discovered the relationship between the rectification of the parabola and the quadrature of the hyperbola. He may have had hints of this idea from Wallis's Proposition 38 where it is clear enough but, as with Neile, his result is expressed

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[68] *Hobbiani puncti dispunctio, or the undoing of Mr Hobs's points: in answer to M. Hobs's ΣΤΙΓΜΑΙ, id est STIGMATA HOBBII*, Wallis 1657.

[69] For further attacks and counter-attacks between Hobbes and Wallis see the bibliography. See also Grant 1996; Probst 1997; Jesseph 1999.

[70] Wallis published all three methods, Neile's, his own, and Brouncker's, in Wallis 1659, 75–123; 91–96; reprinted in Wallis 1693–99, I, 542–569; 550–554.

[71] Van Heuraet 1659; see Van Maanen 1984.

[72] Fermat 1660; see Mahoney 1973, 267–281.

[73] Wallis 1685, 298.

in traditional geometrical language, and this and similar results were likely to have arisen from his own longstanding interest in problems of quadrature rather than in any clues he had picked up from Wallis.[74] Van Heuraet in turn had received only the vaguest reports of Huygens' ideas, and must be given credit for an independent discovery.[75] Such ideas were steadily becoming more widespread in a variety of contexts. Nevertheless it remains true that Wallis was the first to hint at the possibility of a general method of rectification, a problem previously considered by Descartes and others to be impossible.

In questions of quadrature, Wallis's work certainly did have repercussions, and important ones. In 1668 Nicolaus Mercator found the quadrature of the hyperbola by writing its equation as:

$$y = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

and summing the individual terms by Wallis's methods to obtain:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Mercator published his findings in his *Logarithmotechnia* of 1668.[76] Wallis reviewed the book in the *Philosophical transactions* that same year, and referred the reader twice to his own results in the *Arithmetica infinitorum.*[77]

But it was in the hands of Isaac Newton that the *Arithmetica infinitorum* finally came into its own. Newton read the book in the winter of 1664–65 when he was just twenty-two years old, and made extensive notes.[78] His writing did not stop when his reading finished; Newton's train of thought continued uninterrupted where Wallis's had left off, as he saw how to extend and consolidate Wallis's ideas. He recognized the power of Wallis's interpolative methods for handling curves that lay between those whose properties were already known, but he moved far beyond Wallis in introducing an algebraic variable. Thus where Wallis had written simple numerical sums, Newton wrote infinite power series in which the coefficient of each power was defined and clearly visible. For the partial area of a quadrant, for example, using the interpolated values calculated by Wallis, Newton wrote:[79]

$$A = x - \frac{1}{2}\cdot\frac{x^3}{3} - \frac{1}{8}\cdot\frac{x^5}{5} - \frac{1}{16}\cdot\frac{x^7}{7} - \frac{5}{128}\cdot\frac{x^9}{9} - \dots$$

(where, in modern notation, $A(x) = \int_0^x (1-t^2)^{\frac{1}{2}} dt$).

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[74] See Van Maanen 1984, 241–242 for Huygen's formulation of his result and 245–250 for a possible reconstruction of his methods.

[75] Van Maanen 1984, 222–250.

[76] Mercator 1668.

[77] Wallis 1668; see especially 754, 755.

[78] Newton 1664.

[79] Newton 1664, 108.

As Newton's work progressed he also began to see how he could use a method of interpolation different from Wallis's.[80] Where Wallis had regarded his sequences as generated by multipliers, so that, as we have seen, he wrote 1, 3, 6, 10, ... as $1 \times \frac{3}{1} \times \frac{4}{2} \times \frac{5}{3} \times \cdots$, Newton saw that the same sequence could be generated by addition, so that 1, 3, 6, 10, ... could be written as a, $a+b$, $a+2b+c$, $a+3b+3c$, $a+4b+6c, \ldots$ with $a=1$, $b=2$, $c=1$; in other words, with a pattern of constant second differences.[81] Like Wallis, Newton assumed that the overall pattern would hold for any intermediate terms, and because his method was simpler than Wallis's, he could interpolate not just one, but two, three, or more such terms between any two entries of a sequence. He could also extrapolate backwards to negative numbers, something that Wallis had never attempted to do. Thus Newton could find coefficients in the power series expansion of $(1+x)^{p/q}$ for any rational value p/q either positive or negative. In short, building on Wallis's methods and sequences he discovered the coefficients of the general binomial theorem. For Newton this opened up immense possibilities, for now he could express trigonometric and logarithmic quantities by means of infinite series, for example,

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots$$

$$\text{antilog } z = z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots$$

In other words, Newton could treat such quantities as functions of a free variable (though the formal concept of a function did not enter mathematics until some sixty years later). Further, he could integrate and differentiate such functions by operating on the series term by term.

In 1669 Newton wrote up his results in *De analysi per aequationes numero terminorum infinitas*, which he sent privately to Isaac Barrow and John Collins,[82] and he wrote a more extended account in 1676 to Leibniz in two long letters now known as the *Epistola prior* and *Epistola posterior.*[83] In those letters he was explicit about his debt to Wallis,[84] and Wallis was not slow to respond. By 1676 Wallis had completed a large part, possibly the first seventy-two chapters, of *A treatise of algebra.* It was probably Newton's *Epistola posterior* that prompted him to add a further twenty-five chapters in

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[80] Newton 1665; see also Whiteside 1961, Dennis and Confrey 1996, Stedall 2002, 175–180.

[81] Newton 1665, 130.

[82] Newton 1669; though sent to Barrow and Collins in 1669, De analysi remained unpublished until it appeared in Newton 1711.

[83] Newton to Oldenburg, 13 June and 24 October 1676, letters 165 and 188 (and 189) in Turnbull 1959–77, II, 20–47 and 110–163.

[84] Newton 1676b, 111, 130; Newton to Wallis, July 1695, letter 519 in Turnbull 1959–77, IV, 140.

which he outlined the methods and significance of the *Arithmetica infinitorum*, and published substantial extracts from Newton's letters.[85]

Wallis himself, for all the adverse criticism his book had received when it first appeared, had never doubted its worth, and Newton's results were a vindication of his methods. He would have been the first to agree with David Gregory's later accolade:[86]

> The *Arithmetica infinitorum* has ever been acknowledged to be the foundation of all the Improvements that have been made in Geometry since that time.

From a longer perspective it is possible to arrive at a more objective assessment of Wallis's mathematics, but the historical importance of his ideas is not in doubt. Almost two centuries after the *Arithmetica infinitorum* was written, in 1821, Charles Babbage in an unpublished essay entitled 'Of induction' wrote:[87]

> Few works afford so many examples of pure and unmixed induction as the *Arithmetica infinitorum* of Wallis and although more rigid methods of demonstration have been substituted by modern writers this most original production will never cease to be examined with attention by those who interest themselves in the history of analytical science or in examining those trains of thought which have contributed to its perfection.

Because Wallis's text even now gives important insights into the development not only of induction but of so many other seminal ideas of mid seventeenth-century mathematics, this present translation, the first into English, is now offered to a new generation of readers.

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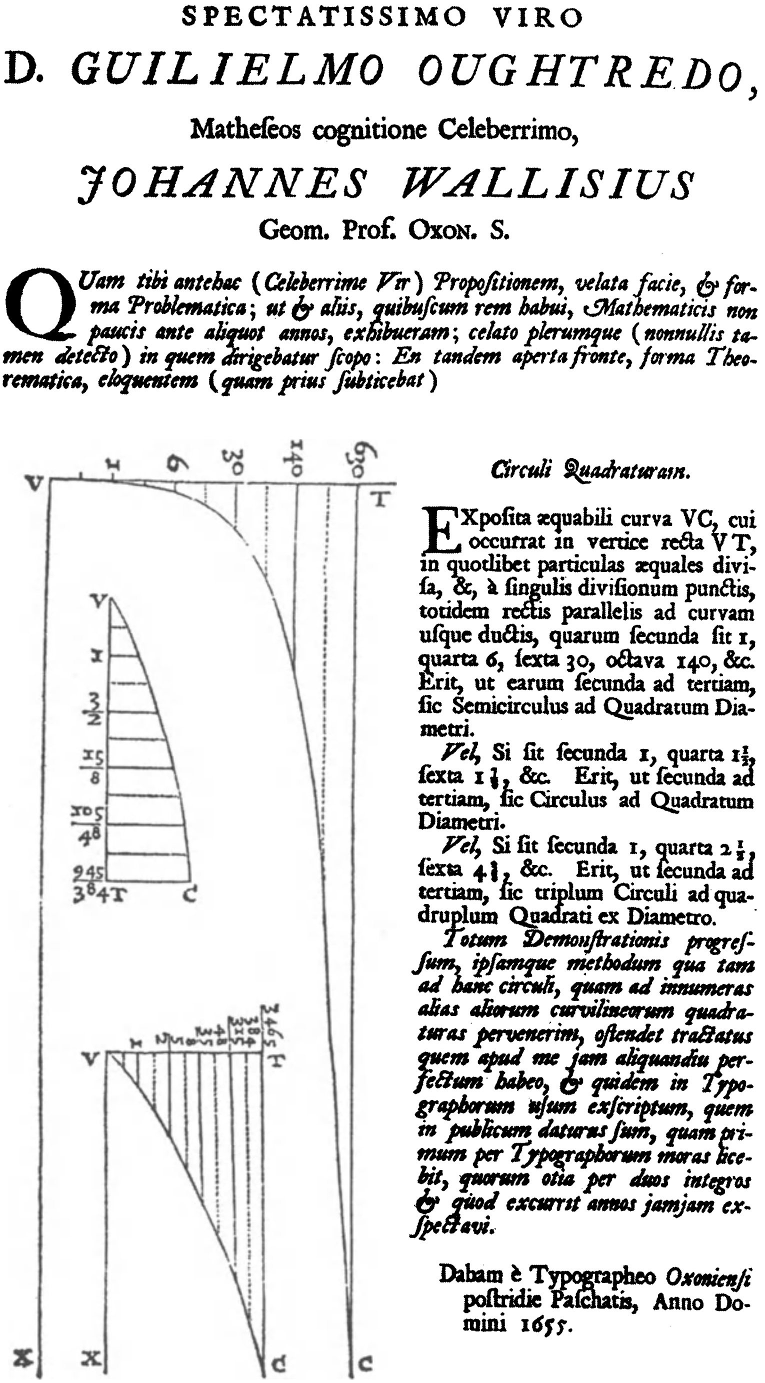
[85] Wallis 1685, 330–346. Wallis published the Epistola prior almost in its entirety together with some supporting material from the Epistola posterior (Turnbull 1959–77, III, 220, note 4, is not quite accurate on this point).

[86] Bodleian Library MS Smith 31, f. 58.

[87] British Library Add MS 37202; Dubbey 1978, 109–114.



SPECTATISSIMO VIRO

D. *GUILIELMO OUGHTREDO*,

Matheſeos cognitione Celeberrimo,

JOHANNES WALLISIUS

Geom. Prof. OXON. S.

Quam tibi antehac (Celeberrime Vir) Propoſitionem, velata facie, & forma Problematica; ut & aliis, quibuſcum rem habui, Mathematicis non paucis ante aliquot annos, exhibueram; celato plerumque (nonnullis tamen detecto) in quem dirigebatur ſcopo: En tandem aperta fronte, forma Theorematica, eloquentem (quam prius ſubticebat)

Circuli Quadraturam.

Expoſita æquabili curva VC, cui occurrat in vertice recta VT, in quotlibet particulas æquales diviſa, &c, à ſingulis diviſionum punctis, totidem rectis parallelis ad curvam uſque ductis, quarum ſecunda ſit 1, quarta 6, ſexta 30, octava 140, &c. Erit, ut earum ſecunda ad tertiam, ſic Semicirculus ad Quadratum Diametri.

Vel, Si ſit ſecunda 1, quarta $1\frac{1}{2}$, ſexta $1\frac{7}{8}$, &c. Erit, ut ſecunda ad tertiam, ſic Circulus ad Quadratum Diametri.

Vel, Si ſit ſecunda 1, quarta $2\frac{1}{2}$, ſexta $4\frac{3}{8}$, &c. Erit, ut ſecunda ad tertiam, ſic triplum Circuli ad quadruplum Quadrati ex Diametro.

Totum Demonſtrationis progreſſum, ipſamque methodum qua tam ad hanc circuli, quam ad innumeras alias aliorum curvilineorum quadraturas pervenerim, oſtendet tractatus quem apud me jam aliquandiu perfectum habeo, & quidem in Typographorum uſum exſcriptum, quem in publicum daturus ſum, quam primum per Typographorum moras licebit, quorum otia per duos integros & quod excurrit annos jamjam exſpectavi.

Dabam è Typographeo *Oxonienſi*
poſtridie Paſchatis, Anno Domini 1655.

Fig. 1. An advertisement of the forthcoming *Arithmetica infinitorum*, Easter 1655.

To the most Distinguished and Worthy gentleman and most Skilled Mathematician,

Dr William Oughtred

Rector of the church of Aldbury in the county of Surrey

Here for you at last (most distinguished gentleman) is now the whole of that work of which I gave hope in that proposition on circle measurement that I gave you in its stead in print last Easter (see Figure 1). For since, by custom, when one puts something out in public, it ought to be dedicated to someone, I thought to seek not only a great gentleman but a great mathematician to whom I might offer it. And therefore I saw that to none other greater than you can that easily be done, who is among mathematicians most deserving, and also by whose writings I readily confess that I have profited: who indeed in your *Clavis mathematicae*, though not a large work, have there taught both briefly and clearly, what we seek in vain in the large volumes of others.[1]

You may find this work (if I judge rightly) quite new. For I see no reason why I should not proclaim it; nor do I believe that others will take it wrongly. For although it is not to be doubted that indeed known propositions are mixed here and there among others (which must necessarily be done, partly so that light would shine from them to others, and so that I would not seem to contrive something that has no relationship to what mathematicians have already discovered or perfected; partly also lest this work itself come out both maimed and crippled, since those things follow immediately from our principles in such a way that, even if they were otherwise unknown, they necessarily here immediately become known; and indeed I have not previously found most of them to stand out in the works of others, even the most distinguished of them, other than those I have arrived at by this method); since, however, this also has much that is new, indeed neither discovered by nor known to others, and it teaches all by a new method, introduced by me for the first time into geometry, and with such clarity (unless I perhaps praise myself too much) that in these more abstruse problems no-one (as far as I know) has used: that is why I would not hesitate to call it new.

Certainly this method of mine takes its beginning where Cavalieri ends his *Method of indivisibles*. Whence the key is given both to the work itself and to

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[1] William Oughtred, *Arithmeticae... quasi clavis est*, London 1631.

its title; for as his was the *Geometry of indivisibles*, so I have chosen to call my method the *Arithmetic of infinitesimals.*[2]

By what means I arrived at it, moreover, it seems less necessary to say, since almost everything found by that method has been written; however, since I judge that it will not be unwelcome to you, I here also briefly bring together the history of the thing.

Around 1650 I came across the mathematical writings of Torricelli (which, as other business allowed, I read in the following year, 1651), where among other things, he expounds the *geometry of indivisibles* of Cavalieri.[3] Cavalieri himself I did not have to hand, and I sought for it in vain at various booksellers. His method, as taught by Torricelli, moreover, was indeed all the more welcome to me because I do not know that anything of that kind was observed in the thinking of almost any mathematician I had previously met;[4] for what holds for most of them concerning the circle (which was usually had by means of polygons with an infinite number of sides, and therefore the circumference by means of an infinite number of infinitely short lines) could also, it seemed to me, with appropriate changes, be usefully adjusted to other problems; and indeed by that means to examine not a little of what is found in Euclid, Apollonius and, especially, throughout Archimedes. Those things, moreover, I thought about as yet only in a disordered way, not yet in the order I would bring them to. For other business has not allowed Mathematicians openly to devote their attention to it, but only to indulge a few spare hours; whence I first felt called to that duty which I now attempt; because nothing before came very close to it.

Once I had perceived that a method of this kind had been obtained, I began to think to myself whether this might not bring some light to the quadrature of the circle, which is known always to have exercised the greatest of men. The hope of doing which, it seemed, was here. The ratio of a cone composed of an infinite number of circles to a cylinder of the same number was already known, namely 1 to 3; moreover all the diameters making a triangle along the axis of the cone, to the same number making a parallelogram along the axis of the cylinder, are (as is known) as 1 to 2. Equally all the circles in a parabolic conoid, to the same number of circles in a cylinder were known to have a ratio of 1 to 2; moreover all the diameters of the former to the diameters of the latter are as 2 to 3. It was also clear that the lines of a triangle are arithmetic proportionals, or as 1, 2, 3, etc. and so the circles of a cone (which are as the squares of the diameters) as 1, 4, 9, etc. In the same way the circles of a parabolic conoid (which are as the squares of the ordinates, that is, in the ratio of the diameters [of the parabola]), are as 1, 2, 3, etc. and therefore their

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[2] Wallis himself translated the title as 'Arithmetick of Infinites; for discussion of the title and its translation see Introduction p. xvii.

[3] Bonaventura Cavalieri, *Geometria indivisibilibus continuorum nova quadam ratione promota*, Bologna 1635; Evangelista Torricelli, *Opera geometrica*, Florence 1644.

[4] Wallis did not know Roberval's 'Traite des Indivisibles', which was not published until 1693. See Introduction p. xiv and note 15.

diameters as $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, etc., indeed as the square roots of their circles. I hoped it might therefore be possible that, from the known ratios of other series of circles, or (which comes down to the same thing) of squares, to the same number of equals, there might be found also the ratio of their diameters or sides to the same number of equals. Moreover if I could find this by some general method, the quadrature of the circle would be sufficiently in sight. For since, as was already known, all the parallel circles in a sphere, to the same number in a cylinder, are as 2 to 3, if thence there could be investigated the ratio of all the diameters of the former to the diameters of the latter, there would be found what was sought: for certainly the diameters of the former constitute a circle, the latter the square of the diameter. Thus a geometric problem is reduced purely to arithmetic.

Therefore I devoted myself to this investigation at the end of that year, 1651, and the beginning of the next, 1652, by that very method that this treatise indicates. I imagined that thence, either it was possible at some time to establish by what means the circle could be squared, or instead that it could indeed not be squared, or that at least something would emerge that would make the work worthwhile.

I therefore began first (so as to start from the more simple cases) with simple series, that is, of quantities in arithmetic proportion, or of their squares, cubes, etc. and then also their square roots, cube roots, etc. and powers composed from these, thus, square roots of cubes etc. or also whatever other composites, whether the power was rational or even irrational. In all of which, the thing indeed came out just as wished for, and more than was hoped for. Whence eventually a general theorem emerged, taught at Proposition 64. But also at the same time there was produced the quadrature not only of the simple parabola, shown by a new method, but also of all higher parabolas, and of their complements, which no-one before, as far as I know, began to address, let alone achieved.[5] And therefore here immediately I felt had enlarged geometry; for since previously the simple parabola was almost the only curved figure whose quadrature was known, there may now be taught by a single proposition the quadrature of all higher parabolas of infinitely many kinds and indeed by one general method. And indeed if the quadrature of one parabola rendered so much fame to Archimedes (so that then all mathematicians since that time placed him as though on the columns of Hercules), I felt it would be welcome enough to the mathematical world if I taught the quadrature also of infinitely many kinds of figures of this sort. But also I saw here the same doctrine widened to conoids and pyramids. For since Archimedes taught correctly only of conoids and spheroids (as also others after him), no mention was ever made of pyramids; I have related everything, whether conoids or pyramids, either erect or inclined, to cylinders and prisms. Not only for those formed from simple parabolas but also for those from all higher parabolas and from their complements, on which so far there has been

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[5] Fermat had done this, but Wallis was ignorant of it, see Introduction p. xiv and note 15.

complete silence from everyone, nor has anyone (as far as I know) anywhere attempted it. But also I saw here that it was possible to derive as a direct consequence an almost complete teaching of spirals; and indeed I have taught the comparison with a circle, not only for the space contained within the usual spirals (as Archimedes did), but also for that contained within other spirals. But also that teaching on the spiral, no less than that on parabolas, was capable of extension, except that I did not wish to digress too much to corollaries.

Passing then to augmented series (as I call them) and those diminished, or altered, which are constituted from sums or differences of two or more other series. And here also the outcome is not at all to be disparaged. That is, that it was not difficult to relate everything to series of equals; in particular I saw that it was no more work than that to relate conoids or spheroids, or even pyramids, not only erect but inclined, to cylinders or prisms. Not only for those arising from simple hyperbolas or ellipses, or also those that can be formed from higher hyperbolas or ellipses in a thousand ways; but I did not consider it necessary to dwell on listing them separately, lest I spread myself too far, especially since anyone can see what can be done by his own efforts from what has been taught.

Moreover, I have continued the investigation with the same success not only for those series, whether augmented or diminished, but also for those which are as the squares, cubes, or any higher power of them, as is to be seen from those propositions which follow afterwards. Where at the same time we made use of the figurate numbers, thus, triangular, pyramidal, etc. (which no-one until now has, except sparingly, made use of, and then almost as a game) and their distinguishing features were unexpectedly uncovered.

But where we next proceeded to other series which were as the square roots, cube roots, etc. of those augmented or diminished series (which have a direct and immediate bearing on the quadratures of the circle, ellipse or hyperbola, and which alone now remain a difficulty) I saw that I was there brought to a standstill, and that I was not able to extricate myself as easily as before. Having tried the thing in various ways, there was nevertheless no way out of it that would satisfy all that was wished for. From that it came about that I believed that ratio that was sought to be of a kind that was not to be expressed in true numbers, nor indeed in surd numbers (as they are commonly called). For I had found some progressions of numbers, between given terms of which, another term was to be interposed, in order to express the sought ratio. Moreover those progressions are of such a kind that they cannot be said to be either arithmetic (where the continual increases are equal) or geometric (where the continual multipliers are equal), but are such that the continual multipliers include arithmetic proportionals, and are therefore yet more complex than geometric progressions. Moreover, although in a geometric progression (where the continual multipliers are equal) it is sometimes the case (for example in 1, 4, 16, 64, etc.) that the means that have to be interpolated between the terms of it are expressible numbers, the same is not possible in every case (thus in 1, 2, 4, 8, etc.), but it may be necessary to indicate an impossible number in some way (thus $\sqrt{2}$, $\sqrt{8}$, etc.); I judged that it will be much less to be hoped for, in a progression yet more complex (where the continual multipliers are continually increasing or decreasing) that this might always be done; and therefore I thought that there must be introduced some other method of notation (than any so far accepted), by which such an impossible number might be indicated.

And so far had I arrived at the beginning of the year 1652, by the time (as I remember) of Lent; at the time, that is, according to our academic constitutions, a series of public lectures is given, and therefore more time away from private investigations.

Moreover, while I stopped here, it seemed good to share the thing with other mathematicians with whom I was friendly, that I might see whether they could be of help in designating the sought quantity. And therefore from the various progressions of this kind that I had taken hold of, I picked out one, which seemed to be the simplest of all (as it progressed in whole numbers) namely that now to be found at Proposition 192 of this treatise, and I brought out the problem almost in this form (for I proposed it in not exactly all the same words, but nevertheless in the same sense): *If any smooth curve touches a line at its vertex, from which line to the curve there are taken lines parallel to the axis, equally spaced from each other, of which the first is 1, the second 6, the third 30, the fourth 140, the fifth 630, etc. what is the size of that which must be interposed between 1 and 6?* Or also arithmetically: *In a series of numbers 1, 6, 30, 140, 630, etc., there is sought the mean term to be placed between 1 and 6?* Moreover, I indicated how those terms arise, from continued multiplication of the numbers $1 \times 4\frac{2}{1} \times 4\frac{2}{2} \times 4\frac{2}{3} \times 4\frac{2}{4}$ etc. or also $1 \times \frac{6}{1} \times \frac{10}{2} \times \frac{14}{3} \times \frac{18}{4}$ etc., of which both the numerators and denominators are arithmetic proportionals. The problem so drawn up I proposed to the minds of the following (among others) the most distinguished gentleman and mathematician Dr Seth Ward, Savilian Professor of Astronomy and my most deserving colleague; Lawrence Rook, then for some time at Oxford but afterwards Professor of Astronomy at Gresham College in London; and Richard Rawlinson, Fellow of The Queen's College, Oxford; and I do not know whether also at the same time (but certainly some time) Robert Wood, Fellow of Lincoln College and Christopher Wren, Fellow of All Souls College (and also some others, whom I refrain from naming). And indeed having revealed to all of them (unless I am mistaken) the mark that I was aiming at, namely, given that quantity that was sought, we would have the complete quadrature of the circle. Moreover, neither I nor any of them (for whom either the answer was not obvious, or there was no leisure after laying down their own problems for them to be at all troubled by mine) satisfied what was wished for. Moreover, some one of them advised that I should consult the *Opus geometricum* of Grégoire de Saint-Vincent (whose name indeed I had not heard before) as he had expounded things of this kind with a bearing on the quadrature of the circle, in a large volume.[6] I heeded this advice; and this book, although it was so large a volume that I did not have leisure to read it the whole of it, I engaged in whenever possible, watching for what I could find out from there that would serve my purpose. Moreover, I found at times the investigations fell out the same way both for him and for me (which was no surprise) though we had arrived there by different methods. For example, what he calls *drawing a plane into a plane*, is what I here and in my *Treatise on conic sections* (the draft of which was conceived and first shaped in the same year, 1652) have called *drawing all the lines in one plane into the respective lines in another.* The reason, moreover, that I did not speak of *drawing a plane into a plane* was that in reality it was not so much a plane into a plane (for thus it would produce a plano-plane, that is, of four dimensions, not a solid), but the width of one into the width of the other, both taken finally to the same altitude; and therefore there emerge three (not four) dimensions. And perhaps some other things. So that I have not taken into my treatise one proposition or demonstration from his that I had not found previously; thus if by chance there happens to be anything common to both, I believed it was not worth the trouble on that account of deleting it from mine, since it is very often bound to happen that where two or more consider treating the same thing, they will sometimes coincide in the same observations. But (although he has astutely made many discoveries, by a method quite different from mine) that which I most sought in him I never found; for he did not follow the thing far enough, nor does he even touch at all on the quadrature of the circle, which he asserts he has found, except at a proposition not very dissimilar to my Proposition 136, where he has arrived at a calculation whence the quadrature of the circle may be found, but has not, however, followed it through, as Dr Huygens showed in his *Exetasis.*[7]

In the autumn of that year (1652), I proposed to the most distinguished gentleman Francis von Schooten, Professor of mathematics at Leiden in the Netherlands, among others, also this problem (concealing the target, however, that it was aimed at), who, having immediately communicated it to the most distinguished gentleman Christiaan Huygens, indicated, in letters written thus not much later, the intricacy and difficulty of the thing (although at first sight it seemed easier), and gave no hope in the meantime that either he nor my Lord Huygens would be free enough to expend more labour in the further investigations of it. From the responses of all of them, I was the more strengthened in that opinion I had previously held, namely that the term sought was neither a rational number, nor any so far accepted surd number, but must be described in new notation, and indeed, if you like, that

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[6] Grégoire De Saint-Vincent, *Opus geometricum quadraturae circuli et sectionum coni*, Antwerp 1647.

[7] Christiaan Huygens, *Theoremata de quadratura hyperbolae, ellipsis et circuli... Quibus subjuncta est Exetasis cyclometriae G. a S. vincentio*, Leiden 1651. Wallis's copy of Huygens' *Theoremata de quadratura* and of his *De circuli magnitudine inventa* of 1654 are both bound in Bodleian Library Savile G.26.

which I have assigned at Proposition 190. But if (as, for example, $\sqrt{2}$, may not be expressed precisely in true numbers, but nevertheless as closely as required, so) we want to express this quantity as closely as required in true numbers (that is, with as much accuracy as one wants, meaning one does not want it cut off), I teach how that may be done at Proposition 191. How that may be exhibited to some extent geometrically is shown in the subsequent propositions. And therefore we seem to have pursued the quadrature of the circle as far as the nature of numbers allows. But whoever requires to show the thing further, it is from there on as though one wanted to express $\sqrt{2}$ in true numbers: which requirement is unfair. Meanwhile I am not ignorant that it is possible to describe that quantity by other methods also, with endless characters, and to arrive in the same way at numbers closely approximate to the true ones by other methods (just as can be said also of surd roots), in which it is not for me to lay down rules to men of mathematics, but I leave them free to those things each prefers to use.

Moreover, having completed the quadrature of the circle, I thought it not worth while to touch separately on other problems related to it: thus, the ratio of the diameter to the circumference, or the sphere to the cube, or the cone or cylinder to the pyramid or prism, and others similar; for anyone can see from this how to gather these together.

Nor did it seem that anything needed to be said separately of the quadrature of the ellipse, which indeed was treated in conjunction with the quadrature of the circle.

The quadrature of the hyperbola as far as I have attained it, I have shown at Proposition 165.

Meanwhile, however, following the thread of the method I teach, I have unexpectedly come across somewhat surprising questions concerning the measurement of figures partly bounded, partly continued to infinity. And in particular what Torricelli showed in one solid figure I have shown can be done in others innumerable, both plane and solid, in Proposition 87 and later at 107. At the same time I teach by what criteria it may be discerned, for proposed figures of this kind continued infinitely, whether they will eventually attain a finite or infinite magnitude. Which observation seems both quite surprising and at the same time pleasing.

Why, moreover, have I not made public more quickly what I already found three years since? The reason was partly that I was frequently called to other business, but especially that the typesetter, more occupied with other publications, only undertook seriously, and carried out lately, the printing of this and other treatises which appear with it. But while those now published were in the press, I was pleased to put out as a foretaste (last spring) a proposition on circle measurement (including also that which I proposed in the form of a problem some years since, as I said above, to various distinguished gentlemen), and you may discover that it was chosen from the three problems that end this treatise. Moreover, since that time (in the month just gone)

Doctor Hobbes produced a book,[8] who had already promised much in geometry, and especially in the quadrature of the circle, and sectioning angles in a given ratio, and other things related to these, and at length he brought his book out publicly, from which it was clear that he had not demonstrated any of these things, nor indeed will he demonstrate them; for the book abounds everywhere with the most disgraceful paradoxes, so that you scarcely at times find anything sensible (which my *Elenchus*,[9] which is also now in the press, will make clear), whence you may easily discern also that the author is not one from whom we may hope that mysteries of this kind are to be unraveled.

For the rest, farewell, honoured old gentleman. And may the most merciful God preserve you happily and make all your doings prosper: so that at length after passing happily and piously through old age, you may exchange this troubled life we now lead for a better life. Which is most ardently to be prayed for.

Your most respectful servant.
John Wallis

Oxford
19 July 1655

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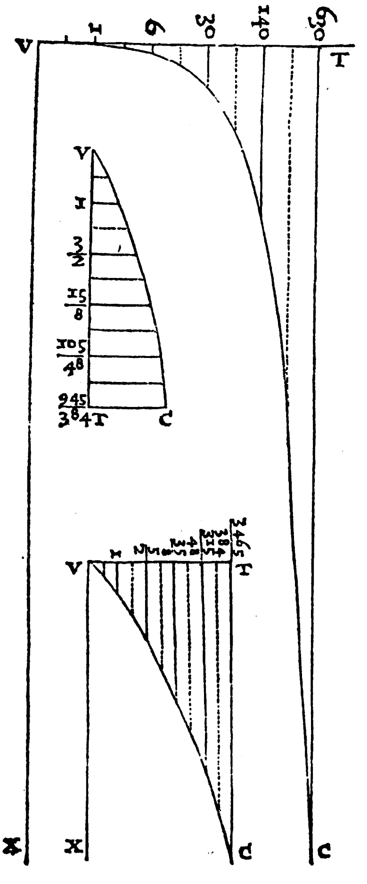
[8] Thomas Hobbes, *Elementorum philosophiae; sectio primo de corpore*, London 1655.

[9] John Wallis, *Elenchus geometria Hobbianae ... refutatio*, Oxford 1655.

To the Most Respected Gentleman
Doctor William Oughtred
most widely famed amongst mathematicians
by John Wallis
Savilian Professor of Geometry at Oxford

That proposition (most famed Gentleman) that I have shown before to you, concealed in shape and in the form of a problem, and also to not a few other mathematicians, to whom I held out the thing some years ago, hiding for the most part (though it was discovered by several) the target it was aimed at: here at last I declare ahead openly, in the form of a Theorem (which was previously buried).



The Quadrature of the Circle

Given a smooth curve *VC* to whose vertex there runs the line *VT*, divided into any number of equal parts, and from each point of the division, the same number of parallel lines, constructed as far as the curve, of which the second is 1, the fourth is 6, the sixth is 30, the eighth is 140, etc. it will be the case that, as the second is to the third, so will be the semicircle to the square of its diameter.

Or if the second is 1, the fourth is $1\frac{1}{2}$, the sixth is $1\frac{7}{8}$, etc. it will be the case that, as the second is to the third, so will be the circle to the square of its diameter.

Or if the second is 1, the fourth is $2\frac{1}{2}$, the sixth is $4\frac{3}{8}$, etc. it will be the case that, as the second is to the third, so will be three times the circle to four times the square of its diameter.

The method of demonstration I have arrived at for all the progressions, both here for the circle and for innumerable other quadratures of other curves, is shown in the treatise that I now have by me, completed for some time, and indeed written out for the use of the printers, and which I will publish, as soon as the delays of the printers allow, on whose leisure I have already awaited for two whole years and more.

Given from the Press at Oxford the day after Easter, the year of our Lord 1655.

From Doctor William Oughtred

A response to the preceding letter (after the book went to press).[10] In which he makes it known what he thought of that method.

Most honoured Sir,

I have with unspeakable delight, so far as my necessary business, the infirmness of my health, and the greatness of my age (approaching now to an end) would permit, perused your most learned papers, of several choice arguments, which you sent me: wherein I do first with thankfulness acknowledge to God, the Father of lights, the great light he hath given you; and next I gratulate you, even with admiration, the clearness and perspicacity of your understanding and genius, who have not only gone, but also opened a way into these profoundest mysteries of art, unknown and not thought of by the ancients. With which your mysterious inventions I am the more affected, because full twenty years ago, the learned patron of sciences, Sir Charles Cavendish, shewed me a written paper sent out of France, in which were some very few excellent new theorems, wrought by the way, as I suppose, of Cavalieri, which I wrought over again more agreeably to my way. The paper, wherein I wrought it, I shewed to many, whereof some took copies, but my own I cannot find. I mention it for this, because I saw therein a light breaking out for the discovery of wonders to be revealed to mankind, in this last age of the world: which light I did salute as afar off, and now at a nearer distance embrace in your prosperous beginnings. Sir, that you are pleased to mention my name in your never dying papers, that is your noble favour to me, who can add nothing to your glory, but only my applause, and prayer that God by you will perfect these happy beginnings so propitiously advanced to his glory. Which is the hearty desire of

Your truly loving friend and honourer,
William Oughtred

August 17 1655

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[10] This letter arrived too late to be included in the first edition of the *Arithmetica infinitorum* but was published in the second edition in 1695. It is reproduced in Stephen Jordan Rigaud, *Correspondence of scientific men of the seventeenth century*, 2 vols, Oxford 1841, I, 87–88, and is included here for completeness.

The Arithmetic of Infinitesimals
or
a New Method of Inquiring into the Quadrature of Curves, and other more difficult mathematical problems

PROPOSITION 1

Lemma

If there is proposed a series,[1] of quantities in arithmetic proportion (or as the natural sequence of numbers)[2] continually increasing, beginning from a point or 0 (that is, nought, or nothing),[3] thus as 0, 1, 2, 3, 4, etc., let it be proposed to inquire what is the ratio of the sum of all of them, to the sum of the same number of terms equal to the greatest.

The simplest method of investigation, in this and various problems that follow, is to exhibit the thing to a certain extent, and to observe the ratios produced and to compare them to each other; so that at length a general proposition may become known by induction.[4]

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[1] Wallis used the Latin word *series* in two ways: (1) to denote a list of terms defined according to some rule; this meaning has been translated as 'sequence', and: (2) to denote a (finite or infinite) collection of such terms, usually (but not necessarily) summed; this meaning has been translated as 'series', even though it does not correspond exactly to the modern mathematical understanding of the word (see Glossary).

[2] Quantities in arithmetic proportion (or arithmetic proportionals) increase or decrease by regular addition of a fixed quantity, thus: $a, a+d, a+2d, a+3d, \ldots$. The sequence of natural numbers $0, 1, 2, 3, \ldots$ is the simplest example.

[3] By allowing his sequences to begin 'from a point or 0', Wallis was implying that the quantities can be taken either from geometry (magnitudes) or from arithmetic (numbers).

[4] *Inductione*, (by induction) is not to be understood here in the modern formal sense of mathematical induction. Wallis used 'by induction' here and throughout simply to mean that a well established pattern could reasonably be assumed to continue.

It is therefore the case, for example, that:

$$\frac{0+1}{1+1}=\frac{1}{2} \qquad \frac{0+1+2=3}{2+2+2=6}=\frac{1}{2}$$

$$\frac{0+1+2+3=6}{3+3+3+3=12}=\frac{1}{2} \qquad \frac{0+1+2+3+4=10}{4+4+4+4+4=20}=\frac{1}{2}$$

$$\frac{0+1+2+3+4+5=15}{5+5+5+5+5+5=30}=\frac{1}{2} \qquad \frac{0+1+2+3+4+5+6=21}{6+6+6+6+6+6+6=42}=\frac{1}{2}$$

And in the same way, however far we proceed, it will always produce the same ratio of one half. Therefore:

PROPOSITION 2

Theorem

If there is taken a series, of quantities in arithmetic proportion (or as the natural sequence of numbers) continually increasing, beginning from a point or 0, either finite or infinite in number (for there will be no reason to distinguish), it will be to a series of the same number of terms equal to the greatest,[5] as 1 to 2.

That is, if the first term, is 0, the second 1 (for otherwise some adjustment must be applied), and the last is l, the sum will be $\frac{l+1}{2}l$ (for in this case the number of terms will be $l+1$). Or (putting m for the number of terms, whatever the second term) $\frac{1}{2}ml$.

PROPOSITION 3

Corollary

Therefore, a triangle to a parallelogram (on an equal base and of equal height) is as 1 to 2.

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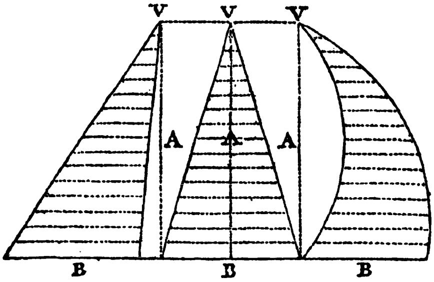
[5] Wallis's reasoning seems to break down immediately at this point, because if his series contains an infinite number of terms increasing indefinitely it can have no greatest term. What he is really thinking of, however, though he does not yet make it clear, is a series with a finite greatest term l, arrived at by m steps of size d, thus $0, d, 2d, 3d, \ldots, md = l$. When m is finite it is clear that the sum of terms is $\frac{1}{2}(m+1)l$, or, to $(m+1)l$ as 1 to 2. Wallis allowed the number of steps m to become infinitely large, by making d arbitrarily small, indeed infinitesimally small, but in such a way that md remains always equal to l and is therefore finite. In that case, Wallis argued ('by induction') that the same ratio of 1 to 2 would still hold.

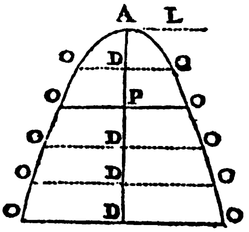


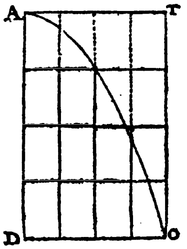
For the triangle consists, as it were,[6] of an infinite number of parallel lines in arithmetic proportion, starting from a point, of which the longest is the base (as we showed in Propositions 1 and 2 of our book *On conic sections*); and the parallelogram consists of the same number of lines equal to the base (as is clear). Therefore the former to the latter is as 1 to 2 (from what has gone before). Which was to be proved.

PROPOSITION 4

Corollary

In the same way, a parabolic pyramid or conoid[7] (whether right or inclined), to a prism or cylinder (on an equal base and of equal height) is as 1 to 2.





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[6] *Triangulum enim constat quasi ex infinitis rectis parallelis* was the phrase to which Thomas Hobbes later objected so strongly ('"as it were" is no phrase of a geometrician'); Hobbes 1656, 46.

[7] A parabola is a curve whose equation in modern notation, in its simplest form, is $y^n = kx$. For the common (or simple) parabola $n = 2$, while for a cubical, biquadratic or supersolid parabola, $n = 3$, 4 or 5, respectively. By *parabola* Wallis always meant the simple parabola; the others he described as *paraboloeides*, translated as 'higher parabolas'. Wallis distinguished also between *right* and *inclined* parabolas (cut from right or inclined cones): in a *right conic* the ordinates are at right angles to the diameter.

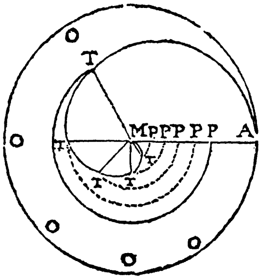
An erect parabolic conoid is the solid formed by rotation of a right parabola around its axis of symmetry (its diameter). A parabolic pyramid is a pyramid with polygonal cross-sections parallel to the base and parabolic cross-sections through the vertex. In Proposition 4 the solid is based on the simple parabola $y^2 = kx$, so that if $x_1, x_2, \ldots$ are arithmetically proportional then so are $y_1^2, y_2^2, \ldots$.

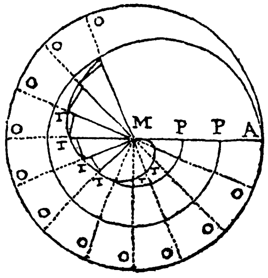
For a parabolic pyramid or conoid consists, as it were, of an infinite number of planes in arithmetic proportion, starting from a point, of which the largest is the base (as we showed in Proposition 9 of *On conic sections*), and the prism or cylinder of the same number of planes equal to the base (as is clear). Therefore the former to the latter is as 1 to 2, by Proposition 2.

PROPOSITION 5

Corollary

In the same way, any spiral line MT[8] (taken from the centre of the spiral) is to the corresponding coterminous arc PT (taken from the beginning of the revolution) as 1 to 2.[9]





Let this spiral line (having completed one revolution) be MTA, and let the centre of the spiral (which is also the centre of what I call the corresponding peripheral arc) be M. The beginning of the revolution is the line MA, by the even circular motion of which (keeping M fixed) there may be supposed described, by its end point A, the

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[8] The words '(*Quam* spuriam *dicimus*)', 'which we call spurious' were added when the *Arithmetica infinitorum* was reprinted in 1695.

[9] In 1695 Wallis added a note at this point to explain that by *spiral* he meant not the Archimedean spiral itself, but the sum of arcs of similar sectors, inscribed inside the Archimedean spiral; this he called the *spurious* spiral. The result stated in Proposition 5 does not hold for the true Archimedean spiral. The first revolution of the Archimedean spiral is equal in length to a half parabola whose base is the greatest radius of the spiral and whose axis is half the circumference of the coterminous circle. This result was discovered by Roberval and published in Mersenne's *Cogitata physico-mathematica* in 1644 (Book II, *De hydraulico*, 129), but Wallis read it there only in 1656 and added a hasty *Scholium* or *Comment* after Proposition 13 to explain his own results.

Wallis failed to understand that the true spiral is generated from a *uniform* motion along the radius, and an *accelerated* motion along a steadily increasing circumference (hence the analogy with the parabola which is similarly generated by uniform motion in one direction and accelerated motion in another) and his failure rendered Propositions 5 to 15 somewhat meaningless. Hobbes, who had discussed the problem with Roberval and understood the correct argument, immediately pointed out Wallis's error, but Wallis persisted in it even in his reply to Hobbes in his *Elenchus* of 1656. For further discussion of this problem see Jesseph 1999, 117–125.

perimeter AOA (which we call the *first circle*, or rather, the *circumference of the first circle*, whichever is the most familiar or useful). While, in the meantime, any point (on the same moving line) may be supposed to move (with the same even motion) from M to A, by its motion describing the spiral line MTA. Thus any straight line MT (from M, the centre of the spiral, to the spiral line as far as constructed) will be to the line MA, as the perimeter arc AO (described in the same time) to the total circumference AOA, or as the angle AMT to four right angles. And therefore also the lines MT, MT, are proportional to the arcs AO, AO, as is clear.

Then, having constructed any number of straight lines MT, MT, etc. making a continuous sequence of angles AMT, TMT, etc. equal to each other (and therefore [the lines MT are] in arithmetic proportion), we may suppose (superimposed on these angles) the same number of similar sectors (or rather, one fewer because a sector may not be inscribed in the first space) inscribing the figure[10] MTM (bounded by the true spiral line MT and the straight line TM). All these sectors together constitute the plane figure (composed from similar sectors), less than the (inscribed) plane figure MTM itself. But the difference is steadily diminished as the number of sectors (inscribed in MTM) becomes larger (as is clear), until in fact, if the sectors are supposed infinite in number, the figure thus inscribed coincides with the figure MTM itself (by that which we showed more generally in Proposition 2 of *On conic sections*) and therefore the arcs of all those sectors coincide with the (spurious) spiral MT.

Moreover, the arcs of those similar sectors (just as their radii) are in arithmetic proportion, that is, as 0, 1, 2, etc., and the angle of any sector is that part of the total angle AMT, which is found from the number of those sectors, or spaces; thus if the sectors are supposed infinite in number, the angle of any one of them will be $\frac{1}{\infty}$ (an infinitesimal, or infinitely small, part) of the whole angle AMT so, that is, that all together are equal to the whole of AMT. (Allow me, moreover, by analogous use of language perhaps, to call this sum of angles also by the name of *angle*, although perhaps it either equals or exceeds two right angles).

Therefore our spiral line MT may be supposed to consist of an infinite number of arcs of sectors in arithmetic proportion (subtending $\frac{1}{\infty}$ of the angle AMT), of which the smallest radius is 0, or a point (of no magnitude), and the greatest is the straight line MT.

Moreover, the corresponding coterminous arc is PT, consisting of the same number of arcs of sectors equal to the greatest, as is clear.

Therefore the sum of the former (that is our spiral line MT) to the sum of the latter (that is the coterminous arc PT) is as 1 to 2, by Proposition 2.

PROPOSITION 6

Corollary

And therefore, (our) spiral line MA, made by one revolution is equal to half the circumference of the first circle, AA.

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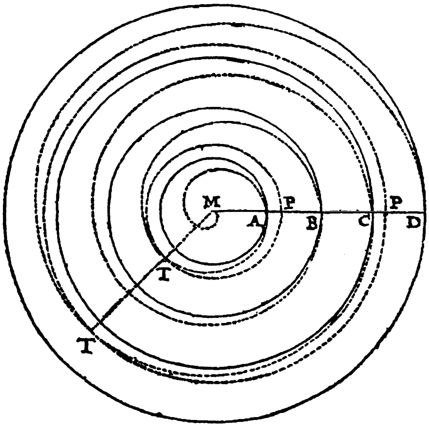
[10] By *figura*, or 'figure', Wallis always meant a plane figure, enclosed by lines and having area. In particular, a *circle* is a plane figure with area, while the line bounding it is the *circumference.*

For the arc coterminous with the spiral line *MA* is the entire circumference of the first circle described by the point *A*. Therefore, by Proposition 5, it is proved.

PROPOSITION 7

Corollary

Also, the spirals described by two, three, four, etc. complete revolutions are equal to half the circumferences of the second, third, fourth, etc. circles taken two, three, four, etc. times.



For while the spiral *MAB* (made by two revolutions) is being described, the point *B* describes the circumference *BB* twice; and while the spiral *MABC* is described, the circumference *CC* is described three times; and the circumference *DD* four times while the spiral *MABCD* is described. And so on, the circumference of the coterminous circle must be multiplied by the number of revolutions, and half of this multiple is equal to the spiral meanwhile described.

PROPOSITION 8

Corollary

But if the spiral is continued beyond one revolution but not for two, it will be equal to half of the circumference of the complete coterminous circle together with half of its continuation beyond the complete circle.

For while the spiral *MAT* is being described (by the combined motion), the arc *PPT* is also described (by the point *P*), that is, the complete circle *PP* plus the adjoined additional length *PT*. Therefore by Proposition 5, it is proved.

PROPOSITION 9

Corollary

Equally, if the spiral is continued through two, three, four or more complete revolutions plus an additional part, it will equal half the circumference of the complete coterminous circle taken two, three, four or more times (as many as the number of complete revolutions) with half of that same addition.

Because while the spiral is being described, the complete circumference of the coterminous circle is described the same number of times, and also the additional part. That is, the corresponding coterminous arc consists of the same number of whole circles (as the number of revolutions) together with the additional part. Therefore the proposition stands, by Proposition 5.

PROPOSITION 10

Corollary

Moreover, these spiral lines made by one, two, three, four, etc. revolutions (thus MA, MAB, $MABC$, $MABCD$) are to each other as the squares of arithmetic proportionals, that is as 1, 4, 9, 16, etc. Or they are as the squares[11] of the straight lines MA, MB, MC, MD etc.

For the straight lines MA, AB, BC, CD, are equal to each other (because of the even motion of the moving point on the line MA extended, progressing as evenly in one revolution as another). Therefore the radii MA, MB, MC, MD, just as the circumferences (described by those radii) A, B, C, D, are to each other as 1, 2, 3, 4. If therefore the circumferences are taken, the first once, the second twice, the third three times, the fourth four times, the multiples (that is $1A$, $2B$, $3C$, $4D$) will be as the square numbers 1, 4, 9, 16, or $1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4$. And therefore so are also the halves of those multiples, that is (by Proposition 5) the spirals MA, MAB, $MABC$, $MABCD$.

Or alternatively, if for the circumference of the first circle we put $A = p$, the second will be $B = 2p$, the third $C = 3p$, the fourth $D = 4p$ and so on; and $1A = 1p, 2B = 2 \times 2p = 4p, 3C = 3 \times 3p = 9p, 4D = 4 \times 4p = 16p$ etc. And (by Proposition 5) the spirals $MA = \frac{1}{2}p, MAB = \frac{2}{2}B = \frac{4}{2}p, MABC = \frac{3}{2}C = \frac{9}{2}p, MABCD = \frac{4}{2}D = \frac{16}{2}p$ etc. and therefore to each other as 1, 4, 9, 16, etc., that is, as the squares of the lines MA, MB, MC, MD, etc. (which are to each other as 1, 2, 3, 4, etc.) Which was to be proved.

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[11] *In duplicata ratione*, literally 'in duplicate ratio' or 'in twice the ratio'. In the Classical geometrical context the 'ratio' (or power) associated with quantities in arithmetic proportion is 1, the 'ratio' associated with their squares is 2 and with their cubes 3. It is not a great step from 'ratio' in this sense to 'index', but Wallis did not make that move formally until Proposition 64. 'In duplicate ratio' is translated here and elsewhere by the more familiar phrase 'as the square of'.

PROPOSITION 11

Corollary

And generally: the segments[12] of this (or any similar) spiral (taken from the centre of the spiral) are to each other as the squares of the coterminous lines.

For while (by the construction of the spiral) the ratio of the lines *MT*, *MT*, is the same as that of the angles *PMT*, *PMT*, (taking angles in the sense indicated above in Proposition 5), the ratio of the arcs *PT*, *PT*, (which [ratio] is composed of those two ratios), and thus of the spirals *MT*, *MT*, (which are half those arcs) will be as the squares of the lines *MT*, *MT*, or as $(MT)^2$, $(MT)^2$.

Thus, for example, if the straight line *MA* (of the first revolution) is denoted by $1r$, and the circumference of the first circle (described by that radius) by $1p$, the spiral *MA* will be $\frac{1}{2}p$. Therefore in the first revolution plus a half, the coterminous line will be $1\frac{1}{2}r = \frac{3}{2}r$, and the circumference of the coterminous circle will be $\frac{3}{2}p$, which multiplied[13] by $\frac{3}{2}$ (the number of revolutions) makes $\frac{3}{2} \times \frac{3}{2} \times p = \frac{9}{4}p$. Half of this, $\frac{9}{2\times 4}p = \frac{9}{8}p$ is the [length of the] spiral described in the same time.

Moreover, I call spirals *similar* if the lines *MA*, *MB*, *MC*, etc. in one are equal to corresponding lines in the other.

PROPOSITION 12

Corollary

But if in dissimilar spirals of this kind (for example, if *MB* in one is equal to *MC* in another) the coterminous lines are equal, then the segments of these spirals are in reciprocal proportion to the corresponding straight lines (that is, *MA* in one and *MA* in the other).

For example, in the first, the spiral *MAB* (described by two revolutions) will be equal to half of its circumference *B* taken twice; and in the second, the spiral *MABC* (described by three revolutions) will be equal to half of its circumference *C* taken three times. And since the circumferences *B* in the first and *C* in the second are supposed equal (because of equal radii), the first spiral *MAB*, and the second *MABC*, will be to each other as 2 to 3 (that is, as one circumference taken twice, to the same

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[12] A 'segment' in Propositions 11 to 18 is to be understood as a portion of length.

[13] *Ducta in*, or, 'drawn into'. The outcome, or 'product', of such a construction is an area delineated by a rectangle or square. As the mathematical paradigm shifted from geometry to arithmetic, *ducta in* came to have the meaning of 'multiplied by', and the 'product' was the result of the multiplication. The geometrical word 'square' is still used for the product of two equal quantities, and Wallis also used 'rectangle' for the product of two unequal quantities (see Proposition 120).

or an equal one taken three times), that is, in reciprocal relation to the corresponding straight lines MA, MA. For the straight line MA in the first is $\frac{1}{2}$ the straight line MB, and the straight line MA in the second is $\frac{1}{3}$ (of the same or an equal straight line) MC. Therefore MA in the second to MA in the first, is as $\frac{1}{3}$ to $\frac{1}{2}$, or as $\frac{2}{6}$ to $\frac{3}{6}$ or as 2 to 3. Therefore the segment MAB of the former spiral, to the segment $MABC$ of the latter, is as the straight line MA in the second to the straight line MA in the first.

The same thing may be shown similarly, whatever the ratio of the corresponding straight lines in the dissimilar spirals.

PROPOSITION 13

Corollary

If, moreover, in dissimilar spirals of this kind the coterminous straight lines are also unequal, then the segments of the spirals will be to each other in a ratio that is composed from the squares of the coterminous lines and the reciprocals of the corresponding straight lines.

Follows from Propositions 11 and 12.

COMMENT[14]

It must be noted in the preceding Propositions concerning spirals (and also in some I shall make in future) that I have made use of the word *spiral* loosely (that there might be no need for lengthy circumlocution on every occasion). For example, for *spiral* (wherever this is compared with a circumference) I would wish there to be understood: *the sum of all the arcs of similar sectors, infinite in number, from which sectors, infinite in number, is constituted the plane figure inscribed in the true spiral*; as we indicated at Proposition 5 (and which evidently we have made use of in this work at Proposition 5, and also Archimedes at Proposition 21 etc. of his *On spiral lines*). Which sum indeed, taking the spiral line itself in the correct sense is always too small, and mostly so around the beginning of the spiral. For although the sum of the infinite number of those sectors may be made equal (according to the method of indivisibles) to the plane figure bounded by a straight line and the spiral itself; one may not, however, obtain that for all the arcs compared with the spiral line itself (strictly speaking).

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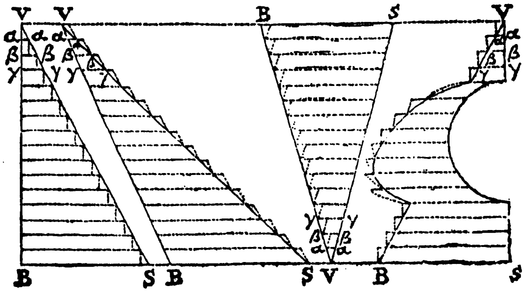
[14] This *Comment* was added after Wallis had discovered the rectification of the true Archimedean spiral in Mersenne's *Cogitata*, in 1656, when most of the *Arithmetica infinitorum* was already printed; see note 9.



For it amounts to the same thing as if, when an infinite number of parallelograms are inscribed in (or circumscribed around) a triangle, it seems that they equal the complete triangle *VBS*, whence one might conclude that the sides (parallel to the line *VB*) of all of them adjacent to the line *VS*, are at the same time equal to *VS* itself; or those (parallel to *VS* itself) adjacent to *VB* are at the same time equal to the whole of *VB*. (Which, though it may sometimes happen to be true, for example, thus in an isosceles triangle, must not, however, be concluded generally.) And indeed I have offered this warning the more strongly because I would see even learned men sometimes inclined to error through plausible possibilities of this kind.[15] That is why, moreover, the genuine spiral has been omitted, and I have compared the spurious spiral to the circumference; the reason being that for the latter but not the former it is possible to assign an equal circumference.

PROPOSITION 14

Corollary

And therefore also the segments of a spiral of this kind, taken from the centre, are to the coterminous lines as the intercepted diameters of a truncated parabola to its ordinates.[16]

That is, as the square, by Proposition 11.

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[15] One such learned man was Wallis himself, see note 9. Wallis took up the same theme again at much greater length in the *Comment* following Proposition 182.

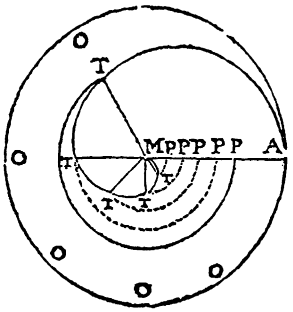
[16] For a parabola with equation $y^2 = kx$, the length of the diameter, or intercepted diameter, at a given point is given by the x-coordinate, while the length of the ordinate is given by the y-coordinate.

A truncated parabola is cut short by the line $x = d$, say, so its final intercepted diameter is d and its final ordinate is $kd^{\frac{1}{2}}$.

PROPOSITION 15

Corollary

Therefore if we suppose that I have unrolled our spiral *MTT* so that it consists of a straight line, and all the straight lines *TM*, *TM*, become parallel to each other, then the former will represent the diameter, and the latter the ordinates, of a parabola. Conversely, if we suppose that the diameter of a parabola is turned in an arc so that the ordinates end at the same point, it will become the spiral; those ordinates will be the coterminous lines and the point will be the centre of the spiral.



COMMENT

This also shows further what we indicated after Proposition 13. That is, our spiral, composed of an infinite number of arcs of similar sectors, cannot properly be said to be the genuine spiral, but less than it. For since it happens in the parabola that the ordinates which are closer to the vertex than that which is equal to the *latus rectum*,[17] are longer than their intercepted diameters; therefore it is not possible to roll up the diameter of the parabola (while keeping it unbroken) in such a way that the ends of the ordinates meet on the vertex itself (indeed because what is now supposed curved cannot be less than the coterminous line, which was formerly an ordinate). Therefore it must be that the true spiral, which turns in the same way, is greater than that supposed formed from the sum of arcs, which is now shown to agree with that formed from the diameter of a parabola, indeed which is everywhere as the squares of the coterminous lines.

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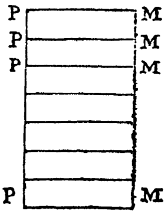
[17] The *latus rectum* of a conic is the total length of the ordinates passing through the focus. For a parabola with equation $y^2 = kx$ (therefore with focus at $(k, 0)$) the *latus rectum* is $2k$.

PROPOSITION 16

Corollary

The [plane] parabola so rolled (that is, the figure contained within our spiral) is half of the same parabola unrolled.



For example, if we suppose that a side *PP* of a parallelogram *PM* is rolled up, in such a way that the points *M* of every line *PM* coincide in the same point, there will be formed from the parallelograms (because all the radii from the common centre *M* are equal) a circular sector (which may be less than, greater than, or equal to a whole circle, according to the ratio of the lines *PP* to *PM* to each other) which sector indeed (that is, the rolled parallelogram) will be half the (unrolled) parallelogram (because in place of the infinite number of parallelograms of which the shown parallelogram is supposed to consist, there arise the same number of triangular sectors having the same bases and heights). In the same way, if the parabola is rolled up as described, so that the other ends of the (previously parallel) ordinates coincide in the same point, the infinite number of parallelograms of which we suppose the plane parabola to be constituted (by what we said in Propositions 2 and 8 of *On conic sections*) become the same number of triangles having the same bases and heights (as the parallelograms); and therefore the area of the parabola so rolled (that is the figure of the spiral) will be half of the same unrolled. Meanwhile it must be noted: if we want the ordinates of the parabola (the boundaries of those parallelograms) to become, in the spiral, those straight lines that bound similar sectors (having everywhere equal angles) we must take, in the parabola, a succession of parallelograms, not indeed of equal height,[18] but whose heights are in arithmetic proportion (thus 1, 3, 5, 7 etc.) by means of which the adjacent ordinates are in arithmetic proportion; (which the ratio of our spiral requires), thus as 1, 2, 3, 4, etc.

And this indeed agrees with what Torricelli says, in Example 8 of those which he sets out in his *Treatise on the hyperbolic solid*;[19] although clearly sought from different principles.

Further, this next must also be noted: just as from rolling a parabola of this kind (contracting the [arcs][20] into a single point) there arises the Archimedean spiral; so

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[18] Note here that the altitude, or height, of a parallelogram is the distance along the diameter of the parabola.

[19] *De dimensione parabolae solidique hyperbolici*, Torricelli 1644, 95–111; 101.

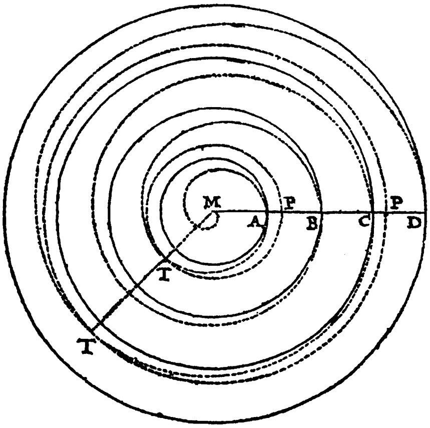
[20] Wallis has mistakenly written 'diameter' here.

from other higher parabolas (or other trilineal plane figures of this kind) by similar rolling, there may arise other different types of spiral, of a thousand different kinds. Some of which we will consider later.[21]

PROPOSITION 17

Corollary

Moreover, those segments of a spiral of this kind which arise from the first, second, third, fourth, etc. revolutions and so on are between themselves in the ratio 1, 3, 5, 7, and so on, in arithmetic progression.



For (by Proposition 10) the spiral lines *MA*, *MAB*, *MABC*, *MABD*, etc. are as 1, 4, 9, 16, etc., therefore the segments of the spirals, *MA, AB* (= *MAB* – *MA*), *BC* (= *MABC* – *MAB*), *CD* (= *MABCD* – *MABC*), etc. are as 1, (4 – 1 =) 3, (9 – 4 =) 5, (16 – 9 =) 7, etc.

PROPOSITION 18

Corollary

And generally, taking any sequence of straight lines *MT*, *MT*, etc. continually making angles *PMT*, *TMT*, etc. equal to each other, the successive intercepted segments (*MT*, *TT*, etc.) of a spiral of this kind will be as 1, 3, 5, 7, etc.

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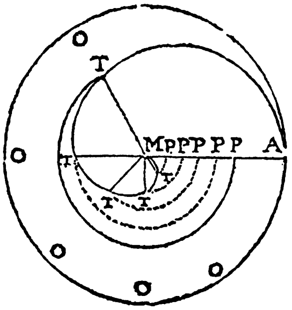
[21] See Proposition 45.



For since the straight lines MT, MT, etc. themselves (because of the equal angles) are as 1, 2, 3, 4 etc. (by the construction of the spiral), and therefore the spiral lines MT, MT, etc. (coterminous with those straight lines) are as the squares of those lines (by Proposition 11), that is as 1, 4, 9, 16, etc., the successive segments MT, TT, etc. themselves will be as 1, $4-1$, $9-4$, $16-9$. Which was to be proved.

COMMENT

All this teaching on the length of the spiral, now given in fourteen successive propositions, is completely missing from the work of Archimedes in his book *On spiral lines*; and I do not know that it has been taught by any other more recent writer since then.

PROPOSITION 19

Lemma

If there is proposed a series, of quantities that are as the *squares* of arithmetic proportionals (or as a sequence of square numbers) continually increasing, beginning from a point or 0 (thus, as 0, 1, 4, 9, etc.), let it be proposed to inquire what is its ratio to a series of the same number of terms equal to the greatest?

The investigation may be done by the method of induction (as in Proposition 1) and we will have:

$$\frac{0+1=1}{1+1=2}=\frac{3}{6}=\frac{1}{3}+\frac{1}{6} \qquad \frac{0+1+4=5}{4+4+4=12}=\frac{1}{3}+\frac{1}{12}$$

$$\frac{0+1+4+9=14}{9+9+9+9=36}=\frac{7}{18}=\frac{1}{3}+\frac{1}{18}$$

$$\frac{0+1+4+9+16=30}{16+16+16+16+16=80}=\frac{3}{8}=\frac{9}{24}=\frac{1}{3}+\frac{1}{24}$$

$$\frac{0+1+4+9+16+25=55}{25+25+25+25+25+25=150}=\frac{11}{30}=\frac{1}{3}+\frac{1}{30}$$

$$\frac{0+1+4+9+16+25+36=91}{36+36+36+36+36+36+36=252}=\frac{13}{36}=\frac{1}{3}+\frac{1}{36}$$

and so on.

The resulting ratio is always greater than one third, or $\frac{1}{3}$. Moreover, the excess continually decreases as the number of terms is increased, thus $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{18}$, $\frac{1}{24}$, $\frac{1}{30}$, $\frac{1}{36}$, etc.; the denominator of the fraction or ratio[22] clearly having been increased, in each place, in sixes (as is clear), so that the excess over one third of the given ratio becomes as one to six times the number of terms after 0. Therefore:

PROPOSITION 20

Theorem

If there is proposed a series, of quantities that are as the squares of arithmetic proportionals (or as a sequence of square numbers) continually increasing, beginning from a point or 0, its ratio to a series of the same number of terms equal to the greatest will exceed one third; and the excess will be the ratio of one, to six times the number of terms after 0; or of the square root of the first term after 0, to six times the square root of the greatest term.

That is (if for the first term after 0 there is put 1, and for the last l),

$$\frac{l+1}{3}l^2 + \frac{l+1}{6l}l^2.$$

Or (denoting the number of terms by m, and the last by l),

$$\frac{m}{3}l^2 + \frac{m}{6m-6}l^2.$$

Clear from the preceding propositions.

Since, moreover, as the number of terms increases, that excess over one third is continually decreased, in such a way that at length it becomes less than any assignable quantity (as is clear); if one continues to infinity, it will vanish completely.[23] Therefore:

PROPOSITION 21

Theorem

If there is proposed an infinite series, of quantities that are as squares of arithmetic proportionals (or as a sequence of square numbers) continually increasing, beginning from a point or 0, it will be to a series of the same number of terms equal to the greatest as 1 to 3.

Clear from what has gone before.

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[22] *Fractionis denominatore, sive consequente rationis*, literally 'the denominator of the fraction, or the consequent term of the ratio'.

[23] ... *ut tandem quolibet assignabili minor evadat, (ut patet;) si in infinitum procedatur, prorsus evaniturus est.*

PROPOSITION 22

Corollary

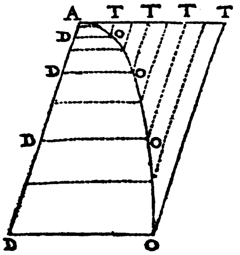
Therefore a cone or pyramid, to a cylinder or prism (on the same or equal base and of equal height), is as 1 to 3.

For we suppose the cone or pyramid to be composed of an infinite number of similar parallel planes, constituting a series of squares of arithmetic proportionals, of which the smallest may be supposed a point, the greatest the base itself; and (by what we said in Proposition 6 of *On conic sections*) the cylinder or prism [is composed] of the same number [of planes] equal to the greatest (as is clear). Therefore the ratio is 1 to 3 by the preceding proposition.

PROPOSITION 23

Corollary

In the same way, the complement of a half parabola (understood as figure *AOT*, which with the half parabola itself completes a parallelogram) is, to the parallelogram *TD* (on the same or equal base and of equal height), as 1 to 3. (And consequently the half parabola itself is to the same parallelogram as 2 to 3.)



For in the figure *AOT*, let the vertex be *A*, the diameter *AT*, the base *TO*, and as many parallels to it as you wish (between base and vertex) *TO, TO*, etc. Since (by Proposition 21 of *On conic sections*) the straight lines *DO*, *DO*, etc. are as the square roots[24] of the lines *AD*, *AD*, etc., conversely *AD*, *AD*, etc., that is, *TO*, *TO*, etc., will be as the squares of the same *DO*, *DO*, etc., that is of *AT*, *AT*, etc. Therefore the whole figure *AOT* (consisting of an infinite number of straight lines *TO*, *TO*, etc., the squares of the arithmetic proportionals *AT*, *AT*, etc.) will be, to the parallelogram of equal height *TD* (consisting of the same number of straight lines equal to the greatest *TO* itself), as 1 to 3, by Proposition 21. (Which was to be proved.) And consequently, the half parabola *AOD* (the remainder of the parallelogram) will be to the same parallelogram as 2 to 3.

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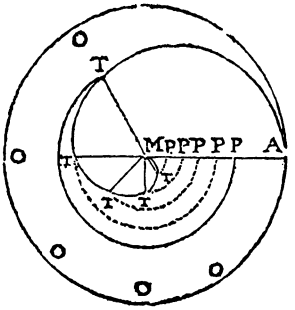
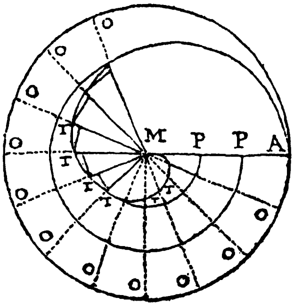
[24] *In subduplicata ratione*, literally 'in half ratio'; see also note to Proposition 10.

PROPOSITION 24

Corollary

In the same way, the plane figure *MTM* contained within the spiral line *MT* (taken from the centre of the spiral *M*) and the straight coterminous line *MT* is, to the corresponding sector *PMT*, as 1 to 3.





For (as we said in Proposition 5) we may suppose the plane figure *MTM* to consist of an infinite number of similar sectors, whose radii are in arithmetic proportion, and therefore the sectors themselves are as squares of arithmetic proportionals (indeed of their sides). Moreover, the sector *PMT* [consists] of the same number of sectors equal to the greatest. And therefore the former figure to the latter will be as 1 to 3, by Proposition 21.

I call by this name *sector* also the sum of any number of sectors, though it may be equal to or even exceed a semicircle (or indeed a whole circle) (just as we also pointed out concerning the name of *angle*, in Proposition 5).

PROPOSITION 25

Corollary

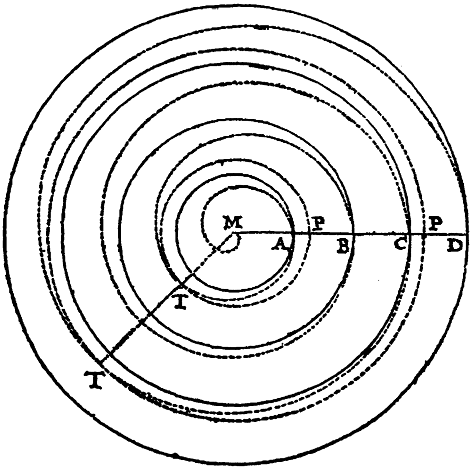
Therefore the plane figure *MTA* described by the first revolution of the spiral is equal to one third of the first circle *AA*.

For the corresponding coterminous sector is the complete circle *AA* itself, the first circle described by the radius *MA* in the same time.

PROPOSITION 26

Corollary

But the plane figures described by the first and second complete revolutions; by the first, second and third; by the first, second, third and fourth; and so on (as many times as any sector is repeated, so many are the revolutions described), will be equal to one third of the second, third, fourth, etc. circles taken two, three, four, etc. times (according to the number of revolutions).



For while the spiral line *MAB* (made by two revolutions) is described by a moving point going from *M* to *B* (on the rotating line *MB*), so at the same time is a plane figure, by the rotating straight line (thus continually increasing). In the same time the second circle is described twice (by the whole rotating line *MB*). Therefore, however many continually increasing sectors (increasing as a sequence of squares) constitute the figure of the bounded spiral, the same number equal to the greatest constitute the circle described twice. So the plane figure thus described, contained in the spiral, will be to the coterminous circle *BB* taken twice as 1 to 3, by Proposition 24.

And equally the figure of the spiral described by the first, second and third revolutions will be, to the third circle taken three times, as 1 to 3. And that described by the first, second, third and fourth revolutions will be, to the fourth circle taken four times, as 1 to 3. And so on.

Here it must be noted that the complete plane figure of the spiral described by the first revolution is repeated in the second revolution; and that described by the second is repeated in the third and so on. Therefore, for example, in four revolutions, the first figure (contained inside the first spiral line, described by the first revolution) is described four times, the second (which lies between the first spiral and the second) three times, the third (which lies between the second spiral and the third) twice, the fourth once. Therefore the first portion is taken four times, the second three times, the third twice and the fourth once, and together equal one third of the fourth circle taken four times, that is, as the number of revolutions. And it may be considered similarly for any number of revolutions, always taking account of the number of revolutions.

PROPOSITION 27

Corollary

If moreover the spiral is continued beyond the first revolution but not as far as a complete second, the plane spiral figure thus described (having taken twice what will be described twice) will equal one third of the complete coterminous circle together with one third of the continuation beyond the whole circle.

For while the spiral figure *MATM* is described, so also is the circular figure *PPTM* described, that is, the complete circle *PP* together with the additional sector *PMT*.

PROPOSITION 28

Corollary

And in the same way, if the spiral is continued through two, three, four or more complete revolutions with an additional part, the spiral figure thus described (any sector being repeated as many times as it is described) will equal one third of so many complete coterminous circles taken two, three, four or more times (that is, as the number of complete revolutions) together with one third of the adjoined additional part or sector.

Because while that spiral figure is described (by the rotation of an increasing straight line) the coterminous circle is described the same number of times (by the rotation of a fixed straight line), and also the additional part beyond.

PROPOSITION 29

Corollary

Also, the spiral figures described by the first; by the first and second; by the first, second and third; by the first, second, third and fourth revolutions, (and so on); (that is, *MAM*, *MABM*, *MABCM*, *MABCDM* etc.), are to each other as the cubes of arithmetic proportionals: 1, 8, 27, 64, etc. or as the cubes[25] of the straight lines *MA*, *MB*, *MC*, *MD*, etc.

For the straight lines *MA*, *MB*, *MC*, *MD*, etc. are as 1, 2, 3, 4, etc. (as has often been said), therefore the first, second, third, fourth, etc. circles (described by these radii) are as 1, 4, 9, 16, etc. (that is, as the squares of the radii). Therefore if the first circle is

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[25] *In triplicata ratione*, literally 'in triplicate ratio'; see also note to Propositions 10 and 23.

denoted by $A = 1c$, the second will be $B = 4c$, the third $C = 9c$, the fourth $D = 16c$. etc. and therefore if the first is taken once, the second twice, the third three times, the fourth four times, etc. we will have $1A = 1c, 2B = 2 \times 4c = 8c, 3C = 3 \times 9c = 27c, 4D = 4 \times 16c = 64c$, etc., and therefore to each other as the cube numbers 1, 8, 27, 64, etc. and therefore also as a third of these, $\frac{1}{3}c$, $\frac{8}{3}c$, $\frac{27}{3}c$, $\frac{64}{3}c$, etc. That is (by Propositions 25 and 26), the spiral figures *MAM*, *MABM*, *MABCM*, *MABCDM*, etc. are also between themselves as the cube numbers 1, 8, 27, 64, etc.

PROPOSITION 30

Corollary

And generally, spiral figures (taken from the centre of the spiral and bounded by the same or similar spiral line) are to each other as the cubes of the coterminous lines.

For (by the construction of the spiral line) the ratio of the straight lines *MT*, *MT*, is the same as that of the angles *PMT*, *PMT*, (taking the name *angles* in the sense of Proposition 5 above and the name *sectors* in the sense of Proposition 24 above). The ratio of the sectors *PMT*, *PMT* to each other (composed from the ratio of the angles and the ratio of the squares of the radii) is the ratio of the cubes of the straight lines *MT*, *MT* to each other. And therefore the ratio to each other of the spiral figures *MTM*, *MTM*, which are one third of those sectors (by Proposition 24), will also be the same.

Thus, for example, if the straight line *MA* (of one revolutions) is said to be $1r$, and the circle described by that radius is said to be $1c$, the spiral figure described in the same time will be $\frac{1}{3}c$. Therefore in one and half revolutions the coterminous straight line will be $1\frac{1}{2}r = \frac{3}{2}r$, the coterminous circle $\frac{3}{2} \times \frac{3}{2} \times c = \frac{9}{4}c$, which multiplied by $\frac{3}{2}$ (the number of revolutions) gives $\frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times c = \frac{27}{8}c$, and one third of this is $\frac{9}{8}c$, the spiral figure described by one complete revolution and half a revolution beyond. And similarly for any number of revolutions.

PROPOSITION 31

Corollary

But if spiral figures of this kind are bounded by dissimilar spiral lines but equal straight lines (that is if *MB* in one spiral is the same as *MC* in another) then those spiral figures on corresponding straight lines (thus *MA* in one and *MA* in another) will be in reciprocal proportion.

For in the first, the figure *MABM* (described by two revolutions) is equal to one third of its circle *B* taken twice. And in the second, the figure *MABCM* (described by three revolutions) is equal to one third of its circle *C* taken three times (by

Propositions 29 and 30),[26] and since it is supposed that circle B in the first is equal to circle C in the second (because they have equal radii), the spiral figures $MABM$ in the first, and $MABCM$ in the second, are to each other as 2 to 3 (that is, as a circle taken twice to the same or an equal circle taken three times), that is, in reciprocal ratio to the corresponding lines MA, MA. For MA in the first is $\frac{1}{2}$ the straight line MB, and MA in the second is $\frac{1}{3}$ of (the equal straight line) MC. Therefore MA in the second, to MA in the first, is as $\frac{1}{3}$ to $\frac{1}{2}$, or $\frac{2}{6}$ to $\frac{3}{6}$, or 2 to 3. And therefore the figure $MABM$ in the first spiral, to $MABCM$ in the second spiral, is as the straight line MA in the second to the straight line MA in the first.

And the same thing can similarly be shown, whatever the ratio of the corresponding lines in the dissimilar spirals.

PROPOSITION 32

Corollary

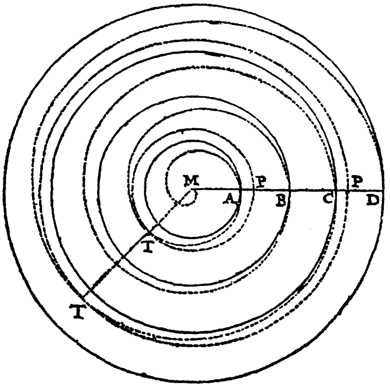
If, moreover, spiral figures of this kind are bounded by dissimilar spiral lines, and at the same time unequal straight lines, they will be to each other in a ratio composed of the ratio of the cubes of the bounding straight lines, and the ratio of the reciprocals of the corresponding straight lines.

Follows from Propositions 30 and 31.

PROPOSITION 33

Corollary

Further, the spiral figures described by the first, second, third, fourth, etc. revolutions are to each other as 1, 7, 19, 37, 61, etc., that is, as the differences of cube numbers whose roots are in arithmetic proportion.



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[26] The Proposition referred to here is actually 26.

For (by Proposition 29) the figures described by the first; by the first and second; by the first, second and third; by the first, second, third and fourth; are as 1, 8, 27, 64, 125, etc. Therefore the figures described by the first, by the second, by the third, by the fourth, etc. are as 1, $8-1$, $27-8$, $64-27$, $125-64$, etc. that is, as 1, 7, 19, 37, 64, etc., that is, as the differences of successive cube numbers. The excesses of these differences, or the differences of the differences, are in arithmetic proportion: for $1+6=7$, $7+12=19$, $19+18=37$, $37+24=61$, etc.

PROPOSITION 34

Corollary

And generally, taking any straight lines *MT*, *MT*, etc. making successive angles *PMT*, *TMT*, etc. equal to each other, the successive spiral figures between these lines are to each other as 1, 7, 19, 37, 61, etc.

For (by Proposition 30) the spiral figures from the centre to these successive lines are as 1, 8, 27, 64, 125, etc. Therefore the figures successively following, contained by these lines, are to each other as 1, $8-1=7$, $27-8=19$, $64-27=37$, $125-64=61$, etc. Or as $\frac{1}{3}c$, $\frac{7}{3}c$, $\frac{19}{3}c$, $\frac{37}{3}c$, $\frac{61}{3}c$, etc.

PROPOSITION 35

Corollary

Finally, the portions of a spiral figure newly described in each revolution (apart from those described in a preceding revolution), that is, contained inside the first spiral, or between the first and second, or between the second and third, or between the third and fourth, etc. are to each other as 1, 6, 12, 18, 24, etc. (by the addition always, after the first place, of sixes). That is, as the differences of the differences of the cube numbers.

Follows from Proposition 33, since 1, $7-1$, $19-7$, $37-19$, $61-37$ etc. are as 1, 6, 12, 18, 24, etc.

COMMENT

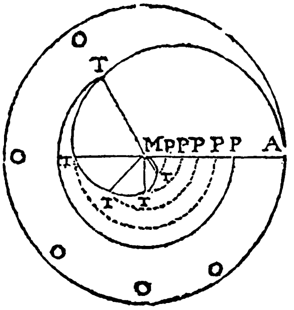
This teaching on the areas of spiral figures, here given in twelve successive propositions, agrees with that given by Archimedes around the end of his book *On spiral lines.* Allow this to follow that a little further.

PROPOSITION 36

Corollary

The complement of the spiral figure (which, that is, with [the spiral figure] itself completes the coterminous [circular] sector) is to the coterminous [circular] sector as 2 to 3.

Follows indeed from Proposition 24. But we may show it otherwise in this way.



We may suppose the figure $PMTT$ (the complement of the spiral figure $MTTM$) to consist of an infinite number of arcs PT, PT, etc. which indeed are as the squares of the arithmetically proportional lines MP, MP, (as we have shown in Proposition 11). Moreover, the coterminous sector MPT consists of the same number of arcs proportional to the same MP, MP, and therefore in arithmetic proportion (as is clear).

Moreover, a series of this kind (the squares of arithmetic proportionals) is $\frac{1}{3}$ of a series of equals (by Proposition 21) and a series of arithmetic proportionals is $\frac{1}{2}$ of the same series of equals (by Proposition 2). Therefore the former to the latter (that is, the complement of the spiral figure to the sector) is as $\frac{1}{3}$ to $\frac{1}{2}$, that is, as 2 to 3.

PROPOSITION 37

Corollary

A special case: the complement of the spiral figure described by one revolution is to the first circle (coterminous to it) as 2 to 3.

For that complement consists of an infinite number of arcs PT, which are as the squares of straight lines in arithmetic proportion, MP, (or as 0, 1, 4, 9, etc.) and the largest of them is the complete circumference A. Moreover, that complete circle consists of the same number of circumferences in arithmetic proportion (as 0, 1, 2, 3, etc.) of which the largest is the same circumference A. Therefore the complement to the circle is as $\frac{1}{3}$ to $\frac{1}{2}$, that is, as 2 to 3.

PROPOSITION 38

Corollary

The spaces lacking in each revolution, between the spiral figures and their respective complete circles, are as 2, 5, 8, 11, 14, etc., arithmetic proportionals.

For (denoting the first circle by c) the spiral figure described by the first revolution will be (by Propositions 29 and 33) $\frac{1}{3}c$, by the second $\frac{7}{3}c$, by the third $\frac{19}{3}c$, by the fourth $\frac{37}{3}c$, etc., with coterminous circles first c, second $4c$, third $9c$, fourth $16c$, etc. Therefore the excess of each circle over its coterminous spiral is for the first $\frac{2}{3}c$, for the second $\frac{5}{3}c$, for the third $\frac{8}{3}c$, for the fourth $\frac{11}{3}c$, etc. For $1c - \frac{1}{3}c = \frac{2}{3}c$, $4c - \frac{7}{3}c = \frac{5}{3}c$, $9c - \frac{19}{3}c = \frac{8}{3}c$, $16c - \frac{37}{3}c = \frac{11}{3}c$, etc.

COMMENT

And indeed it would be easy to adjoin many other propositions similar to these, concerning either the spiral figures themselves or their complements, described either by complete or partial revolutions. But from what has been said, anyone who pleases may easily supplement these, if there seems to be need of it, and so it is not necessary to delay here longer. And I fear lest I have already gone on too much. I add here, however, one or other of the said corollaries (for the sake of those who doubt that it is possible to find some rectilinear figure equal to the circle), that is:

It is clear from what has been said: *any circle is equal to some rectilinear figure.*

For it is clear (from Proposition 25) that there is a spiral figure equal to any circle, and (from Proposition 16) some parabola equal to any spiral, and finally (from Proposition 23) some rectilinear figure equal to any parabola. It follows that there is some rectilinear figure equal to any circle.

Therefore a rectilinear figure and a circle, or a straight line and a curve, are not heterogeneous quantities, but may properly be compared to each other, and indeed may be equal to one another. Although it may be that the diameter and perimeter of a circle are irrational to each other, so neither in true numbers, nor in any way of notation so far accepted, yet their ratio to each other may be forced out.

Further, from what has already been shown there arises also a method *of finding a straight line as close as one wishes*[27] *to a parabola (or higher parabola).*

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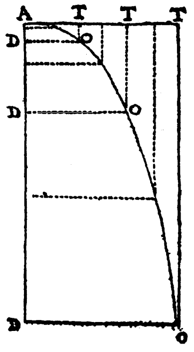
[27] *Aequalem quam proxime*, literally 'very nearly equal', but Wallis uses the phrase *quam proxime* here and elsewhere in a rather stronger sense, to mean 'as close as one wishes' (see also, for example, the *Comment* to Proposition 190).

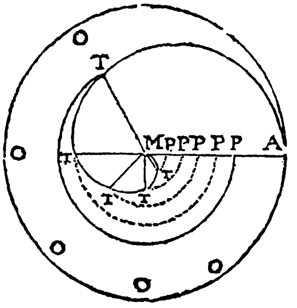


For if an erect half parabola touches at its vertex the line AT, divided into as many equal small parts as you please (any one of which is denoted by a and the number of all of them by n), and on the end of each small part, to that tangent, are the same number of ordinates TO, TO, etc. (therefore parallel to the diameter of the parabola, and making right angles with the tangent) of which the smallest may be said to be 1; then by Proposition 23 they will be to each other as the square numbers 1, 4, 9, 16, etc. and their differences as 1, 3, 5, 7, etc., the odd numbers from 1 onwards (of which the greatest difference is $2n - 1$). The straight lines (ordered along the parabola) connecting the ends of those parallels, (which will therefore in turn be inscribed in the parabola), will be as $\surd(a^2 + 1)$, $\surd(a^2 + 9)$, $\surd(a^2 + 25)$, $\surd(a^2 + 49)$, etc. (because by Euclid I.47 their squares are equal to the squares of the small length a and of the differences between neighbouring parallels, that is, of odd numbers). Indeed, the more of those lines (inscribed in the parabola) there will be, the more nearly the sum of all of them approaches the measure of the parabola. In such a way, however, that the line so composed of all of them is less than the parabola itself.

But if one wants another line, just too long (so that it is agreed that between those bounds one may determine the length of the parabola), this investigation will not be difficult, by means of completing tangents.

And if the curve AOO is supposed not a simple parabola but a cubical or biquadratic parabola, etc. it will be the same process, with appropriate changes, as in the simple parabola. For, taking for the differences of the parallels, not 1, 3, 5, 7, etc., the differences of the square numbers, but 1, 7, 19, 37, etc., the differences of the cube numbers, or 1, 15, 65, 175, etc., the differences of biquadrate numbers, etc., as the nature of the parabola requires, the inscribed lines will be $\surd(a^2 + 1)$, $\surd(a^2 + 49)$, etc. or $\surd(a^2 + 1)$, $\surd(a^2 + 225)$, and so on. As will be clear from what is demonstrated below in Proposition 45.

By almost the same method, *there may be found a straight line as close as one wishes to the true spiral.*



For if (by what was said in Proposition 5) it may be supposed that the spiral figure is inscribed or otherwise constituted from as many similar sectors as you please; then (because of the spiral), the arcs of the sectors, and their right sines and versed sines,[28] as also their radii, will be in arithmetic proportion. Moreover, the successive increase of the radii may be called a. If, therefore, from the beginning of the arc of any sector there is supposed dropped a constructed line to the radius, as far as the foot of the perpendicular, that will be the right sine of that arc, whose square, together with the square of the versed sine universally increased by the amount a, will be equal to the square of the line inscribed in the spiral (by Euclid I.47). If the versed sine is called v, and the diameter of its complete circle d, the square of the right sine (made by multiplying the versed sine, v, by the remainder of the diameter, $d - v$) will therefore be $vd - v^2$; and the square of the versed sine universally increased by the increase (that is, $v + a$) will be $v^2 + 2va + a^2$; and therefore the square of the inscribed lines (composed from these) will be $vd + 2va + a^2$. Since, moreover, (because of equal angles of similar sectors) the versed sine will be everywhere in the same ratio to the diameter, let it be as 1 to m (which ratio will be seen to be greater or less according as the angles of each sector are greater or less). Therefore, as $1 : m = v : d$, we will have $d = vm$; and therefore the square of the lines inscribed in the spiral will be $vd + 2va + a^2 = vvm + 2va + a^2$. Finally, since the arcs of the supposed similar sectors taken in turn, and therefore also the versed sines, are arithmetic proportionals (beginning from 0), they may be called 0, 1, 2, 3, etc., Those inscribed lines are therefore $\sqrt{(0m + 0a + a^2)}$, $\sqrt{(1m + 2a + a^2)}$, $\sqrt{(4m + 4a + a^2)}$, $\sqrt{(9m + 6a + a^2)}$, $\sqrt{(16m + 8a + a^2)}$ and so on. And the more sectors are supposed inscribed in the same spiral figure, the more closely the sum of the lines thus inscribed approaches the spiral line: but, however, it will be always less than the true spiral.

If, moreover, the first of these inscribed lines is omitted and instead of that there is placed, after the last, that which was next to be cut off (which amounts to the same thing as substituting for the figure made from inscribed

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[28] The (right) sine of an arc is half the length of the chord connecting its ends. For an arc subtending an angle 2θ at the centre of a circle of radius, its 'sine' is therefore $r \sin\theta$. The length of the arc itself is $r\theta$. The versed sine is the distance between the centre of the arc and the chord connecting its ends, that is, $r(1 - \cos\theta)$.

sectors a circumscribed), and then a is added, we will have a line summed from all of them, which will be greater than the true spiral, but which more nearly approaches the true value the more sectors are supposed constructed.

PROPOSITION 39

Lemma

If there is proposed a series, of quantities that are as the *cubes* of arithmetic proportionals (or as a sequence of cube numbers) continually increasing, beginning from a point or 0 (that is, as 0, 1, 8, 27, 64, etc.), let it be proposed to inquire what is its ratio to a series of the same number of terms equal to the greatest?

The investigation may be done by the method of induction (as in Propositions 1 and 19). And we will have:

$$\frac{0+1=1}{1+1=2}=\frac{2}{4}=\frac{1}{4}+\frac{1}{4}$$
$$\frac{0+1+8=9}{8+8+8=24}=\frac{3}{8}=\frac{1}{4}+\frac{1}{8}$$
$$\frac{0+1+8+27=36}{27+27+27+27=108}=\frac{4}{12}=\frac{1}{4}+\frac{1}{12}$$
$$\frac{0+1+8+27+64=100}{64+64+64+64+64+64=320}=\frac{5}{16}=\frac{1}{4}+\frac{1}{16}$$
$$\frac{0+1+8+27+64+125=225}{125+125+125+125+125+125=750}=\frac{6}{20}=\frac{1}{4}+\frac{1}{20}$$
$$\frac{0+1+8+27+64+125+216=441}{216+216+216+216+216+216+216=1512}=\frac{7}{24}=\frac{1}{4}+\frac{1}{24}$$

And so on.

The resulting ratio is always greater than one quarter, or $\frac{1}{4}$. Moreover the excess continually decreases as the number of terms is increased, thus $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{20}$, $\frac{1}{24}$, etc.; the denominator of each fraction or ratio clearly having been increased, in each place, in fours (as is clear), so that the excess over one quarter of the resulting ratio becomes as that of one, to four times the number of terms after 0. Therefore:

PROPOSITION 40

Theorem

If there is proposed a series, of quantities that are as the cubes of arithmetic proportionals (or as a sequence of cube numbers) continually increasing, beginning from a point or 0, its ratio to a series of the same number of terms equal to the greatest will exceed one quarter; and the excess will be the ratio of one, to four times the number of terms after 0; or of the cube root of the first term after 0, to four times the cube root of the greatest term.

$$\text{Thus:}\quad \frac{l+1}{4}l^3 + \frac{l+1}{4l}l^3 \text{ or } \frac{m}{4}l^3 + \frac{m}{4l}l^3 = \tfrac{1}{4}ml^3 + \tfrac{1}{4}ml^2$$

Clear from what has gone before.

Since, moreover, as the number of terms increases, that excess over one quarter is continually decreased, in such a way that at length it becomes less than any assignable quantity (as is clear); if one continues to infinity, it will vanish completely. Therefore:

PROPOSITION 41

Theorem

If there is proposed an infinite series, of quantities that are as cubes of arithmetic proportionals (or as a sequence of cube numbers) continually increasing, beginning from a point or 0, it will be to a series of the same number of terms equal to the greatest as 1 to 4.

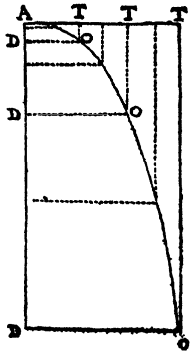
Clear from what has gone before.

PROPOSITION 42

Corollary

Therefore, the complement of half the cubical parabola *AOT* is, to the parallelogram *TD* (on the same or equal base and of equal height), as 1 to 4. (And, consequently, half the cubical parabola itself, to the same parallelogram, as 3 to 4.)

Let half the cubical parabola be *AOD* (of which the diameter is *AD*, the ordinates



DO, *DO*, etc.) and the complement *AOT* (of which the diameter is *AT*, the ordinates *TO*, *TO*, etc.) Therefore since (by Proposition 45 of *On conic sections*) the straight lines *DO*, *DO*, etc. or their equals *AT*, *AT*, etc. are as the cube roots of the lines *AD*, *AD*, etc. or their equals *TO*, *TO*, etc., conversely, *TO*, *TO*, etc. will be as the cubes of the lines *AT*, *AT*, etc. Therefore the whole figure *AOT* (consisting of an infinite number of lines *TO*, *TO*, etc. which are as the cubes of the arithmetically proportional lines *AT*, *AT*, etc.), will be, to the parallelogram *TD* (consisting of the same number of lines equal to the greatest *TO* itself), by what has gone before, as 1 to 4. (Which was to be shown.) And consequently, half the cubical parabola *AOD* (the remainder of the parallelogram) will be to the same parallelogram as 3 to 4.

PROPOSITION 43

Lemma

By the same method may be found the ratio of an infinite series of quantities that are as the fourth powers, fifth powers, sixth powers, etc. of arithmetic proportionals, beginning from a point or 0, to a series of the same number of terms equal to the greatest. That is, for fourth powers, it will be as 1 to 5; for fifth powers, as 1 to 6; for sixth powers, as 1 to 7. And so on.

It will be clear having tried it that the ratios discovered by induction approach continually closer to these, in such a way that the difference at length becomes less than any assignable quantity; and therefore continuing to infinity it vanishes.

I do not attach laborious geometrical demonstrations; which, however, if anyone should require them, he may search out such (at leisure) by the inscription and circumscription of figures, or also by putting forward other demonstrations (such as Archimedes has in Propositions 10 and 11 of *On spiral lines*), by showing that the ratio is neither more nor less than any assigned quantity. To me, what I have produced seems to suffice, following Cavalieri's *Method of indivisibles* (because I find that already to be taken from geometry).

Note, however, those demonstrations I have used, which better represent inscribed figures, since they suppose that the first term is 0. If on the other hand one prefers to represent the figures as circumscribed it may be changed, and one may do it, only the first term is made 1.

It must be noted also, that the ratios sought by induction, for those series which progress as fourth (or higher) powers of arithmetic proportionals are more involved than the preceding ones.

Thus for biquadrates: $\frac{l+1}{5}l^4 + \frac{l+1}{10l}3l^4 + \frac{l+1}{30l^2}l^4 + \frac{-l-1}{30l^3}l^4$.
Or $\frac{m}{5}l^4 + \frac{3m}{10l^2}l^4 + \frac{m}{30l^3}l^4 - \frac{m}{30l^3}l^4 = \frac{1}{5}ml^4 + \frac{3}{10}ml^3 + \frac{1}{30}ml^2 - \frac{1}{30}ml$. (That is, putting the first term 0, the second 1, the greatest l, and the number of terms $m = l + 1$.)

For supersolids: $\frac{l+1}{6}l^5 + \frac{l+1}{3l}l^5 + \frac{l+1}{12l^2}l^5 + \frac{-l-1}{12l^3}l^5$.
Or $\frac{m}{6}l^5 + \frac{m}{3l}l^5 + \frac{m}{12l^2}l^5 - \frac{m}{12l^3}l^5 = \frac{1}{6}ml^5 + \frac{1}{3}ml^4 + \frac{1}{12}ml^3 - \frac{1}{12}ml^2$.

For sixth powers, or squares of cubes:

$$\frac{l+1}{7}l^6 + \frac{5l+5}{14}l^5 + \frac{l+1}{7}l^4 - \frac{l+1}{7}l^3 - \frac{l+1}{42}l^2 - \frac{l+1}{42}l.$$

Or $\frac{1}{7}ml^6 + \frac{5}{14}ml^5 + \frac{1}{7}ml^4 - \frac{1}{7}ml^3 - \frac{1}{42}ml^2 + \frac{1}{42}ml$.
And similarly in those that follow, as will be demonstrated in Proposition 182.

But (which for us here suffices) they continually approach more closely to the required ratio, in such a way that at length the difference becomes less than any assignable quantity.

COMMENT

If, moreover, anyone desires to find ratios of this kind, however intricate, which belong to any higher finite series (thus for seventh powers, eighth powers etc. of arithmetic proportionals), it may be done by the method given below in the *Comment* to Proposition 182.

PROPOSITION 44

Theorem

Therefore if there is considered an infinite series,[29] of quantities beginning from a point or 0, and continually increasing in arithmetic proportion (which I call a series of laterals, or *first powers*) or of their squares, cubes, biquadrates, etc. (which I call a series of *second powers, third powers, fourth powers* etc.)[30] the ratio of the whole series, to a series of the same number of terms equal to the greatest, will be that which follows in this table. That is:

Equals	$\frac{1}{1}$ or	as 1 to	1
First powers	$\frac{1}{2}$		2
Second powers	$\frac{1}{3}$		3
Third powers	$\frac{1}{4}$		4
Fourth powers	$\frac{1}{5}$		5
Fifth powwers	$\frac{1}{6}$		6
Sixth powers	$\frac{1}{7}$		7
Seventh powers	$\frac{1}{8}$		8
Eighth powers	$\frac{1}{9}$		9
Ninth powers	$\frac{1}{10}$		10
Tenth powers	$\frac{1}{11}$		11

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[29] An *infinite series* should be understood in the sense of Proposition 2, that is, an increasing series with a finite greatest term reached by an infinite number of infinitesimally small steps.

[30] Note Wallis's clear distinction between geometric descriptions: *laterals* (or *sides*), *squares*, *cubes*, *biquadrates*, and arithmetic descriptions in terms of *powers*.

And so on. Thus the denominators of the fractions or ratios, are arithmetic proportionals from one; and the common numerator, or the first part of the ratio, is 1.

PROPOSITION 45

Corollary

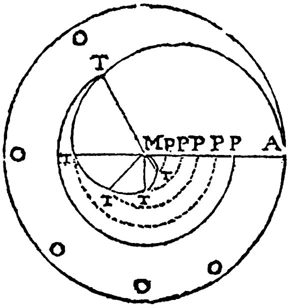
Here we learn the method of finding the area of the complement of a simple parabola, and also of cubical, biquadratic or supersolid parabolas, or those of any higher powers; and consequently also the area of a simple parabola or parabolas of any power. Which I promised to show in Proposition 48 of *On conic sections.*

That is, while the complement of a parabola (or half parabola), is a series of second powers (as we said in Proposition 23), the complement of the cubical parabola (or half of it) is a series of third powers (as we said in Proposition 42), and (for the same reason) the complement of the biquadratic parabola is a series of fourth powers, the complement of a supersolid parabola is a series of fifth powers, and so on. The ratio of these to a circumscribed parallelogram (that is, to a series of the same number of terms equal to the greatest) is 1 to 3, 1 to 4, 1 to 5, 1 to 6, and so on, according to the table in the preceding proposition. And consequently, those same simple, cubical, biquadratic, supersolid parabolas, etc. (which, that is, with their complements are equal to the circumscribing parallelograms) are to the circumscribing parallelograms as 2 to 3, 3 to 4, 4 to 5, 5 to 6 etc.

COMMENT

And here indeed by this means, for innumerable curvilinear figures one may produce equal rectilinear figures. Which in the parabola alone (with greatest admiration) Archimedes showed (and others after him), which we have now shown for parabolas of any power whatever.

Moreover, those things that have been taught for these parabolas, as will soon be shown, may also be accommodated by very easy work to spirals. For if we suppose the line MT to be continually increased, not indeed in the same ratio as the angle PMT (as in the Archimedean spiral) but as the second, third, fourth power, etc. of it, or also as the second, third, fourth roots, etc., or even as the third or fourth power of the second root, or the second or fourth power of the third root, or any others however composed: there will arise some or other kind of spiral, of which, however, the ratio to the circumference or arc (in the sense in which it was explained in the *Comment* to Propositions 13 and 15) will no less become known (as also the ratio of the enclosed plane figure to the circle or sector) than in that of Archimedes.



For example, if the line *MT* is increased as the square of the angle *PMT*, the spiral line *MT* (beginning from the centre) will be to the coterminous arc *PT* as 1 to 3, (that is, the spiral will be as a series of second powers, the arcs as a series of equals). And the enclosed plane figure will be to the coterminous sector as 1 to 5 (that is as a series of fourth powers to series of equals). And similarly, if the line *MT* grows as the third power, fourth power, etc. of the angle *MPT*, the spiral (in the above sense) will be to the coterminous arc as 1 to 4, 5, etc. and the enclosed plane figure to the coterminous sector as 1 to 7, 9, etc. The same, if the lines *MT* increase as the second, third roots, etc. of the angles *PMT*, then the spirals (in the said sense) or the aggregates of increasing similar arcs, to the coterminous arc (produced from the same number of equals) will be as 1 to $1\frac{1}{2}$, $1\frac{1}{3}$, etc. And the enclosed plane figure to the coterminous sector as 1 to $1\frac{2}{2}$, $1\frac{2}{3}$, etc., that is, the former as 2 to 3, 3 to 4, etc., the latter as 2 to 4, 3 to 5, etc. And so in others; whatever the power or root, or however combined from these. Which may all be demonstrated in the same way (with appropriate changes), as in Propositions 5, 24 etc. (at least with the help of certain propositions subsequently introduced). And here indeed this doctrine of spirals may be immensely increased. While, moreover, anyone who wishes may by their own exertion understand enough from what has already been said, and it seems to my mind superfluous labour to expand further on this: let it suffice what I have shown so far.

But here it would even be an easy passage to successfully considering spirals described not only in the plane but in solids, perhaps on the surfaces of cones of spheres, or also of conoids or spheroids, and comparing them to spirals or circles in a cylinder; and the enclosed plane figures of the former to the enclosed plane figures of the latter. Having introduced, however, those propositions that will be pursued below concerning augmented and reduced series. Moreover, all this, if I am not very much in error, I judge can be wholly omitted: since anyone who wishes, from what has been taught here or is to be taught below, may easily deduce it.[31]

At this point in the 1695 edition Wallis inserted a section headed *Monitum* with further comments on spirals; see Wallis 1693–99, I, 385–387.

PROPOSITION 46

Lemma

In the same way (by Proposition 44): given the ratio of one series, of whatever power, (to a series of equals), there may be found the ratio of another series of any other power, (to the same series of equals); by finding, that is, the corresponding term of an arithmetic progression.

For example, if [the sum of] a series of squares, or second powers, is $\frac{1}{3}$ of a series of equals, [the sum of] a series of laterals, or first powers, will be $\frac{1}{2}$ a series of equals: because, as a series of first powers is intermediate between a series of equals and a series of second powers, so 2 (the denominator of the ratio sought for first powers) is the arithmetic mean of 1 and 3 (the denominators of the ratios for equals and second powers). In the same way, while the ratio of a series of cubes or third powers, is $\frac{1}{4}$ or 1 to 4, between that series and a series of equals, two series of powers are interposed; so there must be sought two arithmetic means between 1 and 4, thus 2 and 3, of which the former belongs with first powers, the latter with second powers. And so in other cases.

And similarly, if the ratio belonging to a series of higher powers is sought, it is found by continuing the progression as far as the term sought: thus, if the ratio of a series of fourth powers, to a series of equals, is as 1 to 5, or $\frac{1}{5}$; the ratio of a series of sixth powers will be 1 to 7; because in an arithmetic progression where the fourth term (after one) is 5, the sixth term will be 7, and the same in other cases.

PROPOSITION 47

Lemma

Moreover, this rule is no less effective if there is shown a series of whatever quantities (not even a series of first powers, but) as any other series in the table, and its squares, cubes, etc. are sought.

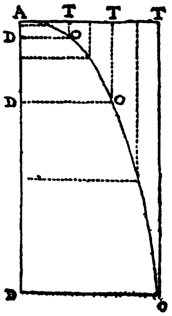
For example, if a series of this kind, of whatever quantities, is understood to be set out as a series of squares (to which in the table [in Proposition 44] is assigned the ratio 1 to 3): to their squares will belong the ratio 1 to 5 (because 1, 3, 5 are in arithmetic proportion) and to their cubes will belong the ratio 1 to 7, and so on, because 1, 3, 5, 7 etc. are arithmetic proportionals, just as unity, root, square, cube etc. are successive powers and geometric proportionals.

Nor is this other than what is to be had in the table; for if the supposed quantities are a series of second powers, whose ratio is $\frac{1}{3}$, their squares will be a series of fourth powers whose ratio is $\frac{1}{5}$; and their cubes will be a series of sixth powers whose ratio is $\frac{1}{7}$; etc. as has been said.

PROPOSITION 48

Corollary

And consequently, a conoid or pyramid generated from the complement of a half parabola (around its own diameter) is to a cylinder or prism on an equal base and of equal height as 1 to 5.



That is, if the complement of half the right parabola *AOT* is revolved keeping the line *AT* in place, so that a right conoid is described;[32] or, more generally, if (according to the method we have indicated in Propositions 5, 6, and 9 of *On conic sections*) around the diameter or axis *AT* the ordinates become circles, or any similar planes, of which the radii, or lines similarly placed, have the same ratio between them as the lines *TO*, *TO*, etc., so that the conoid or pyramid, whether right or inclined, is completed: I say that the conoid or pyramid is to a cylinder or prism on the same base and of equal height as 1 to 5. For since all the lines *TO*, *TO*, etc. are a series of second powers (to which belongs the ratio 1 to 3), any similar planes similarly constructed on these lines, will be between themselves as the squares of these lines; or as the squares of the lines *TO*, *TO*. And the ratio belonging to the series of those lines (that is, a series of second powers) is 1 to 3; therefore to the series of planes there belongs the ratio 1 to 5: because, that is, 1, 3, 5, are arithmetic proportionals (as unity, root, and square are geometric proportionals). And indeed, if the lines *TO*, *TO*, etc. are a series of second powers, their squares (or planes proportional to the squares) will be a series of fourth powers, to which in the table belongs the ratio 1 to 5.

PROPOSITION 49

Corollary

And similarly, if from the complement of half a cubical parabola there is generated (around its own diameter) a conoid or pyramid, this will be to a cylinder or prism (on the same or equal base and of equal height) as 1 to 7.

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[32] Wallis or his printers gave the wrong diagram in this Proposition: he needed a *right* parabola as in the *Comment* to Proposition 38, and as given here.

For since the lines TO, TO, etc. (in the complement of half a cubical parabola) are a series of third powers, to which in the table belongs the ratio 1 to 4, to the series of their squares (or of planes proportional to squares) belongs the ratio 1 to 7, because 1, 4, 7, are arithmetic proportionals. Or also, because the planes are a series of sixth powers, to which in the table is assigned the ratio 1 to 7.

PROPOSITION 50

Corollary

And equally, if from the complement of any other half parabola (thus biquadratic, supersolid, etc.) is generated (around its diameter) any conoid or pyramid, it will have to a cylinder or prism (on an equal base and of equal height) a known ratio (thus 1 to 9, 1 to 11 etc.).

For since the lines of these complements are series of fourth powers, fifth powers, etc., and therefore have in the table the assigned ratios 1 to 5, 1 to 6, etc., series of their squares (or of planes proportional to squares) will have the ratios 1 to 9, 1 to 11, etc., because 1, 5, 9, or 1, 6, 11, etc. are arithmetic proportionals. Or also, since the lines are series of fourth powers, fifth powers, etc., similar planes, similarly positioned to those lines, will be series of eighth powers, tenth powers, etc. to which belong the ratios 1 to 9, 1 to 11, etc.

COMMENT

Therefore by this method a huge number of solid figures contained by curved surfaces may be reduced to others contained by plane surfaces; and not only conical bodies (as the Ancients taught) but also many other conoids, may be reduced to a cylinder. Which I do not know that anyone else has shown before now.

PROPOSITION 51

Lemma

According to the same rule (Propositions 46 and 47) if there is proposed a series of any quantities, corresponding to any series in the table, their square roots, cube roots, etc. or any intermediate powers, may be investigated in the same way.

For example, if there is proposed an infinite number of squares[33] (or any similar planes) corresponding to a series of fourth powers, (to which is assigned in the

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[33] Here the *squares* are to be understood as geometrical objects, since Wallis goes on to compare them with similar *planes*.

table the ratio 1 to 5), the series of sides (or of lines similarly placed in those [planes]) will have the ratio 1 to 3 (to a series of equals); because 1, 3, 5, are arithmetic proportionals. Or also, because where the planes are a series of fourth powers, their sides will be a series of second powers, to which is assigned in the table the ratio 1 to 3.

Thus, if there is proposed an infinite number of cubes (or any similar solids) corresponding to a series of sixth powers, to which in the table corresponds the ratio 1 to 7, to the sides of those cubes (or to lines similarly placed in them) belongs the ratio 1 to 3; and to the squares of these sides (or to planes similarly placed in those cubes) the ratio 1 to 5; because, of the two arithmetic means between 1 and 7, the smaller is 3, the larger 5, (for 1, 3, 5, 7, are arithmetic proportionals). Moreover, I interpose two arithmetic means between 1 and 7, because we assume the same number of geometric means between unity and a cube, that is, the side and the square; for unity, side, square, cube are geometric proportionals. And indeed, if the cubes are a series of sixth powers, the sides will be a series of second powers; and the squares of the sides, a series of fourth powers; to which in the table belong the ratios 1 to 3, 1 to 5.

But if the proposed quantities in the same series of sixth powers are squares, (or any similar planes), to their sides will belong the ratio 1 to 4, because between 1 and 7 the arithmetic mean is 4, just as between unity and a square the geometric mean is the root or side. And indeed if the squares are a series of sixth powers, their sides will be a series of third powers, to which in the table belongs the ratio 1 to 4.

PROPOSITION 52

Corollary

And besides, from the known ratios of conoids and pyramids, mentioned in Propositions 48, 49 and 50, to a cylinder or prism (on an equal base and of equal height), there may be known the ratios of those planes from which they are constituted, to a circumscribing parallelogram. Indeed, the complement of a half parabola is as 1 to 3; the complements of half a cubical, biquadratic, supersolid parabola etc. are as 1 to 4, 1 to 5, 1 to 6, etc.

For if those conoids or pyramids are known to be series of fourth, sixth, eighth, tenth powers, etc. and to those belong the ratios 1 to 5, 1 to 7, 1 to 9, 1 to 11, etc., then to their sides (which are therefore series of second, third, fourth, fifth powers, etc.) belong the ratios 1 to 3, 1 to 4, 1 to 5, 1 to 6, etc. Because 1, 3 5, and in the same way 1, 4, 7, and 1, 5, 9, and 1, 6, 11, etc. are arithmetic proportionals.

PROPOSITION 53

Lemma

This understood, it opens an avenue to the investigation of the ratios (to a series of quantities equal to the greatest) that series of this kind, of square roots, cube roots, biquadratic roots, etc. of numbers or arithmetic proportionals, beginning from a point or 0, may be said to have. (Thus $\sqrt{0}$, $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, etc., $\sqrt{\ }^{3}0$, $\sqrt{\ }^{3}1$, $\sqrt{\ }^{3}2$, $\sqrt{\ }^{3}3$, etc., $\sqrt{\ }^{4}0$, $\sqrt{\ }^{4}1$, $\sqrt{\ }^{4}2$, $\sqrt{\ }^{4}3$, etc.) Which I call series of second roots, third roots, fourth roots etc.[34]

For example, if there is proposed an infinite number of squares of this kind which are arithmetic proportionals, or as a series of first powers, to which in the table is assigned the ratio 1 to 2, then to their sides (that is, to a series of second roots) belongs the ratio 1 to $1\frac{1}{2}$ (or 2 to 3); because 1, $1\frac{1}{2}$, 2, are arithmetic proportionals.

Similarly, if there is supposed an infinite number of cubes of this kind which are arithmetic proportionals, or as a series of first powers, to which belongs in the table the ratio 1 to 2, then to the cube roots of those (that is, to a series of third roots) belongs the ratio 1 to $\frac{1}{3}$ (or 3 to 4), and to their fourth roots, the ratio 1 to $1\frac{2}{3}$ (or 3 to 5). Because clearly 1, $1\frac{1}{3}$, $1\frac{2}{3}$, 2 are arithmetic proportionals, just as unity, root, square, cube are geometrical proportionals.

And in the same way, if there are understood to be an infinite number of biquadrates, supersolids, etc. which are as a series of first powers, to which belongs the ratio 1 to 2, then to their fourth roots, fifth roots, etc. belong the ratios 4 to 5, 5 to 6, etc. or 1 to $1\frac{1}{4}$, 1 to $\frac{1}{5}$, etc. because 1, $1\frac{1}{4}$, $1\frac{2}{4}$, $1\frac{3}{4}$, 2, and in the same way 1, $1\frac{1}{5}$, $1\frac{2}{5}$, $1\frac{3}{5}$, $1\frac{4}{5}$, 2, etc. are arithmetic proportionals. Therefore:

PROPOSITION 54

Theorem

If there is understood to be an infinite series, of quantities beginning from a point or 0, and continually increasing, as the square roots, cube roots, biquadratic roots, etc. of numbers in arithmetic proportion (which I call series of *second roots*, *third roots*, *fourth roots*, etc.), then the ratio of all of them, to a series of the same number of terms equal to the greatest, will be that which follows in this table, that is:

Second roots	$\frac{2}{3}$	or	as 1 to	1
Third roots	$\frac{3}{4}$			$1\frac{1}{3}$
Fourth roots	$\frac{4}{5}$			$1\frac{1}{4}$
Fifth roots	$\frac{5}{6}$			$1\frac{1}{5}$
Sixth roots	$\frac{6}{7}$			$1\frac{1}{6}$
Seventh roots	$\frac{7}{8}$			$1\frac{1}{7}$
Eighth roots	$\frac{8}{9}$			$1\frac{1}{8}$
Ninth roots	$\frac{9}{10}$			$1\frac{1}{9}$
Tenth roots	$\frac{10}{11}$			$1\frac{1}{10}$
And so on.				

Clear from what has gone before.

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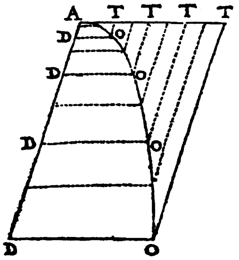
[34] At this point Wallis writes $\sqrt{c}$, $\sqrt{qq}$, for cube root, fourth root. Later (see Proposition 73) he changes to the notation given here, $\sqrt{\ }^{3}$, $\sqrt{\ }^{4}$.

PROPOSITION 55

Corollary

Therefore half a plane parabola (or also a whole parabola) is to the circumscribed parallelogram as 2 to 3. (And consequently its complement is to the same parallelogram as 1 to 3.)

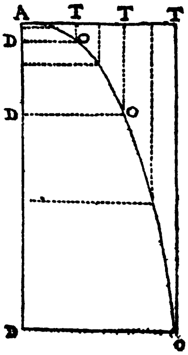


For a plane half parabola (or also a whole parabola) is an infinite series of second roots (by Proposition 8 of *On conic sections*). The parallelogram, moreover, is a series of the same number of terms equal to the greatest. Therefore the former to the latter is as 1 to $1\frac{1}{2}$, or as 2 to 3 (and consequently, its complement, that is, the remainder of the parallelogram, as 1 to 3).

PROPOSITION 56

Corollary

In the same way, half a plane cubical parabola (or also a whole cubical parabola), is to the circumscribed parallelogram, as 3 to 4 (and consequently, its complement is to the same parallelogram as 1 to 4).



For since (by Proposition 45 of *On conic sections*) the ordinates in a cubical parabola are as the third roots of the diameters (or of the distances from the vertex), the plane constituted from all those is a series of third roots, which, to a series of the same number of terms equal to the greatest (that is, to the circumscribed parallelogram), is as 1 to $1\frac{1}{3}$, or as 3 to 4. (And consequently, its complement, that is, the remainder of the parallelogram, is to the same parallelogram as 1 to 4.)

PROPOSITION 57

Corollary

In the same way, the ratio of a half (or whole) biquadratic or supersolid parabola, or a parabola of any higher power, to the circumscribed parallelogram will be known; thus as 4 to 5, 5 to 6, etc. (And consequently, their complements will also have a known ratio to the same parallelograms; thus as 1 to 5, 1 to 6, etc.)

For those planes are series of fourth roots, fifth roots, etc. and therefore, to a series of equals, as 4 to 5, 5 to 6, etc.; and consequently, their complements (which are series of fourth powers, fifth powers, etc.) as 1 to 5, 1 to 6, etc.

COMMENT

Therefore also by this table, one may find the area[35] of any parabola, cubical parabola, biquadratic parabola, or one of any higher power, and also of their complements: which I promised in Proposition 48 of *On conic sections* and have shown above at Proposition 45 of this.

PROPOSITION 58

Lemma

Finally, with the help of these rules (Proposition 46): if there is proposed any infinite series of this kind, of quantities beginning from a point or 0, and continually increasing, in the ratio of any power (not just any simple

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[35] This is the first time in the text that Wallis uses *area* as an absolute quantity (rather than expressing it as a ratio).

power, but also a composite), then its ratio to a series of the same number of terms equal to the greatest may be investigated. Thus the squares, cubes, biquadrates, etc. of second roots, third roots, fourth roots, etc. or also of second powers, third powers, fourth powers, etc. Or square roots, cube roots, biquadratic roots, etc. of second powers, third powers, fourth powers, etc. or of second roots, third roots, fourth roots, etc. Or also any other series in whatever way composed.

For example, since a series of third roots (thus $\surd^3 0$, $\surd^3 1$, $^3 2$, $\surd^3 3$, etc.) has a ratio (to a series of the same number of terms equal to the greatest) which is 3 to 4, or 1 to $1\frac{1}{3}$, their squares (which are also the same as cube roots of second powers, thus $\surd^3 0$, $\surd^3 1$, $\surd^3 4$, $\surd^3 9$, etc.) will have a ratio, to the same number of terms equal to the greatest, which is 1 to $1\frac{2}{3}$, or 3 to 5. Because, that is, 1, $1\frac{1}{3}$, $1\frac{2}{3}$, or $\frac{3}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, are arithmetic proportionals.

Equally, a series of cubes of fourth roots, or (which amounts to the same thing) biquadratic roots of a series of cubes or third powers, will have to a series of equals the ratio 4 to 7. For since a series of fourth roots has a ratio in the table of 1 to $1\frac{1}{4}$, or 4 to 5, their cubes will have a ratio (to a series of the same number of terms equal to the greatest) as 1 to $1\frac{3}{4}$, or 4 to 7. Because, that is, 1, $1\frac{1}{4}$, $1\frac{2}{4}$, $1\frac{3}{4}$, or $\frac{4}{4}$, $\frac{5}{4}$, $\frac{6}{4}$, $\frac{7}{4}$, are arithmetic proportionals, just as unity, root, square, cube, etc. are geometric proportionals.

And similarly in powers more compounded than this: thus square roots of cubes of a series of fifth roots. For to a series of fifth roots belongs the ratio of 1 to $1\frac{1}{5}$, or 5 to 6; therefore to their cubes belongs the ratio 1 to $1\frac{3}{5}$, or 5 to 8 (because, that is, 1, $1\frac{1}{5}$, $1\frac{2}{5}$, $1\frac{3}{5}$, or $\frac{5}{5}$, $\frac{6}{5}$, $\frac{7}{5}$, $\frac{8}{5}$, are arithmetic proportionals); and to their square roots, the ratio 1 to $1\frac{3}{10}$, or 10 to 13 (because, that is, $1\frac{3}{10}$ is the arithmetic mean between 1 and $1\frac{3}{5}$, for 1, $1\frac{3}{10}$, $1\frac{6}{10}$ $(=1\frac{3}{5})$ or $\frac{10}{10}$, $\frac{13}{10}$, $\frac{16}{10}$ $(=\frac{8}{5})$ are arithmetic proportionals). Or also since the square roots of fifth roots are a series of tenth roots, to which belongs the ratio 10 to 11, or 1 to $1\frac{1}{10}$, the cubes of these will have the ratio which is 10 to 13, or 1 to $1\frac{3}{10}$. Because 1, $1\frac{1}{10}$, $1\frac{2}{10}$, $1\frac{3}{10}$, or $\frac{10}{10}$, $\frac{11}{10}$, $\frac{12}{10}$, $\frac{13}{10}$, are four terms in arithmetic proportion.

And in the same way, in series of other powers however composed, their ratio to series of equals may be investigated. And therefore:

PROPOSITION 59

Theorem

If there is understood to be an infinite series, of quantities beginning from a point or 0, and continually increasing according to any power composed from simple powers (as mentioned in Propositions 44 and 45), the ratio of all of them, to a series of the same number of terms equal to the greatest, will be that which follows in this table. That is:

Of a series of

Roots

	Equals	First powers	Second powers	Third powers	Fourth powers	Fifth powers	Sixth powers	Seventh powers	Eighth powers	Ninth powers	Tenth powers
	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$
Square	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	$\frac{2}{9}$	$\frac{2}{10}$	$\frac{2}{11}$	$\frac{2}{12}$
Cube	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{3}{9}$	$\frac{3}{10}$	$\frac{3}{11}$	$\frac{3}{12}$	$\frac{3}{13}$
Biquadrate	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	$\frac{4}{9}$	$\frac{4}{10}$	$\frac{4}{11}$	$\frac{4}{12}$	$\frac{4}{13}$	$\frac{4}{14}$
Supersolid	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{5}{10}$	$\frac{5}{11}$	$\frac{5}{12}$	$\frac{5}{13}$	$\frac{5}{14}$	$\frac{5}{15}$
Sixth	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{9}$	$\frac{6}{10}$	$\frac{6}{11}$	$\frac{6}{12}$	$\frac{6}{13}$	$\frac{6}{14}$	$\frac{6}{15}$	$\frac{6}{16}$
Seventh	$\frac{7}{7}$	$\frac{7}{8}$	$\frac{7}{9}$	$\frac{7}{10}$	$\frac{7}{11}$	$\frac{7}{12}$	$\frac{7}{13}$	$\frac{7}{14}$	$\frac{7}{15}$	$\frac{7}{16}$	$\frac{7}{17}$
Eighth	$\frac{8}{8}$	$\frac{8}{9}$	$\frac{8}{15}$	$\frac{8}{11}$	$\frac{8}{12}$	$\frac{8}{13}$	$\frac{8}{14}$	$\frac{8}{15}$	$\frac{8}{16}$	$\frac{8}{17}$	$\frac{8}{18}$
Ninth	$\frac{9}{9}$	$\frac{9}{10}$	$\frac{9}{11}$	$\frac{9}{12}$	$\frac{9}{13}$	$\frac{9}{14}$	$\frac{9}{15}$	$\frac{9}{16}$	$\frac{9}{17}$	$\frac{9}{18}$	$\frac{9}{19}$
Tenth	$\frac{10}{10}$	$\frac{10}{11}$	$\frac{10}{12}$	$\frac{10}{13}$	$\frac{10}{14}$	$\frac{10}{15}$	$\frac{10}{16}$	$\frac{10}{17}$	$\frac{10}{18}$	$\frac{10}{19}$	$\frac{10}{20}$

And so on

And so on

PROPOSITION 60

Corollary

Therefore parabolic conoids and pyramidoids, which, that is, are generated from simple or cubical, biquadratic, supersolid parabolas, etc. are to the circumscribed cylinders or prisms (or to any others on an equal base and of equal height) as 2 to 4, 3 to 5, 4 to 6, 5 to 7, etc.

For since those plane parabolas are series of lines that are as second roots, third roots, fourth roots, fifth roots, etc. or as square roots, cube roots, biquadratic roots, supersolid roots, etc. of first powers, the conoids and pyramidoids thus generated are series of planes that are as the squares of these lines, and therefore as square roots, cube roots, biquadratic roots, supersolid roots, etc. of second powers, to which in the table are assigned those ratios 2 to 4, 3 to 5, 4 to 6, 5 to 7, etc.

PROPOSITION 61

Corollary

But here also there becomes known the method of quadrature not only for the simple parabola but also for all parabolas (and their complements), not only those in which the ordinates progress according to any simple power (of which I have spoken in Propositions 55, 56 and 57, and the same in Propositions 23 and 45) but also according to any power composed from simple powers. Thus if the ordinates are as the squares of third roots, fifth roots, seventh roots, etc. of the diameters, or cubes of fourth roots, fifth roots, etc., then they will have ratios to the circumscribed parallelogram which are 3 to 5, 5 to 7, 7 to 9, etc., or 4 to 7, 5 to 8, etc. And their complements (of which the ordinates are therefore as square roots of third powers, fifth powers, seventh powers, etc. of the diameters, or third roots of fourth powers, fifth powers, etc.) will have the ratios 2 to 5, 2 to 7, 2 to 9, etc. or 3 to 7, 3 to 8, etc. And similarly for the rest, according to the continuation of the preceding table, at Proposition 59.

For if the ordinates are as the squares of the cube roots of the diameters, that plane will be a series of lines which are to each other as squares of cube roots (or cube roots of squares) of numbers in arithmetic proportion, or as cube roots of second powers; to which in the table belongs the ratio 3 to 5.

And the complement of this will have ordinates that are as the square roots of the cubes of its diameters (which may be proved by such argument as was used in Proposition 23) and therefore that plane will be a series of square roots of cubes, or third powers, to which is assigned in the table the ratio 2 to 5.

And it is to be considered the same way in other cases.

COMMENT

And therefore by this method, yet other curved figures (besides those we indicated at Propositions 45 and 57) may be reduced to equal rectilinear figures. That is, all parabolas however generated, and their complements.

PROPOSITION 62

Corollary

And thence is clear also the method of reducing to equal cylinders or prisms, all parabolic conoids and pyramidoids (not only those mentioned in Proposition 60, where the ordinates of the plane figures progress as any simple powers, but also) those generated by any parabola of this kind (as mentioned in Proposition 61) whose ordinates progress as any series of composite powers.

For example, if the ordinates of the parabola are as the cube roots squared (or the cube root of the squares) of the diameters, its plane will be an infinite series of lines that are as cube roots of second powers: and therefore the conoids or pyramidoids will be series of the same number of planes, which are as squares of the same lines, therefore as cube roots of fourth powers, and therefore (according to the table in Proposition 59) to the circumscribed cylinder or prism as 3 to 7.

In the same way, if the ordinates of the parabola are as the fourth roots of the cubes of the diameters, then the planes of the conoid or pyramidoid will be as the fourth roots of the sixth powers of those same diameters (or, which amounts to the same thing, square roots of cubes), and therefore that conoid or pyramidoid (constituted from a series of these planes) will be to the circumscribed cylinder or prism as 4 to 10, or 2 to 5.

And in the same way for others according to the continuation of the table.

PROPOSITION 63

Corollary

In the same way, the conoids and pyramidoids generated by the complements of those same half parabolas may be reduced to equal cylinders or prisms.

For example, if the complement of a half parabola has ordinates that are as the square roots of the cubes of the diameters, the plane will be an infinite series of lines that are as square roots of cubes, or third powers, and thence the conoid or pyramidoid generated from this will be a series of the same number of planes that are as the squares of the same lines, and therefore as the square roots of the sixth power of the diameters, (or, which amounts to the same thing, as the cubes of the diameters) and will therefore be to the circumscribed cylinder or prism as 2 to 8, or 1 to 4.

In the same way, if the complement of a half parabola has ordinates that are as the cube roots of the fourth power of the diameters, the planes of the conoid or pyramidoid will be as the cube roots of the eighth power of those same diameters, and therefore as the cube roots of eighth powers; and that conoid or pyramidoid to the circumscribed cylinder or prism as 3 to 11.

And it may be considered for others in the same way according to the previously shown table.

COMMENT

We have shown therefore, by what method all parabolas of whatever kind, and their complements, may be reduced to parallelograms; and their conoids or pyramidoids to cylinders or prisms. And therefore we have solved numerous problems that no one (as far as I know) has taken up before, still less followed through.

Moreover, it seems appropriate to collect everything so far from all the preceding tables (at Propositions 44, 54 and 59) in this general theorem (which indeed belongs with the rule at Proposition 46), that is:

PROPOSITION 64

Theorem

If there is considered an infinite series, of quantities beginning from a point or 0, continually increasing according to any power either simple or composite, then the ratio of all of them, to a series of the same number of terms equal to the greatest, is that of unity to the index of that power increased by one.

I set the indices of first powers, second powers, third powers, fourth powers, etc. (or laterals, squares, cubes, biquadrates, etc.) to be 1, 2, 3, 4, etc.; I set the indices of second roots, third roots, fourth roots, etc. (or square roots, cube roots, biquadratic roots, etc. of first powers, or arithmetic proportionals) to be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.[36] I form the composed index of any composite power from the indices of the composing powers. Thus, cubes of second powers (or squares of third powers) have index $6 = 2 \times 3$; cube roots of second roots (or square roots of third roots) have index $\frac{1}{6} = \frac{1}{2} \times \frac{1}{3}$; cubes of square roots of fifth powers will have index $\frac{15}{2} = 3 \times \frac{1}{2} \times 5$.

Moreover, the ratios assigned to these powers (in the tables) are of the same kind. Thus to first powers, second powers, third powers, fourth powers, etc. 1 to 2, 1 to 3, 1 to 4, 1 to 5, etc., that is 1 to $1+1$, 1 to $2+1$, 1 to $3+1$, 1 to $4+1$, etc. To second roots, third roots, fourth roots, etc., 2 to 3, 3 to 4, 4 to 5, etc. or 1 to $1\frac{1}{2}$, 1 to $1\frac{1}{3}$, 1 to $1\frac{1}{4}$, etc., that is 1 to $\frac{1}{2}+1$, 1 to $\frac{1}{3}+1$, 1 to $\frac{1}{4}+1$, etc. To squares of third powers (or sixth powers), 1 to 7, that is 1 to $6+1$. To square roots of third powers, 2 to 5, or 1 to $\frac{5}{2}$, that is 1 to $\frac{3}{2}+1$. To cube roots of second roots (or sixth roots) 6 to 7, or 1 to $\frac{7}{6}$, that is 1 to $\frac{1}{6}+1$. To cubes of square roots of fifth powers (or square roots of fifteenth powers), 2 to 17, or 1 to $\frac{17}{2}$, that is 1 to $\frac{15}{2}+1$. (And so on for the rest.) Which the theorem confirms. And if the index is supposed irrational, thus $\sqrt{3}$, the ratio will be as 1 to $1+\sqrt{3}$ etc.

PROPOSITION 65

Theorem

From the known ratio of any series to a series of equals, may be known the ratio of any series to any other.

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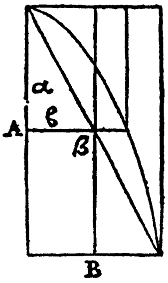
[36] Although this is the first time Wallis has formally defined the concept of a fractional index, he has already used the idea implicitly; see for example the *Comment* to Proposition 45 where the fractions associated with square roots and cube roots are taken to be $\frac{1}{2}$ and $\frac{1}{3}$ (or, in classical language, subduplicate and subtriplicate ratios).

For example, a parabola to a triangle (that is, a series of second roots to a series of first powers) is as $\frac{2}{3}$ to $\frac{1}{2}$, or as 4 to 3. The complement of a half parabola to a triangle, or also a cone to a parabolic conoid (that is, a series of second powers to a series of first powers) is as $\frac{1}{3}$ to $\frac{1}{2}$, or 2 to 3. A half parabola to its complement (that is, a series of second roots to a series of second powers) is as $\frac{2}{3}$ to $\frac{1}{3}$, or as 2 to 1. Thus a parabola to a cubical parabola is as $\frac{2}{3}$ to $\frac{3}{4}$, or as 8 to 9; and the conoid of the former to the conoid of the latter, as $\frac{2}{4}$ to $\frac{3}{5}$, or as 5 to 6. A cubical parabola to a biquadratic parabola is as $\frac{3}{4}$ to $\frac{4}{5}$, or as 15 to 16; and the conoid of the former to the conoid of the latter, as $\frac{3}{5}$ to $\frac{4}{6}$, or as 9 to 10. And so on in other cases.

It is to be understood that both the bases and the heights are the same or equal (or at least reciprocal); for if they have different bases or heights or both, the ratio of one series to the other is composed from the ratios of the bases and of the heights and from the ratios that belong to each series. Thus if a parabola has base B and height A, and a triangle has base ϐ and height α, then the parabola to the triangle will be as $\frac{2}{3}AB$ to $\frac{1}{2}\alpha$ϐ, or $4AB$ to 3αϐ, and similarly in other cases. In the same way, if the triangle has base B, height A, and the parabola has base β, height α, the parabola to the triangle will be as $\frac{2}{3}\alpha\beta$ to $\frac{1}{2}AB$, or $4\alpha\beta$ to $3AB$.



The proof is clear. For since parabola AB is $\frac{2}{3}$ parallelogram AB, and triangle αϐ is $\frac{1}{2}$ parallelogram αϐ, the former to the latter is as $\frac{2}{3}AB$ to $\frac{1}{2}\alpha$ϐ. And similarly in other cases.

PROPOSITION 66

Theorem

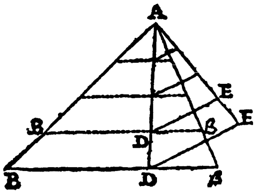
From the known quantity for any complete series, may be known the quantity for that series truncated.

Thus if triangle AB is $\frac{1}{2}$ parallelogram AB, and triangle αϐ is $\frac{1}{2}$ parallelogram αϐ, then the residual trapezium will be $\frac{1}{2}AB - \frac{1}{2}\alpha$ϐ. In the same way, the parabola AB is $\frac{2}{3}$ of the circumscribed parallelogram AB, and parabola $\alpha\beta$ is $\frac{2}{3}$ of the circumscribed parallelogram $\alpha\beta$, so the residual portion is $\frac{2}{3}AB - \frac{2}{3}\alpha\beta$. And the same in other cases.

PROPOSITION 67

Corollary

If a triangle is cut by any number of lines parallel to the base and equally spaced (cutting off portions of equal altitude), then the cut-off triangles (between the vertex and the cutting lines) are as 1, 4, 9, 16, etc., square numbers. The spaces between those lines are as 1, 3, 5, 7, etc., arithmetic proportionals.

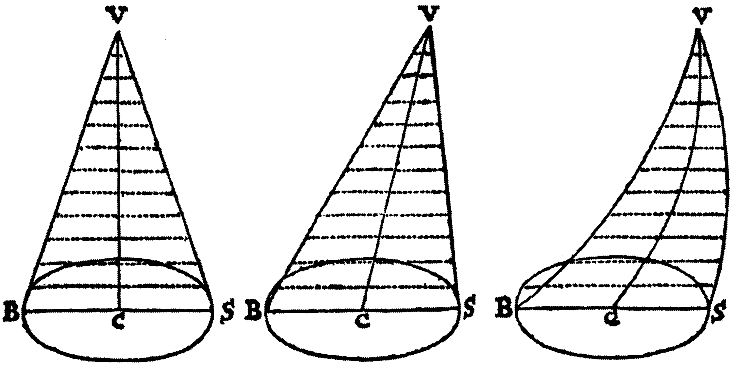


Because both the heights and the bases of the cut-off triangles are arithmetic proportionals, therefore the planes are as squares of arithmetic proportionals, or as square numbers 1, 4, 9, 16, etc. And therefore the spaces between as 1, $3 = 4 - 1$, $5 = 9 - 4$, $7 = 16 - 9$, etc.

PROPOSITION 68

Corollary

If a cone is cut by any number of planes parallel to the base and equally spaced (cutting off portions of equal altitude), the cut-off cones (between the vertex and the cutting planes) are as 1, 8, 27, 64, etc., cube numbers. The portions between are as 1, 7, 19, 37, etc., differences of cube numbers (and similarly for pyramids.)

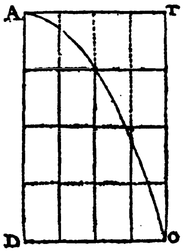


Because since the altitudes of the cut-off cones are arithmetic proportionals, and because so also are the diameters of the bases, therefore the bases are as squares of those, and the cones (which are composed from the altitudes and the bases) are as the cubes of the altitudes, or as 1, 8, 27, 64, etc. And therefore the portions between as 1, $7 = 8 - 1$, $19 = 27 - 8$, $37 = 64 - 27$, etc.

PROPOSITION 69

Corollary

If a parabola is cut by any number of lines (parallel to the base and equally spaced, cutting off portions of equal altitude), the cut-off parabolas (between the vertex and the cutting lines) will be as $1\sqrt{1}$, $2\sqrt{2}$, $3\sqrt{3}$, $4\sqrt{4}$, etc. or as $\sqrt{1}$, $\sqrt{8}$, $\sqrt{27}$, $\sqrt{64}$, etc., square roots of cube numbers. And the spaces between as the differences of the roots.

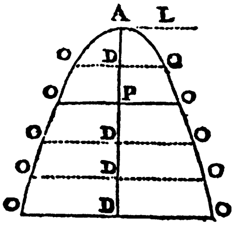


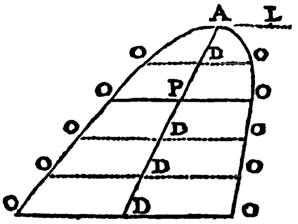
For the bases (or rather the ordinates of the parabolas) are as the square roots of the altitudes.

PROPOSITION 70

Corollary

If a parabolic conoid is cut by any number of planes (parallel to the base and equally spaced, cutting off portions of equal altitude), the conoids thus cut off (between the vertex and the cutting planes) are as 1, 4, 9, 16, etc., square numbers. And the portions between as 1, 3, 5, 7, etc., arithmetic proportionals. (And similarly for pyramids.)





That is, as was said of the triangle in Proposition 67, for the bases of the cut-off conoids are as the squares of the semi-diameters, that is, of ordinates of the parabola, and therefore proportional to the altitudes.

PROPOSITION 71

Corollary

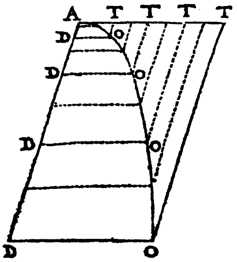
If the complement of a half parabola is cut by any number of lines (parallel to the base and equally spaced, cutting off portions of equal altitude), the complements thus cut off (between the vertex and the cutting lines) will be as 1, 8, 27, 64, etc., cube numbers. And the portions between as 1, 7, 19, 37, etc., differences of cube numbers.

That is, as was said above of the cone at Proposition 68, for the bases, that is, the ordinates of the complement, are as the squares of the altitudes.

PROPOSITION 72

Corollary

If also a conoid generated by the complement of a half parabola is cut by any number of planes (parallel to the base and equally spaced, cutting off portions of equal altitude), the cut-off conoids (between the vertex and the cutting planes) are as 1, 32, 243, 1024, etc., supersolid numbers. And the portions between as 1, 31, 211, 781, etc., the differences of supersolids. (And similarly for pyramids.)



For the bases of the cut-off conoids are as the squares of their semi-diameters, and therefore as fourth powers of the altitudes (for the semi-diameters of the bases themselves, or the ordinates of the complement of the half parabola, are as squares of the altitudes). And therefore the cut-off solid figures are as the fifth powers of the altitudes, or rather the power composed from those of the bases and of the altitudes.

COMMENT

And it must be considered similarly for other figures of this kind (whether plane or solid) cut in this way: always having regard for the degree or powers of the series that pertain.

PROPOSITION 73

Theorem

If any two series (or also more) are multiplied term by term (that is the first term of one by the first of the other, the second by the second, etc.) there will be produced another series of the same kind, which will have an index that is the sum of the indices of the multiplied series. Moreover, its ratio to a series of terms equal to its greatest will be that which the preceding tables (or also Proposition 64) indicated.

For example, if a series of squares or second powers (with index 2) is multiplied term by term by a series of cubes or third powers (with index 3) it will produce a series of fifth powers (with index $5 = 2 + 3$), which will therefore have, to a series of terms equal to the greatest, the ratio 1 to 6 ($= 5 + 1$). Thus if a series of second powers is multiplied term by term by a series of third powers, it will produce a series of fifth powers.

$0a$	$1a$	$4a$	$9a$	$16a$	etc.
$0b$	$1b$	$8b$	$27b$	$64b$	etc.
$0ab$	$1ab$	$32ab$	$243ab$	$1024ab$	etc.

In the same way, if a series of second powers (with index 2) is multiplied term by term by a series of third roots (with index $\frac{1}{3}$), there will be produced a series of cube roots of seventh powers (with index $\frac{7}{3} = 2 + \frac{1}{3}$) which, to a series of the same number of terms equal to the greatest, is as 1 to $\frac{10}{3}$ ($= \frac{7}{3} + 1$), or as 3 to 10. Thus if there are multiplied term by term

the series	$0a$,	$1a$,	$4a$,	$9a$,	etc.
by the series	$\sqrt{}^{3}0b$,	$\sqrt{}^{3}1b$,	$\sqrt{}^{3}2b$,	$\sqrt{}^{3}3b$,	etc.
that is, the series	$\sqrt{}^{3}0a^3$,	$\sqrt{}^{3}1a^3$,	$\sqrt{}^{3}64a^3$,	$\sqrt{}^{3}729a^3$,	etc.
by the series	$\sqrt{}^{3}0b$,	$\sqrt{}^{3}1b$,	$\sqrt{}^{3}2b$,	$\sqrt{}^{3}3b$,	etc.
it will produce the series	$\sqrt{}^{3}0a^3b$,	$\sqrt{}^{3}1a^3b$,	$\sqrt{}^{3}128a^3b$,	$\sqrt{}^{3}2187a^3b$,	etc.

In the same way, if a series of second roots (with index $\frac{1}{2}$) is multiplied term by term by a series of fifth roots (with index $\frac{1}{5}$), there will be produced a series of tenth roots of seventh powers (with index $\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$) and therefore it will have, to a series of terms equal to the greatest, a ratio that is 1 to $\frac{17}{10}$ ($= \frac{7}{10} + 1$), or 10 to 17. Thus if there are multiplied term by term

the series	$\surd^{2}0a$,	$\surd^{2}1a$,	$\surd^{2}2a$,	$\surd^{2}3a$,	etc.
by the series	$\surd^{5}0b$,	$\surd^{5}1b$,	$\surd^{5}2b$,	$\surd^{5}3b$,	etc.
that is, the series	$\surd^{10}0a^{5}$,	$\surd^{10}1a^{5}$,	$\surd^{10}32a^{5}$,	$\surd^{10}243a^{5}$,	etc.
by the series	$\surd^{10}0b^{2}$,	$\surd^{10}1b^{2}$,	$\surd^{10}4b^{2}$,	$\surd^{10}9b^{2}$,	etc.
it will produce	$\surd^{10}0a^{5}b^{2}$,	$\surd^{10}1a^{5}b^{2}$,	$\surd^{10}128a^{5}b^{2}$,	$\surd^{10}2187a^{5}b^{2}$,	etc.

And this holds similarly in other multiplications of this kind.

PROPOSITION 74

Corollary

Therefore, where the sums of the indices of series multiplied term by term are the same, there the indices of the series produced will also be the same.

For example, if a series of third powers is multiplied term by term by a series of third powers, or a series of second powers by a series of fourth powers, or a series of first powers by a series of fifth powers, or a series of equals by a series of sixth powers, it will produce a series of sixth powers. Because, that is, in each case the sum of the indices is 6 (for $3 + 3 = 2 + 4 = 1 + 5 = 0 + 6 = 6$). And similarly in other cases.

PROPOSITION 75

Corollary

If all the lines DB of a parallelogram ADB are set perpendicular one by one to the lines $D\beta$ of a triangle $AD\beta$ (of the same height),[37] the rectangles produced will be a series of first powers (of the same kind as the planes of the parabolic cone, by Proposition 11 of *On conic sections*), for which if there is substituted the same number of squares (or any other similar plane figures) equal to them, there will be constituted a parabolic pyramid. And the sides of those squares (or similar figures), or the mean proportionals between the lines so multiplied,[38] DE, constitute a parabola, or half parabola.

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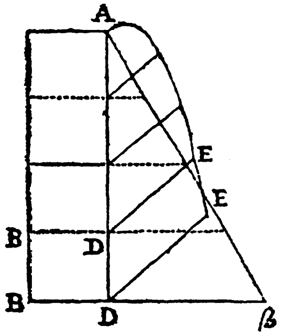
[37] '*Si Parallelogrami rectae omnes, in rectas Triangulis respective ducantur*; ...'. As pointed out in the note to Proposition 11, the verb *ducere* (*in*) was used to describe the construction of a perpendicular, the 'product' being the square or rectangle so defined. Such a construction gives the geometrical equivalent of multiplication in arithmetic.

[38] A literal translation of '... *inter rectas sic multiplicatas*'; Wallis is now blurring the classical distinction between geometry (which deals with lines) and arithmetic (which deals with numbers).

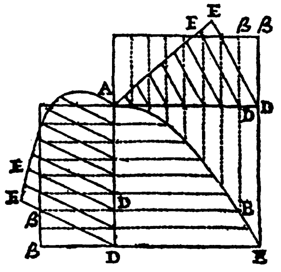


(It is to be understood that those planes, or mean proportionals, which thus emerge, are supposed positioned on some line, like ordinates, as the nature of the constituted figures requires.[39] Which may also be understood in whatever comes afterwards.)

For since the lines of the parallelogram are a series of equals (with index 0) and the lines of the triangle a series of first powers (with index 1), there is produced by multiplication the same series of first powers (since $0 + 1 = 1$), of the same kind as the planes of a parabolic cone or parabolic pyramid (by Propositions 9 and 11 of *On conic sections*), and the mean proportionals (or sides of similar planes) will be a series of second roots (or rather square roots of first powers), of the same kind as the lines of a parabola, by Proposition 8 of *On conic sections.*

PROPOSITION 76

If the lines βD of a parallelogram $AD\beta$ are set perpendicular one by one to the lines DB of a half parabola ADB of equal altitude, the rectangles produced will be a series of second roots; and the mean proportionals a series of fourth roots (of the same kind as the lines of a biquadratic parabola DE).[40]



That is, a series of equals (with index 0) multiplied term by term by a series of second roots (with index $\frac{1}{2}$) produces the same series of second roots (since $\frac{1}{2} + 0 = \frac{1}{2}$), and the mean proportionals (or rather square roots of second roots) will be a series of fourth roots.

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[39] Here the planes, or products, are represented by a single line, or ordinate, of the same magnitude.

[40] The lines DE are those in the lower left part of the diagram. The lines on the upper right illustrate Proposition 77, which follows.

PROPOSITION 77

Corollary

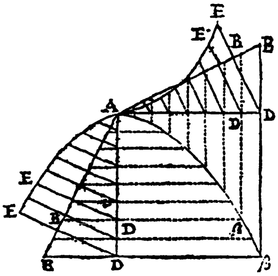
If the lines $D\beta$ of a parallelogram $AD\beta$ are set perpendicular one by one to the lines DB of the complement of a half parabola, the rectangles produced will be a series of second powers; and the mean proportionals a series of first powers (constituting triangle ADE).

That is, a series of equals, thus multiplied by a series of second powers, gives also a series of second powers (since $0 + 2 = 2$), of which the square roots are a series of first powers.

PROPOSITION 78

Corollary

If the lines DB of a triangle are set perpendicular one by one to the lines of a half parabola $D\beta$, the rectangles produced will be a series of square roots of third powers, and the mean proportionals the fourth roots of third powers, DE.



That is, a series of first powers, thus multiplied by a series of second roots, will give a series of square roots of third powers (since $1 + \frac{1}{2} = \frac{3}{2}$), of which the square roots are fourth roots of third powers.

PROPOSITION 79

Corollary

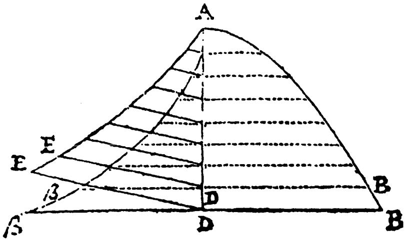
If the lines DB of a triangle are set perpendicular one by one to the lines $D\beta$ of the complement of a half parabola, the rectangles produced are a series of third powers (since $1 + 2 = 3$). And the mean proportionals are the square roots of third powers, DE.

Demonstrated as in the preceding propositions.

PROPOSITION 80

Corollary

If the lines of the complement of a half parabola are set perpendicular one by one to the lines of a half parabola, the rectangles produced will be a series of square roots of fifth powers (since $2 + \frac{1}{2} = \frac{5}{2}$), and the mean proportionals a series of fourth roots of fifth powers. As is obvious.



COMMENT

It may be considered in the same way for any figures, whether plane or solid, which arise from multiplication of this kind. So if the lines of one triangle are set perpendicular one by one to the lines of another triangle (whether similar or dissimilar, only of equal altitude), there will arise a pyramid, but the mean proportionals again constitute a triangle. The lines of a half parabola set perpendicular one by one to the lines of another produce a parabolic pyramid, but the mean proportionals a half parabola. And so on in other cases.

PROPOSITION 81

Corollary

If all the terms of a series are divided one by one by the terms of another series, the quotients will form another series, of which the index may be found by subtracting the index of the dividing series from the index of the divided series, for what remains will be the index of the series arising from the division, or of the quotient. Moreover, the ratio of the series thus produced, to a series of the same number of terms equal to its greatest, will be that which the preceding tables (or Proposition 64) indicate.

For example, if a series of biquadrates or fourth powers (with index 4) is divided by a series of cubes or third powers (with index 3) the quotients will be a series of sides or first powers, with index $1 = 4 - 3$.

If a series of third powers is divided by a series of first powers, there will arise a series of second powers, with index $2 = 3 - 1$.

And if a series of second powers is divided by a series of second powers, there will arise a series of equals, with index $0 = 2 - 2$. And so on for the rest.

The proof is clear from Proposition 73. Because, that is, a series of third powers multiplied term by term by a series of first powers gives rise to a series of fourth powers. And a series of first powers thus multiplied by a series of second powers gives rise to a series of third powers. And a series of second powers multiplied in the same way by a series of equals produces a series of second powers. And so on in all the rest. For what is composed by multiplication may be resolved by division.[41]

PROPOSITION 82

Corollary

Therefore, where the excesses of the degrees or indices of series to be divided, over those of the dividing series, are the same then the indices of the quotients will be the same.

For example, if a series of sixth powers is divided by a series of fourth powers, or a series of fifth powers by a series of third powers, or a series of fourth powers by a series of second powers, or a series of third powers by a series of first powers, or a series of second powers by a series of equals, there will arise a series of second powers. Because, that is, in each case, the series divided exceeds the dividing series by a degree of two (for $6 - 4 = 5 - 3 = 4 - 2 = 3 - 1 = 2 - 0 = 2$). Therefore (from what has gone before) the resulting series will have the same index. And the same in other cases.

PROPOSITION 83

Corollary

If a pyramid (or a series of second powers) is applied plane by line to a triangle[42] of equal altitude (that is, the planes of the former to the lines of the latter) it will produce a triangle (since, that is, $2 - 1 = 1$). If it is applied to the complement of a half parabola it will produce a parallelogram (since $2 - 2 = 0$). If to a half parabola, the plane arising will be a series of square

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[41] Wallis used *composition* and *resolution* for inverse processes such as addition and subtraction; or multiplication and division; or raising to powers and taking roots. Other writers, however, used the terms as equivalents of *synthesis* and *analysis*.

[42] '*Si Pyramis ad Triangulum respective applicetur*, ...'. The verb *applicare* (*ad*), literally 'to lay to', was used for the geometrical construction of setting an area against a line (or a solid against an area), the geometrical equivalent of division in arithmetic (see also the notes to Propositions 11 and 75).

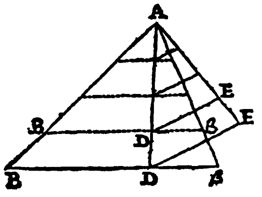
roots of third powers (since $2 - \frac{1}{2} = 1\frac{1}{2} = \frac{3}{2}$). If to a series of equals, it will produce the complement of a half parabola (since $2 - 0 = 2$). And thus in other cases.

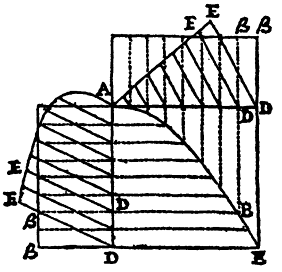
Clear from Proposition 81.

PROPOSITION 84

Corollary

Or, if from the respective lines of a first triangle ADB, and a second ADE, are taken third proportionals,[43] there will be produced a third triangle $AD\beta$. If from respective lines of the complement of a half parabola ADB, and triangle ADE, there will be produced parallelogram $AD\beta$. If indeed from parallelogram $AD\beta$ and triangle ADE, there will be produced the complement of the half parabola ADB.





Follows from what has gone before. For the squares of the lines in the triangle constitute a pyramid. And similarly for the rest. And the first part is shown in the first figure, the second and third in the next.

PROPOSITION 85

Corollary

If a parabolic pyramid (that is, a series of first powers) is applied plane by line to a triangle of equal altitude it will produce a parallelogram (since $1 - 1 = 0$). If to a parallelogram, it will produce a triangle (since $1 - 0 = 1$). If to a half parabola, it will produce a half parabola (since $1 - \frac{1}{2} = \frac{1}{2}$). If to half a cubical parabola, the plane arising will consist of cube roots of second powers (since $1 - \frac{1}{3} = \frac{2}{3}$). And equally in other applications of the same kind.

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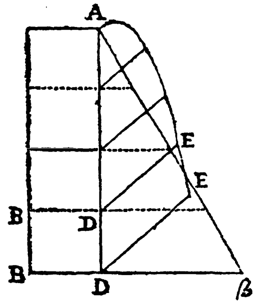
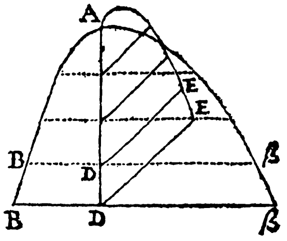
[43] The third proportional of two (ordered) quantities x and y is y^2/x (since $x : y = y : y^2/x$).





Obvious from Proposition 81.

PROPOSITION 86

Corollary

Similarly, if from the respective lines of a triangle $AD\beta$ and a half parabola ADE, are taken third proportionals, there will be produced a parallelogram ADB; if from the respective lines of parallelogram ABD, and half parabola ADE, there will be produced triangle $AD\beta$; if from the respective lines of a first parabola ADB, and a second ADE, there will be produced a third parabola $AD\beta$; and thus in other cases.

Follows from what has gone before. For the squares of the lines in a half parabola constitute a parabolic pyramid. Shown in the preceding figure.

COMMENT

And it may be considered in the same way for other plane by line applications of solid figures to planes. It suffices to have indicated a few by way of example, in imitation of which innumerable others become possible.

PROPOSITION 87

Corollary

If there is proposed any of the aforementioned series, to be divided by another of higher degree or power, it will not be possible to produce any of the series already mentioned (since it is not possible to take the index of a higher power from the index of a lower power, or rather a greater from a smaller), but clearly another kind of series, that is, one whose terms are in reciprocal proportion to the corresponding terms of another series, which has index equal to the excess of the index of the dividing series over the index of the divided series.

Moreover, the series thus arising may be called *reciprocal* series, and they have negative indices.

For example, if a series of second powers is to be divided by a series of third powers, or a series of first powers by a series of second powers, or a series of equals by a series of first powers (where the dividing series is one degree higher than the series to be divided, and so the index of the dividing series is one more than the index of the divided series, thus $3 - 2 = 2 - 1 = 1 - 0 = 1$), the terms of the series arising will be in reciprocal proportion to the corresponding terms of a series of first powers. Thus if there are divided term by term

the series	$0a^2$,	$1a^2$,	$4a^2$,	$9a^2$,	$16a^2$,	etc.
by the series	$0a^3$,	$1a^3$,	$8a^3$,	$27a^3$,	$64a^3$,	etc.
or the series	$0a$,	$1a$,	$2a$,	$3a$,	$4a$,	etc.
by the series	$0a^2$,	$1a^2$,	$4a^2$,	$9a^2$,	$16a^2$,	etc.
or the series	1,	1,	1,	1,	1,	etc
by the series	$0a$,	$1a$,	$2a$,	$3a$,	$4a$,	etc.

there will be produced the series:

$$\frac{1}{0a}, \quad \frac{1}{1a}, \quad \frac{1}{2a}, \quad \frac{1}{3a}, \quad \frac{1}{4a}, \quad \text{etc.}$$

whose terms are in reciprocal proportion to the corresponding terms in a series of first powers

$$\frac{0a}{1}, \quad \frac{1a}{1}, \quad \frac{2a}{1}, \quad \frac{3a}{1}, \quad \frac{4a}{1}, \quad \text{etc. as is obvious.}$$

That is,

$$\frac{1}{2a} : \frac{1}{3a} = \frac{3a}{1} : \frac{2a}{1}, \quad \text{and thus everywhere.}$$

In the same way, if a series of first powers is to be divided by a series of third powers (or what amounts to the same thing) a series of equals by a series of second powers, the series arising will be reciprocal to a series of second powers. Thus

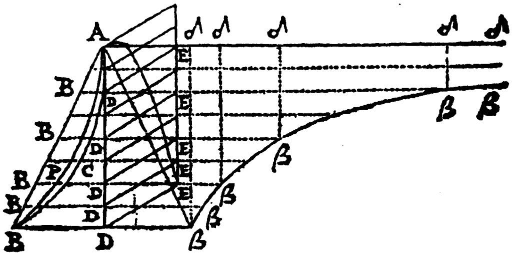
$$\frac{1}{0a^2}, \quad \frac{1}{1a^2}, \quad \frac{1}{4a^2}, \quad \frac{1}{9a^2}, \quad \frac{1}{16a^2}, \quad \text{etc.}$$

And this holds in the same way in all such divisions of this kind.

PROPOSITION 88

Corollary

If an infinite number of (parallel) planes of a parallelepiped are applied to the same number of lines of a triangle of equal altitude, (or if from the respective lines of the triangle, and parallelogram, are taken third proportionals), the series of lines arising will be reciprocal to a series of first powers, which lines are indeed in reciprocal proportion to their distances from the vertex (or, if you like, to their intercepted diameters).



For let there be a parallelepiped, in which the infinite planes are equal to the squares of the same number of lines of a parallelogram *ADE*, which, if applied to the lines of triangle *ADB*, will produce lines (third proportionals to the lines of the triangle and of the parallelogram), together constituting figure $AD\beta$, which will be in reciprocal proportion both to the corresponding lines of the triangle (since, that is, together with them they form equal rectangles) and to their intercepted diameters, or distances from the vertex, which (since the cut-off triangles are similar) are proportional to those lines of the triangle.

PROPOSITION 89

Corollary

The same holds if the planes of a pyramid (equal to the squares of the lines of a triangle *ADE*) are applied to the same number of lines of the complement of half a cubical parabola *ADBC*.

Clear from Proposition 87. For (as in the preceding Proposition) the series of first powers from which the triangle is constituted, is one degree higher than the series of equals from which the parallelepiped is constituted. Thus (in this Proposition) the series of third powers in the complement of half the cubical parabola is one degree higher than the series of second powers from which the pyramid is constituted. In either case, therefore, there arises a series reciprocal to a series of first powers.

PROPOSITION 90

Corollary

The same holds if the planes of a parabolic pyramid (that is, a series of first powers) equal to the squares of the lines of a half parabola ADE, are applied one by one to the lines of the complement of a half parabola $ADBP$ (that is, to a series of second powers).

For here also the index of the dividing series exceeds by one the index of the series to be divided.

PROPOSITION 91

Corollary

The plane figure constituted from a series of lines proportional to reciprocals of first powers, is infinite. Which is also similarly true of all reciprocal series.

For since the first term in a series of first powers is 0, the first term in a reciprocal series is ∞ or infinity (just as, in division, if the divisor is 0, the quotient will be infinite). And therefore the line $A\delta$, and the curve $\beta\beta$ do not meet unless after an infinite distance (that is, never).

For the same reason, the same curve $\beta\beta$ and line AD (however far either is continued) also do not meet (unless after an infinite distance), for the distance $D\beta$ will not vanish before there are infinitely many lines DB. And therefore:

PROPOSITION 92

Corollary

The curve $\beta\beta$ has two asymptotes, the lines $A\delta$, AD. Which is also true of other curves of this kind bounding a reciprocal series of lines.

That is, the lines [$A\delta$, AD] approach continually closer to the curve, in such a way that at last their distance becomes less than any assignable quantity (as is easily proved from what has been said), nor, however, do they ever meet, as has already been shown. And the same may equally be shown of any other curves of this kind.

PROPOSITION 93

Corollary

The lines $D\beta$, $D\beta$, etc., proportional to reciprocals of first powers, continually decrease from infinity ($A\delta = \infty$) (in the same ratio as the respective lines DB, *DB*, continually increase as a series of first powers from a point $A = 0$), until there is reached a minimum (as in the series of first powers there is reached a maximum). Which is also true in other reciprocal series.

Clear on account of the reciprocal proportion.

PROPOSITION 94

Corollary

In the figure $AD\beta\beta$ (from the reciprocals of first powers) the inscribed parallelograms $AD\beta$, $AD\beta$, etc. are equal to each other.

For they have reciprocal bases and altitudes by Proposition 88.

PROPOSITION 95

Corollary

And therefore the curve $\beta\beta$ itself is a hyperbola, of which the centre is A, the asymptotes AD, $A\delta$.

By Proposition 12 of Book II of Apollonius.

PROPOSITION 96

Corollary

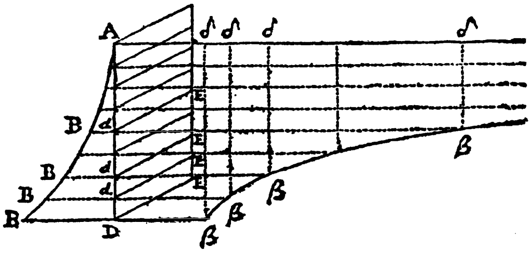
If a musical chord AD is variously divided at points D, D, etc., it produces sounds proportional to the lines $D\beta$, $D\beta$, etc.

For (from musical principles) the same chord (evenly and equally tense) produces sounds in reciprocal proportion to the lengths. Therefore if the chords are as AD, AD, etc. the sounds will be as $D\beta$, $D\beta$, etc. by Proposition 88.

PROPOSITION 97

Corollary

If the planes of a parallelepiped (or rather a series of equals), equal to the squares of the lines of a parallelogram ADE, are applied one by one to the lines of the complement of a half parabola ADB (that is, a series of second powers), there will be produced a series of lines proportional to the reciprocals of second powers. If it is supposed that a plane figure $AD\beta$ is constituted from these, it will be infinite; and the curve bounding those [lines] will have two asymptotes, the lines AD, $A\delta$.



Proved in the same way as various preceding propositions on series reciprocal to series of first powers.

PROPOSITION 98

Corollary

The same holds if the planes of a pyramid are thus applied to the lines of the complement of half a biquadratic parabola; or the planes of parabolic pyramid to the lines of the complement of a cubical parabola.

For in those cases also, the index of the series dividing exceeds by two the index of the series to be divided.

PROPOSITION 99

Corollary

In a series of this kind, of reciprocals of second powers, the lines $D\beta$, $d\beta$, etc. are in reciprocal ratio to the squares of the diameters (or distances from the vertex), (or as $(dA)^2$, $(DA)^2$, etc.).[44]

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[44] Wallis used Oughtred's notation dAq for squares, and $dAq \cdot DAq :: D\beta \cdot d\beta$ for ratio, throughout Propositions 99 and 100.

Therefore, that is, the reciprocal proportionals are the lines DB, dB, etc., which are in direct ratio to the squares of the diameters AD, Ad, etc. Thus

$$(dA)^2 : (DA)^2 = dB : DB = D\beta : d\beta$$

Therefore
$$\frac{(dA)^2}{(DA)^2} = \frac{D\beta}{d\beta}.$$

PROPOSITION 100

Corollary

In the plane figure $AD\beta\beta$ constituted from a series of lines which are reciprocals of second powers, the inscribed parallelograms ($AD\beta$, $Ad\beta$) are in reciprocal proportion to the intercepted diameters (DA, dA).

For (by Euclid VI.23) they are as $DA \times D\beta$ to $dA \times d\beta$.
And moreover (by what has gone before)

$$d\beta = \frac{(DA)^2}{(dA)^2} D\beta.$$

And

$$dA \times d\beta = \frac{(DA)^2}{dA} D\beta.$$

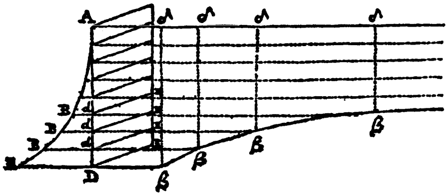
Therefore also

$$\begin{aligned} DA \times D\beta : dA \times d\beta &= DA \times D\beta : \frac{(DA)^2}{dA} D\beta \\ &= dA \times DA \times D\beta : (DA)^2 \times D\beta \\ &= dA : DA. \end{aligned}$$

PROPOSITION 101

Corollary

But if the plane figure $AD\beta\beta$ is constituted from a series of lines that are reciprocals of third powers (which, that is, are third proportionals of the lines of the complement of half a cubical parabola ADB and of a parallelogram ADE) those lines ($D\beta$, $d\beta$, etc.) will be in reciprocal ratio to the cubes of the diameters (DA, dA, etc.), (that is, as $(dA)^3$, $(DA)^3$, etc.). And the inscribed parallelograms ($AD\beta$, $Ad\beta$, etc.) will be in reciprocal ratio to the squares of the diameters (or as $(dA)^2$, $(DA)^2$, etc.).



For (by the construction) $D\beta : d\beta = dB : DB = (dA)^3 : (DA)^3$.
And therefore also $d\beta = \frac{(DA)^3}{(dA)^3} D\beta$.
Therefore the parallelograms

$$\begin{aligned} AD\beta : Ad\beta &= DA \times D\beta : dA \times d\beta \\ &= DA \times D\beta : dA \times \frac{(DA)^3}{(dA)^3} D\beta. \\ &= DA \times D\beta : \frac{(DA)^3}{(dA)^2} D\beta \\ &= (dA)^2 \times DA \times D\beta : (DA)^3 \times D\beta \\ &= (dA)^2 : (DA)^2. \end{aligned}$$

Which was to be proved.

COMMENT

And it may be considered in the same way for other plane figures of this kind constituted from any *reciprocal series* of lines; as also for the inscribed parallelograms themselves (or rectangles, or oblique angled figures, as the condition of the figure requires).

Which, moreover, leads to the area of these plane figures constituted from reciprocal series, which may be sought in almost the same way as above for direct series. Moreover, where direct series have indices 1, 2, 3, etc. as they ascend by so many degrees above a series of equals, so indeed will these (reciprocal to those) have their indices contrary and negative, $-1, -2, -3$, etc., descending below a series of equals by as many degrees. Moreover, just as the former continually increase from 0, ciphra, or nothing, the latter, on the contrary, continually decrease from ∞, or infinity; and in the former a greatest term, in the latter a least term, concludes the series (which also however may be supposed continued as far as one likes, in the former by increasing, in the latter by decreasing). And therefore, as in the former there is a circumscribed figure (thus a parallelogram or prism), or a series of the same number of terms equal to the greatest, in the latter there is an inscribed figure, or a series of the same number of terms equal to the least, to be had as a common measure, to which the comparison is to be made; in either case making use of that term respectively that concludes the series.

And in the meantime it should not seem surprising to anyone (although possibly unexpected) if I should enquire into the ratio of unbounded figures to another given bounded figure. For any such figures $AD\beta$ of this kind (howsoever extended from line $D\beta$ to whatever boundary, in the manner described in these comments) are supposed continued infinitely from the line $\delta\beta$ (by Proposition 91), but will not, however, on account of that, have either no ratio or always an infinite ratio to a given bounded figure, thus to a parallelogram of equal height described on the same base $D\beta$. Indeed, it seems possible to obtain confirmation of that more easily, since Torricelli has already shown the same thing in one particular solid (which may be called an *acute infinite hyperbola*).[45] But they will not always have a finite ratio, but sometimes either infinite, or also (if this can be said without solecism) greater than infinite. That is, if the lines $\delta\beta$ are shortened by the same ratio as the lines $d\beta$ are lengthened, that ratio will be infinite; where, that is, the lengthening of one is equal to the shortening of the other (and therefore the ratio of a continuous infinite figure composed from both is equal to that of some figure smoothly continued to infinity). But if the lines $\delta\beta$ are shortened by a smaller ratio than the lines $D\beta$ are lengthened, the ratio will be greater than infinity; for then the lengthening of the latter runs ahead of (or more than equals) the shortening of the former. If, moreover, the lines $\delta\beta$ are decreased by a greater ratio than the lines $D\beta$ are increased, the decrease of the former runs ahead of the increase in the latter; and therefore the ratio will be finite, or rather, less than infinite. (And indeed according to this criterion, it may be considered not only for these figures we have already treated, but also for any other infinite figures, whether plane or solid, compared to some bounded figures: which speculation, I believe, will not seem disagreeable.) Moreover, what will happen in each ratio, we will indicate in various following propositions (following the rule in Proposition 64).

PROPOSITION 102

Theorem

If a figure $AD\beta\beta$ has infinite vertex $A\delta$, and continually decreases in width towards the base as far as $D\beta$, according to any reciprocal series of whatever direct series (thus, of those mentioned in Proposition 59) which has index less than 1, it will have to a parallelogram on the same base and of equal height a finite ratio, that is, that of 1 to the index of that reciprocal series increased by 1.

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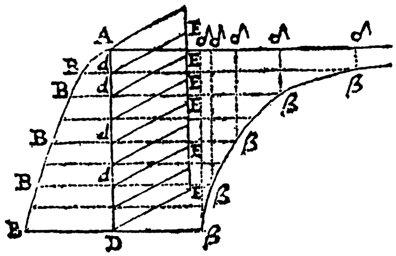
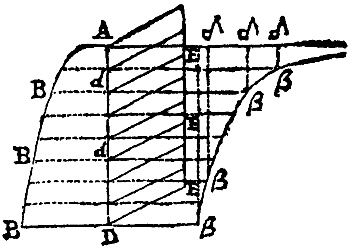
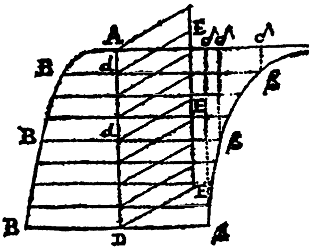
[45] Torricelli 1644, 115–116.







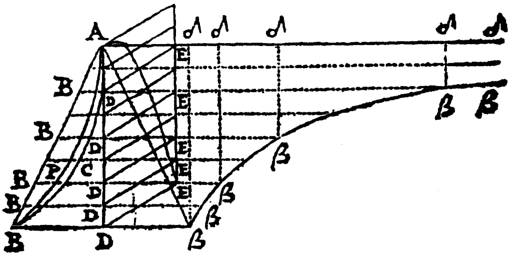
For example, let there be direct series of second roots, third roots, fourth roots, etc. of which the indices are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. (less than 1); the series reciprocal to these will have indices $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$, etc. For if there is supposed a series of equals, with index 0, divided by the former, the series arising from the division will have indices $0 - \frac{1}{2}, 0 - \frac{1}{3}, 0 - \frac{1}{4}$, etc., that is, $-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$, etc. (by Proposition 81), if to which (according to the rule of Proposition 64) there is added 1, they become $-\frac{1}{2} + 1$, $-\frac{1}{3} + 1$, $-\frac{1}{4} + 1$, etc., that is, $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, etc. and therefore the ratio of the whole figure to the inscribed parallelogram (on the same base and of equal height) is as 1 to $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, etc. or as 2 to 1, 3 to 2, 4 to 3, etc.

And in the same way, if there is taken a series reciprocal to a series of cube roots of second powers, or fourth roots of second or third powers, or fifth roots of second, third or fourth powers (of which the indices are $\frac{2}{3}, \frac{2}{4}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$) or to any other such series, whose index is less than 1. Because the negative indices of the reciprocal series, contrary to these (that is, $-\frac{2}{3}, -\frac{3}{4}$, etc.), become positive by the addition of 1; and therefore the ratio of 1 to those indices thus increased will be a finite ratio; or rather, a positive number to a positive.

PROPOSITION 103

Theorem

But if any such figure $AD\beta\beta$ of this kind thus continually decreases as a series which is reciprocal to a direct series having index equal to 1 (that is a series of first powers), it will have to the inscribed parallelogram an infinite ratio, that is, that which is 1 to 0.

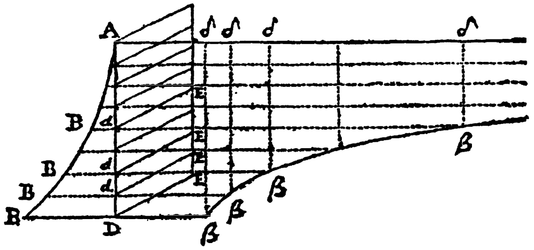


For since a series of first powers has index 1, the series reciprocal to it will have index -1, and therefore (by Proposition 64) the ratio arising will be 1 to $-1 + 1$, that is, 1 to 0.

PROPOSITION 104

Theorem

Finally if any figure $AD\beta\beta$ of this kind thus continually decreases as series which is reciprocal to a direct series having index greater than 1, it will have to the inscribed parallelogram a ratio greater than infinity: of a kind, that is, that a positive number may be supposed to have to a negative number, or less than 0. That is, that of 1 to the index increased by 1.



Thus since the indices of series of second powers, third powers, fourth powers, etc. are 2, 3, 4, etc. (greater than 1) the indices of their reciprocal series will be $-2, -3, -4$, etc., any of which increased by 1 (according to Proposition 64), however, will remain negative, thus $-2 + 1 = -1$, $-3 + 1 = -2$, $-4 + 1 = -3$, etc., and therefore the ratio of 1 to those indices thus increased, thus 1 to -1, 1 to -2, 1 to -3, etc. will be greater than infinity, or 1 to 0, because, that is, the second terms of the ratios are less than 0.

And the same holds if there are taken reciprocals of series of square roots of third powers, fourth powers, fifth powers, etc. (whose indices are $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, etc.) or cube roots of fourth powers, fifth powers, sixth powers, etc. (whose indices are $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{3}$, etc.) or finally of any series whose index is greater than 1. As is obvious.

PROPOSITION 105

Theorem

If any figure $AD\beta\beta$ of this kind having infinite vertex $A\delta$ and finite base $D\beta$, has to the inscribed parallelogram $AD\beta\delta$ a ratio greater than infinity, the same figure $AD\beta\beta$ having infinite vertex AD and finite base $\delta\beta$ will have to the inscribed parallelogram $A\delta\beta D$ a ratio less than infinity (that is, finite). And the other way round, if having considered the former situation, there is a ratio less than infinity, in the latter situation there will be a ratio greater than infinity. If, finally, in one situation there is a simple infinite ratio (that is, neither greater nor less [than infinity]) then in the other situation also there will be a simple infinite ratio.

For example, in a series of reciprocals of second powers, since (by Proposition 99) the lines $D\beta, D\beta$, are as the reciprocals of the squares of the diameters AD, AD, then conversely the lines AD, AD, that is, $\delta\beta, \delta\beta$, will be as the reciprocals of the square roots of the lines $D\beta, D\beta$, that is, the diameters $A\delta, A\delta$; and therefore $\delta\beta, \delta\beta$, etc. are themselves a series of reciprocals of second roots. And the other way round. And (since the same also holds for other series of this kind) what is proposed is clear by Propositions 102 and 104.

But if in a series of reciprocals of first powers, since (by Proposition 88) the lines $D\beta, D\beta$, are in reciprocal proportion to the diameters AD, AD, so also the lines $\delta\beta, \delta\beta$, will be in reciprocal proportion to their diameters $A\delta, A\delta$; and $\delta\beta, \delta\beta$, themselves are likewise a series of reciprocals of first powers. Therefore what was proposed stands, by Proposition 103.

PROPOSITION 106

Theorem

If any reciprocal series is multiplied or divided by another series (whether reciprocal or direct), or also multiplies or divides another, the same laws must be observed as for direct series, as in Propositions 73 and 81.

For example, if a series of reciprocals of second powers (suppose $\frac{1}{1}, \frac{1}{4}, \frac{1}{9}$, etc.) with index -2, is multiplied term by term by a series of reciprocals of third powers (suppose $\frac{1}{1}, \frac{1}{8}, \frac{1}{27}$, etc.) with index -3, it will produce a series of reciprocals of fifth powers ($\frac{1}{1}, \frac{1}{32}, \frac{1}{243}$, etc.) with index $-5 = -2 - 3$, as is obvious.

In the same way, if a series of reciprocals of third powers ($\frac{1}{1}, \frac{1}{8}, \frac{1}{27}$, etc.) with index -3 is multiplied term by term by a series of second powers (1, 4, 9, etc.) with index 2, it will produce a series $\frac{1}{1}, \frac{4}{8}, \frac{9}{27}$, etc., that is, $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$, etc., reciprocals of first powers, with index $-1 = -3 + 2$.

In the same way, if a series of reciprocals of second roots ($\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}$, etc.) with index $-\frac{1}{2}$, is multiplied term by term by a series of squares (1, 4, 9, etc.) with index 2, it will produce the series ($\frac{1}{\sqrt{1}}, \frac{4}{\sqrt{2}}, \frac{9}{\sqrt{3}}$, etc. or $\frac{1}{1}\sqrt{1}$, $\frac{4}{2}\sqrt{2}$, $\frac{9}{3}\sqrt{3}$, etc. or $1\sqrt{1}$, $2\sqrt{2}$, $3\sqrt{3}$, etc. or $\sqrt{1}$, $\sqrt{8}$, $\sqrt{27}$, etc.), square roots of cubes, or third powers, with index $\frac{3}{2} = -\frac{1}{2} + 2$.

Further, if a series of reciprocals of second powers, with index -2, divides a series of reciprocals of first powers, with index -1, it will produce a series of first powers, with index $1 = -1 + 2$, that is, -1 minus -2.

In the same way, if a series of reciprocals of first powers, with index -1, divides a series of reciprocals of second powers, with index -2, it will produce a series of reciprocals of first powers, with index $-1 = -2 + 1$, that is -2 minus -1.

In the same way, if a series of reciprocals of first powers, with index -1, divides a series of second powers, with index 2, it will produce a series of third powers, with index $3 = 2 + 1$, that is, 2 minus -1.

In the same way, if a series of reciprocals of first powers, with index -1, [is divided by][46] a series of second powers, with index 2, it will produce a series of reciprocals of third powers, with index $-3 = -1 - 2$, that is, -1 minus 2.

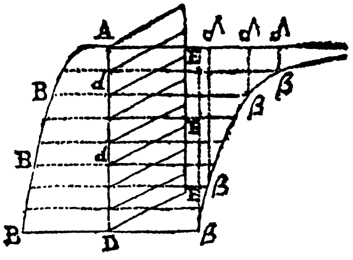
And the same holds in any other series of this kind. And therefore what was proposed stands.

PROPOSITION 107

Corollary

And therefore if from any figure $AD\beta\delta$ of this kind (extended infinitely from one side) corresponding to any series of reciprocals, there is generated (in the manner I have shown in Proposition 9 of *On conic sections* and elsewhere above)[47] an inverse pyramidoid or conoid (or rather, calatoid), it will have to the inscribed cylinder or prism (on the same base and of equal height) that ratio, whether finite or infinite or greater than infinite, that the preceding theorems taught.

Thus, if the plane figure is a series of lines that are reciprocals of third roots, with index $-\frac{1}{3}$, and therefore its ratio to the inscribed parallelogram (by Propositions 64 and 102) is as 1 to $\frac{2}{3}(= -\frac{1}{3} + 1)$, that is, as 3 to 2, then the solid consisting of the same number of planes, which are as the squares of the lines, will be a series of reciprocals of squares of third roots, with index (by Proposition 106) $-\frac{2}{3} = -\frac{1}{3} - \frac{1}{3}$, or $-\frac{1}{3}$ plus $-\frac{1}{3}$, and the ratio of that solid to the inscribed cylinder or prism (on the same base and of equal height) as 1 to $\frac{1}{3} = -\frac{2}{3} + 1$, or as 3 to 1, and in either case a finite ratio. By Propositions 64 and 102.



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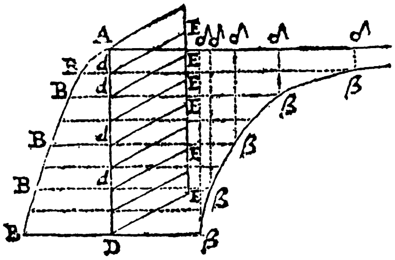
[46] Wallis has mistakenly written 'divides'.

[47] Wallis is now discussing solids of revolution.

If the plane figure is series of reciprocals of second roots, with index $-\frac{1}{2}$, and therefore its ratio to the inscribed parallelogram as 1 to $\frac{1}{2} = -\frac{1}{2} + 1$, or as 2 to 1 (by Proposition 102), then the solid consisting of the same number of planes, which are as the squares of the lines, will be a series of reciprocals of squares of second roots, or (which amounts to the same thing) a series of reciprocals of first powers with index $-\frac{2}{2}$, or $-1 = -\frac{1}{2} - \frac{1}{2}$. And therefore the ratio of this solid to the cylinder or prism (on the same base and of equal height) as 1 to $-1 + 1 = 0$ (by Proposition 103). That is, the former is finite ratio, the latter a simple infinite ratio.



If the plane figure is a series of reciprocals of squares of third roots, with index $-\frac{2}{3}$, and therefore its ratio to the inscribed parallelogram as 1 to $\frac{1}{3} = -\frac{2}{3} + 1$, or as 3 to 1 (by Proposition 102), then the solid consisting of the same number of planes, which are as the squares of those lines, will be a series of reciprocals of biquadrates of third roots, with index $-\frac{4}{3} = -\frac{2}{3} - \frac{2}{3}$, and therefore its ratio to the inscribed cylinder or prism (on the same base and of equal height) as 1 to $-\frac{4}{3} + 1 = -\frac{1}{3}$, or as 3 to -1 (by Proposition 104). That is, the former ratio is finite, the latter greater than infinite.

If the plane figure is a series of reciprocals of first powers, with index -1, and therefore its ratio to the inscribed parallelogram as 1 to $-1 + 1 = 0$ (by Proposition 103), then the solid consisting of planes, which are as the squares of those lines, will be a series of reciprocals of squares of first powers (that is, of second powers) with index -2, and therefore its ratio to the cylinder or prism on the same base and of equal height as 1 to $-2 + 1$, or as 1 to -1 (by Proposition 104). That is, the former is a simple infinite ratio, the latter greater than infinite.

If the plane figure is a series of reciprocals of second powers with index -2, and therefore its ratio to the inscribed parallelogram as 1 to $-2 + 1 = -1$, then the solid consisting of the same number of planes, which are as the squares of those lines, will be a series of reciprocals of squares of second powers, that is, of fourth powers, with index $-4 = -2 - 2$, and its ratio to the inscribed cylinder or prism as 1 to $-4 + 1 = -3$ (by Proposition 104). That is, in both cases the ratio is greater than infinite.

COMMENT

And thus that result (clever indeed and not a little surprising) that Torricelli demonstrated in one solid figure (that is, that an acute hyperbolic solid, infinitely extended, constitutes an equal cylinder) we have demonstrated for innumerable other figures, both plane and solid (by continuation of the six preceding Propositions). Thus, *to show for infinite figures of innumerable different kinds, both plane and solid, equal bounded figures* (or at least, what amounts to the same thing, constituted in a known ratio).

It might have been more skilled perhaps (as I have given so much of the needed report) to have shown by a quicker method than has been arrived at here, some few partial propositions (just as much to be wondered and amazed at) without demonstrations. Which, I wholly suspect, the Ancients at one time often did; who more often seem to have intended that they themselves might be admired rather than that others should understand; at least, that they might show assent to those pronouncements of theirs by force, rather than understand a genuine investigation of the problem. And I believe this to have been the case, because their Analysis (which indeed it is sufficiently clear that they had, from many remains, for not a few of their demonstrations) was almost completely hidden to those who came afterwards (for plainly that part that survives in Diophantus is quite small, if compared with those outstanding discoveries they arrived at). So that mathematicians of the present age (Viète, Oughtred, Harriot, Ghetaldi, Cavalieri, Torricelli, Descartes and other great men) will need either to think anew, or at least revive the old (whether wholly expounded, or completely unknown) in a new way; who indeed by their success have shown that our analysis of the present day, is certainly equal, or rather without doubt supercedes, that of the Ancients, hidden by so much superstition.

Indeed, I prefer by freely philosophizing, to open those springs, that with the same work the reader may begin to see both the demonstrations of the propositions and the method by which I have arrived at them; whence he may also by his own efforts investigate innumerable others of the same kind, which I (lest I become tedious) readily pass over, content by this to have indicated them, whence others may produce at will others similar to mine.

Indeed, it is possible both to add much to the foregoing and to interpolate much throughout, which may be easily deduced from the principles already taught. Indeed, since those things I have already taught seem to me abundantly sufficient, that both they themselves may be clearly enough understood, and also that they seem to comprise a satisfactorily complete treatment of series (whether simple, compound, or reciprocal to either), I appear to be hastening towards the explanation of *conjoint* series (whether in the form of binomes or apotomes).

PROPOSITION 108

Theorem

If a series of equals is reduced term by term by a series of first powers (thus, if the first term of the latter is taken from the first of the former, the second from the second, etc.), the [series of] differences will be half of the whole [series of equals]. But if it is augmented in a similar way, the series of sums will be one and a half times the given series of equals.

It is to be understood that the last terms of the equals and of the first powers are the same, or equal (which is also to be understood in whatever follows). If they are unequal, however, it will not be difficult to find the ratios that arise; which it is sufficient to have pointed out, since anyone may show it by their own effort.

Suppose, for example, that any of the equal terms, and the greatest of the first powers, is R. An infinitely small part of it may be called $a = R/\infty$, and the number of all the terms (or the altitude of the figure) A.[48]

$$\begin{array}{llll} \text{Differences:} & R-0a & \text{Sums:} & R+0a \\ & R-1a & & R+1a \\ & R-2a & & R+1a \\ & R-3a & & R+2a \\ & \text{etc.} & & \text{etc.} \end{array}$$

If the terms are continued to infinity, as far as:

$$R-R \qquad\qquad R+R$$

then the sums of the differences and sums will be:

$$AR-\tfrac{1}{2}AR \qquad\qquad AR+\tfrac{1}{2}AR$$

For the sum of all the equals will be AR (as is obvious). The sum of the first powers will be half of that, or $\frac{1}{2}AR$, (by Proposition 2). $AR - \frac{1}{2}AR = \frac{1}{2}AR$, and $AR + \frac{1}{2}AR = \frac{3}{2}AR$. That is, to the series of equals (AR), the former is $\frac{1}{2}$ and the latter $\frac{3}{2}$, just as asserted. That is, the former will be to series of equals as $\frac{1}{2}$ to 1, or 1 to 2; the latter as $\frac{3}{2}$ to 1, or 1 to $\frac{2}{3}$, or 3 to 2.

PROPOSITION 109

Corollary

Therefore, if from a parallelogram there is taken a triangle (on the same or equal base and of equal height) the remainder (which indeed is itself also

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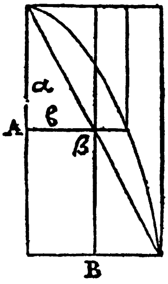
[48] Wallis wants to approach R by an infinite number of small steps, which would suggest that $A = \infty$, so there is a serious contradiction here. The problem arises from Wallis's concept of an area as a sum of lines: for him the area of a rectangle with base R is equivalent to R taken infinitely many times. But if the altitude of the rectangle is A, its area is AR, leading Wallis to state that A is equivalent to 'infinitely many times', or 'the number of all the terms'.

a triangle inverted) will be half the parallelogram. But if the triangle is added, the sum (that is, the trapezium) will be one and a half times the parallelogram.



Clear from what has gone before; for the parallelogram is a series of equals and the triangle a series of first powers.

PROPOSITION 110

Corollary

In the same way, if a cylinder is hollowed out parabolically, it becomes half the complete cylinder (which same is also true of a similarly hollowed prism).

That is, if from the cylinder (that is, a series of equals) there is taken away a parabolic conoid (on the same base and of equal height) which indeed is a series of first powers (by Proposition 4 or 60), what is left will be half the total, by Proposition 108.

And the same happens if from a prism there is taken a parabolic pyramid.

PROPOSITION 111

Theorem

If a series of equals is reduced by series of second powers, third powers, fourth powers, etc. [the sums of] the differences will be two-thirds, three-quarters, four-fifths, etc. of the whole [series of equals]. But if it is augmented in a similar way, the sums will be four-thirds, five-fourths, six-fifths, etc. [of the series of equals].

For if the terms:

$$\begin{array}{lll} R^2 \mp 0a^2 & R^3 \mp 0a^3 & R^4 \mp 0a^4 \\ R^2 \mp 1a^2 & R^3 \mp 1a^3 & R^4 \mp 1a^4 \\ R^2 \mp 4a^2 & R^3 \mp 8a^3 & R^4 \mp 16a^4 \\ R^2 \mp 9a^2 & R^3 \mp 27a^3 & R^4 \mp 81a^4 \end{array}$$

are continued to:

$$\begin{array}{lll} R^2 \mp R^2 & R^3 \mp R^3 & R^4 \mp R^4 \\ \hline \end{array}$$

then the sums will be (by Proposition 44):

$$AR^2 \mp \tfrac{1}{3}AR^2 \qquad AR^3 \mp \tfrac{1}{4}AR^3 \qquad AR^4 \mp \tfrac{1}{5}AR^4$$

that is, the sums of the differences will be:

$$1 - \tfrac{1}{3} = \tfrac{2}{3} \qquad 1 - \tfrac{1}{4} = \tfrac{3}{4} \qquad 1 - \tfrac{1}{5} = \tfrac{4}{5}$$

and the sums of the sums will be:

$$1 + \tfrac{1}{3} = \tfrac{4}{3} \qquad 1 + \tfrac{1}{4} = \tfrac{5}{4} \qquad 1 + \tfrac{1}{5} = \tfrac{6}{5}$$

PROPOSITION 112

Corollary

Therefore if from a parallelogram there is taken the complement of half of a parabola, cubical parabola, biquadratic parabola, etc. the remainders (that is, the half parabola, cubical parabola, biquadratic parabola, etc.) will be $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, etc. of the whole parallelogram. But if the complements are added to the same parallelogram, the sums will be $\frac{4}{3}$, $\frac{5}{4}$, $\frac{6}{5}$, etc. of the parallelogram.

Clear from what has gone before.

PROPOSITION 113

Corollary

In the same way, a cylinder excavated by a cone (or a prism by a pyramid) comprises two thirds of the whole. And it may be considered similarly for other excavated figures (with appropriate changes).

Clear from Proposition 111, in fact, the subtraction of a series of second powers from a series of equals.

PROPOSITION 114

Theorem

If a series of equals is reduced by a series of second roots, third roots, fourth roots, fifth roots, etc. [the sums of] the differences will be one-third, one-quarter, one-fifth, etc. of the whole. But if thus augmented, [the sums of] the sums will be five-thirds, seven-fourths, nine-fifths, etc. or twice, minus one-third, one-quarter, one-fifth, etc.

For if the terms:

$$\begin{array}{lll} \sqrt{R} \mp \sqrt{0a} & \sqrt[3]{R} \mp \sqrt[3]{0a} & \sqrt[4]{R} \mp \sqrt[4]{0a} \\ \sqrt{R} \mp \sqrt{1a} & \sqrt[3]{R} \mp \sqrt[3]{1a} & \sqrt[4]{R} \mp \sqrt[4]{1a} \\ \sqrt{R} \mp \sqrt{2a} & \sqrt[3]{R} \mp \sqrt[3]{2a} & \sqrt[4]{R} \mp \sqrt[4]{2a} \\ \sqrt{R} \mp \sqrt{3a} & \sqrt[3]{R} \mp \sqrt[3]{3a} & \sqrt[4]{R} \mp \sqrt[4]{3a} \end{array}$$

are continued to:

$$\begin{array}{lll} \sqrt{R} \mp \sqrt{R} & \sqrt[3]{R} \mp \sqrt[3]{R} & \sqrt[4]{R} \mp \sqrt[4]{R} \\ \hline \end{array}$$

then the sums will be (by Proposition 54):

$$A\sqrt{R} \mp \tfrac{2}{3}AR \qquad A\sqrt[3]{R} \mp \tfrac{3}{4}A\sqrt[3]{R} \qquad A\sqrt[4]{R} \mp \tfrac{4}{5}A\sqrt[4]{R}$$

that is, the sums of the differences will be:

$$1 - \tfrac{2}{3} = \tfrac{1}{3} \qquad 1 - \tfrac{3}{4} = \tfrac{1}{4} \qquad 1 - \tfrac{4}{5} = \tfrac{1}{5}$$

and the sums of the sums will be:

$$1 + \tfrac{2}{3} = \tfrac{5}{3} \qquad 1 + \tfrac{3}{4} = \tfrac{7}{4} \qquad 1 + \tfrac{4}{5} = \tfrac{9}{5}$$

PROPOSITION 115

Corollary

Therefore, if from a parallelogram there is taken a parabola, cubical parabola, biquadratic parabola, etc. the remainders will be one-third, one-quarter, one-fifth, etc. of the whole. But if they are added, the sums will be twice the parallelogram, less one-third, one-quarter, one-fifth, etc.

Follows from what has gone before.

PROPOSITION 116

Theorem

It may be considered in the same way for any other series for which the index is known by Propositions 59 or 64, subtracted from or added to a series of equals.

Thus if the terms:

$$\begin{array}{lll}
\sqrt{R^3} \mp \sqrt{0a^3} & \sqrt[3]{R^2} \mp \sqrt[3]{0a^2} & \sqrt[3]{R^4} \mp \sqrt[3]{0a^4} \\
\sqrt{R^3} \mp \sqrt{1a^3} & \sqrt[3]{R^2} \mp \sqrt[3]{1a^2} & \sqrt[3]{R^4} \mp \sqrt[3]{1a^4} \\
\sqrt{R^3} \mp \sqrt{8a^3} & \sqrt[3]{R^2} \mp \sqrt[3]{4a^2} & \sqrt[3]{R^4} \mp \sqrt[3]{16a^4} \\
\sqrt{R^3} \mp \sqrt{27a^3} & \sqrt[3]{R^2} \mp \sqrt[3]{9a^2} & \sqrt[3]{R^4} \mp \sqrt[3]{81a^4}
\end{array}$$

are continued to:

$$\begin{array}{lll}
\sqrt{R^3} \mp \sqrt{R^3} & \sqrt[3]{R^2} \mp \sqrt[3]{R^2} & \sqrt[3]{R^4} \mp \sqrt[3]{R^4} \\
\hline
\end{array}$$

the sums will be:

$$\begin{array}{lll}
A\sqrt{R^3} \mp \tfrac{2}{5}A\sqrt{R^2} & A\sqrt[3]{R^2} \mp \tfrac{3}{5}A\sqrt[3]{R^2} & A\sqrt[3]{R^4} \mp \tfrac{3}{7}A\sqrt[3]{R^4}
\end{array}$$

that is, the differences will be:

$$\begin{array}{lll}
1 - \tfrac{2}{5} = \tfrac{3}{5} & 1 - \tfrac{3}{5} = \tfrac{2}{5} & 1 - \tfrac{3}{7} = \tfrac{4}{7}
\end{array}$$

and the sums:

$$\begin{array}{lll}
1 + \tfrac{2}{5} = \tfrac{7}{5} & 1 + \tfrac{3}{5} = \tfrac{8}{5} & 1 + \tfrac{3}{7} = \tfrac{10}{7}
\end{array}$$

And similarly (with appropriate changes) in any others whatever.

PROPOSITION 117

Theorem

If there is proposed a series of equals reduced by a series of first powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of them.

Instead of the notation 1a, 2a, 3a, etc. (used in the preceding propositions) one may now substitute a, b, c, etc. by which the process of the operation may be better perceived.

Series	Squares	Cubes
$R-0$	$R^2-0R+00$	$R^3-0R^2+00R-000$
$R-a$	$R^2-2aR+a^2$	$R^3-3aR^2+3a^2R-a^3$
$R-b$	$R^2-2bR+b^2$	$R^3-3bR^2+3b^2R-b^3$
$R-c$	$R^2-2cR+c^2$	$R^3-3cR^2+3c^2R-c^3$
etc. to		
$R-R$	$R^2-2RR+R^2$	$R^3-3RR^2+3R^2R-R^3$
$AR-\frac{1}{2}AR$	$AR^2-\frac{2}{2}AR^4+\frac{1}{3}AR^4$	$AR^3-\frac{3}{2}AR^3+\frac{3}{3}AR^3-\frac{1}{4}AR^6$
that is:		
$1-\frac{1}{2}=\frac{1}{2}$	$1-\frac{2}{2}+\frac{1}{3}=\frac{1}{3}$	$1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}=\frac{1}{4}$
or:		
$\frac{1}{2}$	$\frac{1\times 2}{2\times 3}$	$\frac{1\times 2\times 3}{2\times 3\times 4}$

And so on, by continually multiplying numbers in arithmetic proportion (as the degree of the power requires), from 1 and 2, continually increasing by one.

And indeed, these are nothing but series of the same number of first powers, second powers, third powers, fourth powers, etc. reversed.[49]

PROPOSITION 118

Theorem

If there is proposed a series of equals reduced by a series of second powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of them. That is:

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[49] *Inversae*, here translated as 'reversed', means 'decreasing instead of increasing'.

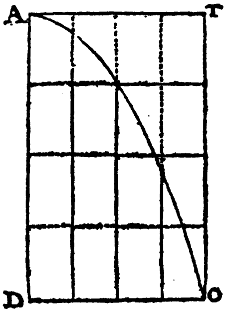
Series	Squares	Cubes
$R^2 - 00$	$R^4 - 00R^2 + 00$	$R^6 - 00R^4 + 00R^2 - 00$
$R^2 - a^2$	$R^4 - 2a^2R^2 + a^4$	$R^6 - 3a^2R^4 + 3a^4R^2 - a^6$
$R^2 - b^2$	$R^4 - 2b^2R^2 + b^4$	$R^6 - 3b^2R^4 + 3b^4R^2 - b^6$
$R^2 - c^2$	$R^4 - 2c^2R^2 + c^4$	$R^6 - 3c^2R^4 + 3c^4R^2 - c^6$
etc. to		
$R^2 - R^2$	$R^4 - 2R^2R^2 + R^4$	$R^6 - 3R^2R^4 + 3R^4R^2 - R^6$
Sum:		
$AR^2 - \frac{1}{3}AR^2$	$AR^4 - \frac{2}{3}AR^4 + \frac{1}{5}AR^4$	$AR^6 - \frac{3}{3}AR^6 + \frac{3}{5}AR^6 - \frac{1}{7}AR^6$
that is:		
$1 - \frac{1}{3} = \frac{2}{3}$	$1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$	$1 - \frac{3}{3} + \frac{3}{5} - \frac{1}{7} = \frac{48}{105}$
or:		
$\frac{2}{3}$	$\frac{2 \times 4}{3 \times 5}$	$\frac{2 \times 4 \times 6}{3 \times 5 \times 7}$

And so on, by continually multiplying numbers in arithmetic proportion (as far as the degree of the power requires), from 2 and 3, continually increasing by twos.

PROPOSITION 119

Corollary

And therefore a conoid (or pyramidoid) generated by a half (or whole) parabola around one of its ordinates, will be to a cylinder (or prism) of the same base and height as 8 to 15.



That is, as the squares of the differences of a series of equals reduced by second powers (to the same number of terms equal to the greatest). For revolving the half parabola ADO around its ordinate DO (or even other lines, as we said in Proposition 9 of *On conic sections*) as an axis, there is formed a conoid (or pyramidoid) with vertex O. The constituent planes of that conoid (or pyramidoid) will be as the squares of a series of equals reduced by second powers. (For the lines in the half parabola ADO parallel to the line AD are equals reduced by the second powers found in the complement ATO, as is clear from what was said in Proposition 23.) And therefore to a series of the same number of terms equal to the greatest (that is, the cylinder or prism) they are as 8 to 15, by what has gone before.

COMMENT

And it may be considered in the same way for conoids or pyramidoids generated around an ordinate of any higher parabola, with the help of the following propositions. Thus, for a cubical parabola the ratio will as 9 to 14, for a biquadratic parabola as 32 to 45, for a supersolid parabola as 25 to 33, etc. as in the table in Proposition 126.

PROPOSITION 120

Corollary

Thence, if an infinite series of equals reduced by a series of first powers is multiplied term by term by the same series of equals augmented by the same series of first powers, the sum of the rectangles,[50] (or squares or any similar figures equal or even proportional to them) will have a known ratio to the sum of the same number of terms equal to the greatest.

And the same happens if the squares of a reduced series are multiplied by the squares of an augmented series, or cubes of the former by cubes of the latter, and so on.

That is, they will produce ratios as in Proposition 118. For:

$$\begin{array}{lcc} & R-a & (R-a)^2 = R^2 - 2aR + a^2 \\ \text{times} & R+a & (R+a)^2 = R^2 + 2aR + a^2 \\ \cline{2-3} \text{makes} & R^2 - a^2 & R^4 - 2a^2R^2 + a^4 \end{array}$$

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[50] Since Wallis is speaking of multiplication of series, the Latin *rectangulorum* would here more naturally be translated as 'of the products'. I have kept the more literal translation 'of the rectangles', because Wallis goes on to compare these 'rectangles' with squares or other geometrical figures.

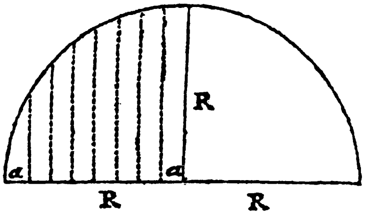
$$\begin{array}{ll} & (R-a)^3 = R^3 - 3aR^2 + 3a^2R - a^3 \\ \text{times} & \underline{(R+a)^3 = R^3 + 3aR^2 + 3a^2R + a^3} \\ \text{makes} & \qquad R^6 - 3a^2R^4 + 3a^4R^2 - a^6 \\ \text{etc.} & \end{array}$$

and so for each term of any power, as may be shown by multiplication.

PROPOSITION 121

Corollary

Therefore the circle to the square of the diameter (or also any ellipse to its circumscribed parallelogram) will have the same ratio as a series of square roots of differences, of an infinite series of equals reduced by a series of second powers, to that same series of equals.



For if the radius of the circle is taken to be R (of which an infinitely small part is $R/\infty = a$) and on it stand an infinite number of perpendiculars, or right sines, filling the quadrant of the circle, those perpendiculars are the mean proportionals between the segments of the diameter (as is well known), that is:

between	$R+0$	$R+1a$	$R+2a$	$R+3a$ etc.
and	$R-0$	$R-1a$	$R-2a$	$R-3a$ etc.
whose product is	R^2-00	R^2-1a^2	R^2-4a^2	R^2-9a^2 etc.
the mean proportionals are:	$\surd(R^2-00)$	$\surd(R^2-1a^2)$	$\surd(R^2-4a^2)$	$\surd(R^2-9a^2)$ etc.

Therefore the ratio of the sum of those square roots, to the same number of terms equal to the greatest (that is, the radius), is that of the quadrant of the circle (consisting of the former) to the square of the radius (consisting of the latter). And therefore also of the whole circle to the square of the diameter. Which was to be shown.

And the same may be easily shown of any ellipse (with appropriate changes) since its ordinates are also mean proportionals (between segments of the transverse diameter). Proportionals, and indeed sometimes equals, as is known from the teaching on conics.

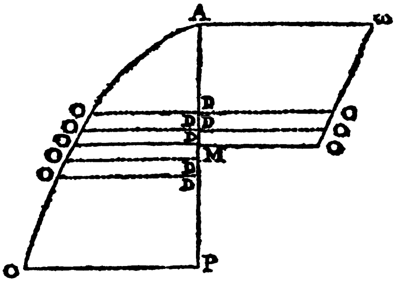
COMMENT

Moreover, the ratio this proposition points to (that is the circle to the square of its diameter) is that of 1 to the number intermediate between 1 and $\frac{3}{2}$ in the second sequence across in the table of Proposition 127. And the method of finding that number (or any other interpolated between the numbers of any such series) is to be investigated from here on.

PROPOSITION 122

Corollary

And hence, if we take an infinite number of lines of any half parabola, set perpendicular one by one to the lines of its continuation, placed in inverse position to the same height, the solid that arises, consisting of an infinite number of those rectangles (or of squares equal to those rectangles) will be to the parallelepiped on the same base and of equal height, as the circle to the square of the diameter. (And indeed, the mean proportionals will be as the square roots of the ordinates of the circle or ellipse.)



Suppose the line MO (parallel to the base) cuts any half parabola APO into two segments of equal height, and let the length of the line MO be $\surd R$. The remaining ordinates in the upper segment, ascending, will be $\surd(R-a)$, $\surd(R-2a)$, $\surd(R-3a)$, etc. and in the lower segment, descending, will be $\surd(R+a)$, $\surd(R+2a)$, $\surd(R+3a)$, etc. (since the squares of the ordinates of a parabola are in arithmetic proportion). Therefore if we suppose that the half parabola thus divided is replicated, so that point P coincides with point A, and the whole segment MPO is transferred to the position $MA\omega$ (so that the ordinates of the lower segment correspond to the ordinates of the upper segment the other way round) the rectangles ODo, ODo, etc. will be $\surd(R^2-0)$, $\surd(R^2-a^2)$, $\surd(R^2-4a^2)$, $\surd(R^2-9a^2)$, etc. as will be clear by multiplication:

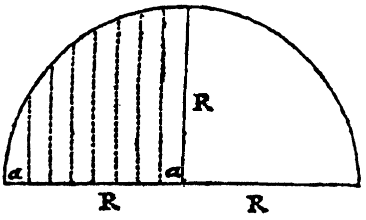
$$\begin{array}{ccccc}
\surd(R-0) & \surd(R-1a) & \surd(R-2a) & \surd(R-3a) & \text{etc.} \\
\surd(R+0) & \surd(R+1a) & \surd(R+2a) & \surd(R+3a) & \text{etc.} \\
\hline
\surd(R^2-0) & \surd(R^2-1a^2) & \surd(R^2-4a^2) & \surd(R^2-9a^2) & \text{etc.}
\end{array}$$

Therefore the sum of all these, to the greatest ($\surd(R^2 - 0) = \surd R^2 = R$) taken together (that is, the proposed solid to a parallelepiped of the same base and height) is as the circle to the square of the diameter, by what has gone before. And therefore also the mean proportionals will be as the square roots of the ordinates in the circle or ellipse, as is clear.

PROPOSITION 123

Corollary

In the same way, a sphere (or spheroid or elliptic pyramid) to a circumscribed cylinder (or prism) is as an infinite series of equals reduced by a series of second powers, to a series of the same number of terms equal to the greatest. That is as 2 to 3.

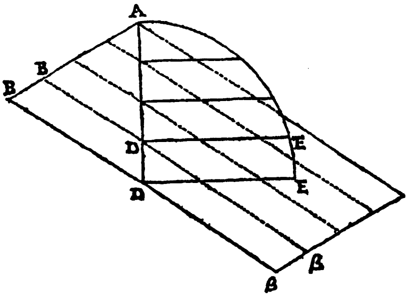


Follows from Proposition 121. For if the mean proportionals of the segments of the diameter, filling the quadrant of the circle (or ellipse), are now assumed to become the same number of radii of other circles parallel to each other, filling a hemisphere (or hemispheroid), (or the similarly placed lines of any similar planes constituting half an elliptic pyramidoid), those circles (or planes) will be as the squares of their radii (or of the similarly placed lines). That is, as $R^2 - 0, R^2 - a^2, R^2 - 4a^2$, $R^2 - 9a^2$, etc. (for those lines are $\surd(R^2 - 0), \surd(R^2 - a^2), \surd(R^2 - 4a^2), \surd(R^2 - 9a^2)$, etc. by Proposition 121). Therefore the sum of all these to the sum of all equal to the greatest, is as 2 to 3 by Proposition 118.

PROPOSITION 124

Corollary

In the same way, if the lines of a triangle *ADB* are set perpendicular one by one to the lines of a trapezium $AD\beta$ (of equal altitude and, with the triangle itself, completing the parallelogram), the rectangles produced will be equal to the same number of similar planes of an elliptic conoid (or pyramidoid). And the mean proportionals *DE*, *DE*, etc. will be ordinates in the (circle or) ellipse.



The demonstration appears easily from what has been said at Propositions 121 and 123. For the segments of the line $B\beta$ in this figure amount to the same thing as the segments of the diameter in that.

If, moreover, the lines AD, DB, are equal, and the lines AD, DE are perpendicular to each other, those lines DE, DE, will be ordinates of a circle. But if [AD is] less [than DB] then certainly an ellipse. The portion AE, moreover, whether of a circle or ellipse, is greater or less than the quadrant according to whether DB is greater than or less than $D\beta$.

PROPOSITION 125

Theorem

If there is proposed a series of equals reduced by a series of third powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of those.

Series	Squares	Cubes
$R^3 - 000$	$R^6 - 00R^3 + 00$	$R^9 - 00R^6 + 00R^3 - 00$
$R^3 - a^3$	$R^6 - 2a^3R^3 + a^6$	$R^9 - 3a^3R^6 + 3a^6R^3 - a^9$
$R^3 - b^3$	$R^6 - 2b^3R^3 + b^6$	$R^9 - 3b^3R^6 + 3b^6R^3 - b^9$
$R^3 - c^3$	$R^6 - 2c^3R^3 + c^6$	$R^9 - 3c^3R^6 + 3c^6R^3 - c^9$
etc. to		
$R^3 - R^3$	$R^6 - 2R^3R^3 + R^6$	$R^9 - 3R^3R^6 + 3R^6R^3 - R^9$
Sum:		
$AR^3 - \frac{1}{4}AR^3$	$AR^6 - \frac{2}{4}AR^6 + \frac{1}{7}AR^6$	$AR^9 - \frac{3}{4}AR^6 + \frac{3}{7}AR^6 - \frac{1}{10}AR^9$
that is:		
$1 - \frac{1}{4} = \frac{3}{4}$	$1 - \frac{2}{4} + \frac{1}{7} = \frac{18}{28}$	$1 - \frac{3}{4} + \frac{3}{7} - \frac{1}{10} = \frac{162}{280}$
or:		
$\frac{3}{4}$	$\frac{3 \times 6}{4 \times 7}$	$\frac{3 \times 6 \times 9}{4 \times 7 \times 10}$

And so on, by continually multiplying numbers in arithmetic proportion (as far as the degree of the power requires), from 3 and 4, continually increasing by threes.

PROPOSITION 126

Theorem

In the same way, if there is proposed a series of equals reduced by a series of fourth powers, fifth powers, sixth powers, etc. the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of them.

Thus:

$$1-\tfrac{1}{5}=\tfrac{4}{5}$$
$$1-\tfrac{2}{5}+\tfrac{1}{9}=\tfrac{32}{45}$$
$$1-\tfrac{3}{5}+\tfrac{3}{9}-\tfrac{1}{13}=\tfrac{384}{585}$$
$$1-\tfrac{4}{5}+\tfrac{8}{9}-\tfrac{4}{13}+\tfrac{1}{17}=\tfrac{6144}{9945}$$

or

$$\frac{4}{5},\quad \frac{4\times 8}{5\times 9},\quad \frac{4\times 8\times 12}{5\times 9\times 13},\quad \frac{4\times 8\times 12\times 16}{5\times 9\times 13\times 17}$$

In the same way:

$$1-\tfrac{1}{6}=\tfrac{5}{6}$$
$$1-\tfrac{2}{6}+\tfrac{1}{11}=\tfrac{50}{66}$$
$$1-\tfrac{3}{6}+\tfrac{3}{11}-\tfrac{1}{16}=\tfrac{750}{1056}$$
$$1-\tfrac{4}{6}+\tfrac{6}{11}-\tfrac{4}{16}+\tfrac{1}{21}=\tfrac{15000}{22176}$$

or

$$\frac{5}{6},\quad \frac{5\times 10}{6\times 11},\quad \frac{5\times 10\times 15}{6\times 11\times 16},\quad \frac{5\times 10\times 15\times 25}{6\times 11\times 16\times 21}$$

And so in any others you please; that is, by continual multiplication of numbers in arithmetic proportion (as far as the degree of the power requires), from 4 and 5, or 5 and 6, or 6 and 7, etc. continually increasing by fours, fives, sixes, etc. (according to the index of the subtracted series). As will be clear by induction. In this way:

A series of equals reduced by a series of	Ratio to a series of terms equal to the greatest				
	Differences	Squares	Cubes	Biquadrates	Supersolids
First powers	$\frac{1}{2}$·	$\frac{1}{2}\times\frac{2}{3}=\frac{2}{6}$·	$\frac{2}{6}\times\frac{3}{4}=\frac{6}{24}$·	$\frac{6}{24}\times\frac{4}{5}=\frac{24}{120}$·	$\frac{24}{120}\times\frac{5}{6}=\frac{120}{720}$·
Second powers	$\frac{2}{3}$·	$\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$·	$\frac{8}{15}\times\frac{6}{7}=\frac{48}{105}$·	$\frac{48}{105}\times\frac{8}{9}=\frac{384}{945}$·	$\frac{384}{945}\times\frac{10}{11}=\frac{3840}{10395}$·
Third powers	$\frac{3}{4}$·	$\frac{3}{4}\times\frac{6}{7}=\frac{18}{28}$·	$\frac{18}{28}\times\frac{9}{10}=\frac{162}{280}$·	$\frac{162}{280}\times\frac{12}{13}=\frac{1944}{3640}$·	$\frac{1944}{3640}\times\frac{15}{16}=\frac{29160}{58240}$·
Fourth powers	$\frac{4}{5}$·	$\frac{4}{5}\times\frac{8}{9}=\frac{32}{45}$·	$\frac{32}{45}\times\frac{12}{13}=\frac{384}{585}$·	$\frac{384}{585}\times\frac{16}{17}=\frac{6144}{9945}$·	$\frac{6144}{9945}\times\frac{20}{21}=\frac{122880}{208845}$·
Fifth powers	$\frac{5}{6}$·	$\frac{5}{6}\times\frac{10}{11}=\frac{50}{66}$·	$\frac{50}{66}\times\frac{15}{16}=\frac{750}{1056}$·	$\frac{750}{1056}\times\frac{20}{21}=\frac{15000}{22176}$·	$\frac{15000}{22176}\times\frac{25}{26}=\frac{275000}{576576}$·

And so on

And so on

And so on.

That is, if the index of the reduced series is denoted by a, its ratio to a series of terms equal to the greatest, thus:

	Reduced series	Squares	Cubes
have the ratio	$\frac{a}{a+1}$	$\frac{a}{a+1}\times\frac{2a}{2a+1}$	$\frac{a}{a+1}\times\frac{2a}{2a+1}\times\frac{3a}{3a+1}$ etc.
or	$\frac{a}{a+1}$	$\frac{2a^2}{2a^2+3a+1}$	$\frac{6a^3}{6a^3+11a^2+6a+1}$ etc.

to unity, or that unity has to

	$\frac{a+1}{a}$	$\frac{a+1}{a}\times\frac{2a+1}{2a}$	$\frac{a+1}{a}\times\frac{2a+1}{2a}\times\frac{3a+1}{3a}$ etc.
or	$\frac{a+1}{a}$	$\frac{2a^2+3a+1}{2a^2}$	$\frac{6a^3+11a^2+6a+1}{6a^3}$ etc.

And this same will hold if the reduced series is a series of roots.

For example, if from a series of equals there is taken a series of second roots with index $\frac{1}{2}$. For if one puts $a=\frac{1}{2}$,

$$\text{then } \frac{a+1}{a}=3$$

$$\text{and } \frac{a+1}{a}\times\frac{2a+1}{2a}=3\times 2=6$$

$$\text{and } \frac{a+1}{a}\times\frac{2a+1}{2a}\times\frac{3a+1}{3a}=3\times 2\times 1\tfrac{2}{3}=10 \text{ etc.}$$

Moreover, in this kind of subtraction, the [sums of] differences, squares, cubes, etc. are to a series of terms equal to the greatest as 1 to 3, 6, 10, etc.

Similarly, if there is taken away a series of fourth roots with index $\frac{1}{4}$, then $a=\frac{1}{4}$, and

$$\frac{a+1}{a}=5, \quad \frac{2a+1}{2a}=3, \quad \frac{3a+1}{3a}=2\frac{1}{3}, \quad \frac{4a+1}{4a}=2 \text{ etc.}$$

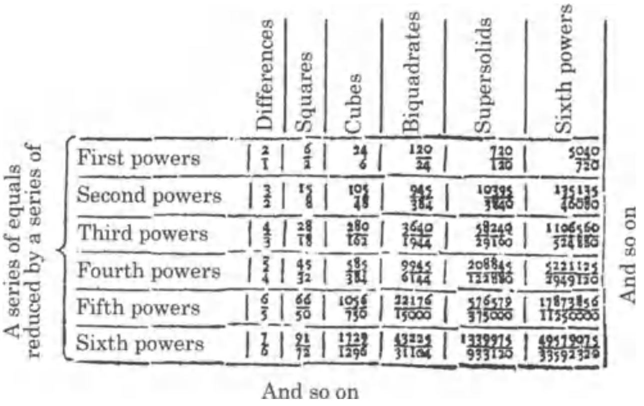
And $5\times 3=15$, $15\times 2\frac{1}{3}=35$, $35\times 2=70$, etc. Moreover, in this kind of subtraction, the series of differences, squares, cubes, biquadrates, etc. are as 1 to 5, 15, 35, 70, etc. And similarly in others of this kind, as will also be shown further below.

Meanwhile one may put some of the preceding propositions together in a table, adjoined to the following proposition. That is:

PROPOSITION 127

Theorem

If there is proposed an (infinite) series of equals reduced by a (similar) series of first powers, second powers, third powers, etc. [the sums of] the differences themselves, and [of] their squares, cubes, etc. will have ratios to the proposed series of equals as 1 to the numbers indicated in the adjoined Table. That is:



A series of equals reduced by a series of

	Differences	Squares	Cubes	Biquadrates	Supersolids	Sixth powers
First powers	$\frac{2}{1}$	$\frac{6}{2}$	$\frac{24}{6}$	$\frac{120}{24}$	$\frac{720}{120}$	$\frac{5040}{720}$
Second powers	$\frac{3}{2}$	$\frac{15}{8}$	$\frac{105}{48}$	$\frac{945}{384}$	$\frac{10395}{3840}$	$\frac{135135}{46080}$
Third powers	$\frac{4}{3}$	$\frac{28}{18}$	$\frac{280}{162}$	$\frac{3640}{1944}$	$\frac{58249}{29160}$	$\frac{1106560}{524880}$
Fourth powers	$\frac{5}{4}$	$\frac{45}{32}$	$\frac{585}{384}$	$\frac{9945}{6144}$	$\frac{208845}{122880}$	$\frac{5221125}{2949120}$
Fifth powers	$\frac{6}{5}$	$\frac{66}{50}$	$\frac{1056}{750}$	$\frac{22176}{15000}$	$\frac{576579}{375000}$	$\frac{17873856}{11250000}$
Sixth powers	$\frac{7}{6}$	$\frac{91}{72}$	$\frac{1729}{1296}$	$\frac{43225}{31104}$	$\frac{1339975}{933120}$	$\frac{49579075}{33592320}$

And so on

And so on

Follows from what has gone before.

COMMENT

Truly it may be investigated in the same way, what are the ratios of series of apotomes[51] of square roots, cube roots, etc. to a series of the same number of terms equal to the greatest of them. The work may be done as the need arises. For nothing else is lacking for the quadrature of the circle and ellipse. As is already clear from Proposition 121, and as will further be clear from various propositions following.

PROPOSITION 128

Theorem

If there is proposed a series of equals reduced by a series of second roots, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of them.

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[51] *Apotomes* are quantities of the form $\sqrt{a} - \sqrt{b}$.

That is:

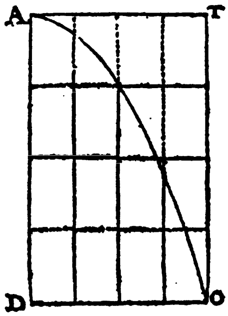
Series	Squares	Cubes
$\sqrt{R}-\sqrt{a}$	$R-2\sqrt{aR}+a$	$R\sqrt{R}-3R\sqrt{a}+3a\sqrt{R}-a\sqrt{a}$
$\sqrt{R}-\sqrt{b}$	$R-2\sqrt{bR}+b$	$R\sqrt{R}-3R\sqrt{b}+3b\sqrt{R}-b\sqrt{b}$
$\sqrt{R}-\sqrt{c}$	$R-2\sqrt{cR}+c$	$R\sqrt{R}-3R\sqrt{c}+3c\sqrt{R}-c\sqrt{c}$
etc. to		
$\sqrt{R}-\sqrt{R}$	$R-2\sqrt{RR}+R$	$R\sqrt{R}-3R\sqrt{R}+3R\sqrt{R}-R\sqrt{R}$
$A\sqrt{R}-\frac{2}{3}A\sqrt{R}$	$AR-\frac{4}{3}AR+\frac{1}{2}AR$	$AR\sqrt{R}-\frac{6}{3}AR\sqrt{R}+\frac{3}{2}AR\sqrt{R}-\frac{2}{5}AR\sqrt{R}$
$1-\frac{2}{3}=\frac{1}{3}$	$1-\frac{4}{3}+\frac{1}{2}=\frac{1}{6}$	$1-\frac{6}{3}+\frac{3}{2}-\frac{2}{5}=\frac{1}{12}$
$\frac{1}{1+2}$	$\frac{1}{1+2+3}$	$\frac{1}{1+2+3+4}$

And so on; that is, the ratio of 1 to the triangular numbers, or to a sum of numbers in arithmetic proportion from 1 continually increasing by one (as far as the degree of the power requires).

PROPOSITION 129

Corollary

And therefore the conoid (or pyramidoid) generated by the complement of a half parabola around one of its ordinates is to a cylinder (or prism) of the same base and height as 1 to 6.



That is, as the squares of differences of a series of equals reduced by a series of second roots, to the same number of terms equal to the greatest. For since in the complement of a half parabola *AOT*, the lines parallel to its diameter *AT* are differences of equals reduced by second roots (the ordinates of the half parabola *AOD*), if that complement *AOT* is turned around *TO* itself as axis (or also around others, as has been said elsewhere) there is formed a conoid (or by analogy a pyramid) with vertex *O*; and the circles described by such turning (or similar planes in similar positions) will be as the squares of those lines (parallel to *AT*). That is, as squares of differences, of a series of equals reduced by second roots; and therefore as 1 to 6 by what has gone before.

COMMENT

And it may be considered in the same way for cones and pyramids generated by the complement of any higher parabola about one of its ordinates, according to the following propositions. That is, with appropriate changes, as the degree of the parabola will require. Thus for a cubical parabola as 1 to 10, for a biquadratic parabola as 1 to 15, for a supersolid parabola as 1 to 21, etc. according to the table in Proposition 131.

PROPOSITION 130

Theorem

If there is proposed a series of equals reduced by a series of third roots, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of them. That is:

Series	Squares	Cubes
$R-\sqrt[3]{a}$	$\sqrt[3]{R^2}-2\sqrt[3]{aR}+\sqrt[3]{a^2}$	$\sqrt[3]{R^3}-3\sqrt[3]{aR^2}+3\sqrt[3]{a^2R}-\sqrt[3]{a^3}$
$R-\sqrt[3]{b}$	$\sqrt[3]{R^2}-2\sqrt[3]{bR}+\sqrt[3]{b^2}$	$\sqrt[3]{R^3}-3\sqrt[3]{bR^2}+3\sqrt[3]{b^2R}-\sqrt[3]{b^3}$
$R-\sqrt[3]{c}$	$\sqrt[3]{R^2}-2\sqrt[3]{cR}+\sqrt[3]{c^2}$	$\sqrt[3]{R^3}-3\sqrt[3]{cR^2}+3\sqrt[3]{c^2R}-\sqrt[3]{c^3}$
etc. to		
$R-\sqrt[3]{R}$	$\sqrt[3]{R^2}-2\sqrt[3]{RR}+\sqrt[3]{R^2}$	$\sqrt[3]{R^3}-3\sqrt[3]{RR^2}+3\sqrt[3]{R^2R}-\sqrt[3]{R^3}$
$A\sqrt[3]{R}-\frac{3}{4}A\sqrt[3]{R}$	$A\sqrt[3]{R^2}-\frac{6}{4}A\sqrt[3]{R^2}+\frac{3}{5}A\sqrt[3]{R^2}$	$AR-\frac{9}{4}AR+\frac{9}{5}AR-\frac{1}{2}AR$
$1-\frac{3}{4}=\frac{1}{4}$	$1-\frac{6}{4}+\frac{3}{5}=\frac{1}{10}$	$1-\frac{9}{4}+\frac{9}{5}-\frac{1}{2}=\frac{1}{20}$
or:		
$\frac{1}{1+3=4}$	$\frac{1}{4+6=10}$	$\frac{1}{10+10=20}$

And so on; by continually adding triangular numbers, or sums of arithmetic proportionals, there may be had the denominator of a ratio in which the numerator is 1.

Ratio to a series of terms equal to the greatest

A series of equals reduced by a series of

	Differences	Squares	Cubes	Biquadrates	Supersolids	Sixth powers
First roots	$\frac{1}{1+1}=\frac{1}{2}$.	$\frac{1}{2+1}=\frac{1}{3}$.	$\frac{1}{3+1}=\frac{1}{4}$.	$\frac{1}{4+1}=\frac{1}{5}$.	$\frac{1}{5+1}=\frac{1}{6}$.	$\frac{1}{6+1}=\frac{1}{7}$.
Second roots	$\frac{1}{1+2}=\frac{1}{3}$.	$\frac{1}{3+3}=\frac{1}{6}$.	$\frac{1}{6+4}=\frac{1}{10}$.	$\frac{1}{10+5}=\frac{1}{15}$.	$\frac{1}{15+6}=\frac{1}{21}$.	$\frac{1}{21+7}=\frac{1}{28}$.
Third roots	$\frac{1}{1+3}=\frac{1}{4}$.	$\frac{1}{4+6}=\frac{1}{10}$.	$\frac{1}{10+10}=\frac{1}{20}$.	$\frac{1}{20+15}=\frac{1}{35}$.	$\frac{1}{35+21}=\frac{1}{56}$.	$\frac{1}{56+28}=\frac{1}{84}$.
Fourth roots	$\frac{1}{1+4}=\frac{1}{5}$.	$\frac{1}{5+10}=\frac{1}{15}$.	$\frac{1}{15+20}=\frac{1}{35}$.	$\frac{1}{35+35}=\frac{1}{70}$.	$\frac{1}{70+56}=\frac{1}{126}$.	$\frac{1}{126+84}=\frac{1}{210}$.
Fifth roots	$\frac{1}{1+5}=\frac{1}{6}$.	$\frac{1}{6+15}=\frac{1}{21}$.	$\frac{1}{21+35}=\frac{1}{56}$.	$\frac{1}{56+70}=\frac{1}{126}$,	$\frac{1}{126+126}=\frac{1}{252}$.	$\frac{1}{252+210}=\frac{1}{462}$.
Sixth roots	$\frac{1}{1+6}=\frac{1}{7}$.	$\frac{1}{7+21}=\frac{1}{28}$.	$\frac{1}{28+56}=\frac{1}{84}$.	$\frac{1}{84+126}=\frac{1}{210}$,	$\frac{1}{210+252}=\frac{1}{462}$.	$\frac{1}{462+462}=\frac{1}{924}$,

And so on

And so on

PROPOSITION 131

Theorem

In the same way, if there is proposed a series of equals reduced by a series of fourth roots, fifth roots, etc. the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of the same number of terms equal to the greatest of them.

For as in the subtraction of a series of second roots, the denominators of the ratios arise by continual addition of numbers in arithmetic proportion, $1 + 2 = 3$, $1 + 2 + 3 = 3 + 3 = 6$, $1 + 2 + 3 + 4 = 6 + 4 = 10$, $1 + 2 + 3 + 4 + 5 = 10 + 5 = 15$, $1 + 2 + 3 + 4 + 5 + 6 = 15 + 6 = 21$, etc. so in the subtraction of third roots, the denominators arise by continual addition of those numbers (1, 3, 6, 10, 15, 21, etc.) found in the method for subtraction of second roots, that is, $1 + 3 = 4$, $4 + 6 = 10$, $10 + 10 = 20$, $20 + 15 = 35$, $35 + 21 = 56$, etc., or $1 + 1 + 2 = 1 + 3 = 4$, $4 + 1 + 2 + 3 = 1 + 3 + 6 = 10$, $10 + 1 + 2 + 3 + 4 = 1 + 3 + 6 + 10 = 20$, $20 + 1 + 2 + 3 + 4 + 5 = 1 + 3 + 6 + 10 + 15 = 35$, $35 + 1 + 2 + 3 + 4 + 5 + 6 = 1 + 3 + 6 + 10 + 15 + 21 = 56$, etc. Thence from the numbers already found (1, 4, 10, 20, 35, 56, etc.) by continual addition, there arise the denominators of the ratios for the subtraction of fourth roots (that is, $1 + 4 = 5$, $5 + 10 = 15$, $15 + 20 = 35$, $35 + 56 = 91$, etc.). And from these again by continual addition, there arise the denominators of the ratios for subtraction of the next series (fifth roots). And so on, by this method.

COMMENT

And here we have met on the way an unexpected investigation of figurate numbers (as they are usually called). For all the numbers (here and in the following tables) made by this kind of addition are figurate numbers, that is, laterals, triangular numbers, pyramidal numbers, etc. Which, since it is obvious to anyone, it is sufficient to have pointed out.

It is also evident (in either table) that the sequences of numbers thus found are just the same horizontally as vertically.

Moreover, from what has been said it is possible to bring together a summary of some of the preceding propositions (that is, concerning series of equals reduced by series of roots) in one table, which I adjoin to the next proposition. That is:

PROPOSITION 132

Theorem

If there is proposed an infinite series of equals reduced by a similar series of first powers (or, if one likes, first roots, which amounts to the same thing), second roots, third roots, etc. then [the sums of] the differences themselves, and [of] their squares, cubes, biquadrates, etc. will have ratios to the corresponding series of equals as 1 to the numbers indicated in the adjoined table. That is:

A series of equals reduced by a series of

	Equals	Differences	Squares	Cubes	Biquadrates	Supersolids	Sixth powers	Seventh powers	Eighth powers	Ninth powers	Tenth powers
Nulls	1	1	1	1	1	1	1	1	1	1	1
First roots	1	2	3	4	5	6	7	8	9	10	11
Second roots	1	3	6	10	15	21	28	36	45	55	66
Third roots	1	4	10	20	35	56	84	120	165	220	286
Fourth roots	1	5	15	35	70	126	210	330	495	715	1001
Fifth roots	1	6	21	56	126	252	462	792	1287	2002	3003
Sixth roots	1	7	28	84	210	462	924	1716	3003	5005	8008
Seventh roots	1	8	36	120	330	792	1716	3432	6435	11440	19448
Eighth roots	1	9	45	165	495	1287	3003	6435	12870	24310	43758
Ninth roots	1	10	55	220	715	2002	5005	11440	24310	48620	92378
Tenth roots	1	11	66	286	1001	3003	8008	19448	43758	92378	184756

And so on

And so on

Follows from what has gone before. Moreover, any intermediate number in the table is the sum of two next to it, one from above, the other moved to the right.

It must also be noted that the same ratio is produced for squares of differences if there are taken third roots, as for cubes of differences if there are taken second roots; the same for sixth powers if there are taken seventh roots, and for seventh powers if there are taken sixth roots; and so everywhere, as though by reciprocation of powers, as is clear from inspection of the table.

But other similarities also sometimes happen, thus for supersolids if there are taken first roots, and for the differences themselves if there are taken fifth roots, but also for squares if there are taken second roots. The same for ninth powers if there are taken first roots, and for the differences themselves if there are taken ninth roots, but also for cubes if there are taken second roots, and for squares if there are taken third roots. The same for eighth powers if there are taken sixth roots, and for sixth powers if there are taken eighth roots, but also for twelfth powers if there are taken fifth roots, and for fifth powers if there are taken twelfth roots; and so on elsewhere as is clear from the table.

PROPOSITION 133

Theorem

If there is proposed a series of first powers reduced by a series of second powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of equals. Thus:

Series	Squares	Cubes
$aD^2 - a^2$	$a^2D^2 - 2a^3D + a^4$	$a^3D^3 - 3a^4D^2 + 3a^5D - a^6$
$bD^2 - b^2$	$b^2D^2 - 2b^3D + b^4$	$b^3D^3 - 3b^4D^2 + 3b^5D - b^6$
$cD^2 - c^2$	$c^2D^2 - 2c^3D + c^4$	$c^3D^3 - 3c^4D^2 + 3c^5D - c^6$
etc. to		
$DD - D^2$	$D^2D^2 - 2D^3D + D^4$	$D^3D^3 - 3D^4D^2 + 3D^5D - D^6$
$\frac{1}{2}AD^2 - \frac{1}{3}AD^2$	$\frac{1}{3}AD^4 - \frac{2}{4}AD^4 + \frac{1}{5}AD^4$	$\frac{1}{4}AD^6 - \frac{3}{5}AD^6 + \frac{3}{6}AD^6 - \frac{1}{7}AD^6$
$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}AD^2$	$\frac{1}{3} - \frac{2}{4} + \frac{1}{5} = \frac{1}{30}AD^4$	$\frac{1}{4} - \frac{3}{5} + \frac{3}{6} - \frac{1}{7} = \frac{41}{140}AD^6$
$\frac{1}{2 \times 3} = \frac{1}{6}$	$\frac{1 \times 2}{3 \times 4 \times 5} = \frac{2}{60} = \frac{1}{30}$	$\frac{1 \times 2 \times 3}{4 \times 5 \times 6 \times 7} = \frac{6}{840} = \frac{1}{140}$
$\frac{1}{2 \times 3} = \frac{1}{6}$	$\frac{1}{2 \times 3} \times \frac{4}{4 \times 5} = \frac{1}{30}$	$\frac{1}{2 \times 3} \times \frac{4}{4 \times 5} \times \frac{9}{6 \times 7} = \frac{1}{140}$

And so on, by continually multiplying the numerators by square numbers, and the denominators by pairs of consecutive arithmetic proportionals.

PROPOSITION 134

Corollary

Therefore if a series of equals reduced by a series of first powers is multiplied term by term by a series of first powers, the sum of the rectangles[52] (or of squares or any similar figures, equal or even proportional to them) will have a known ratio to the sum of the same number of equals.

And the same happens if the squares of the former series are multiplied by the squares of the latter, the cubes of the former by the cubes Of the latter, etc.

That is, they produce the same ratios as in the preceding proposition. As if to say:

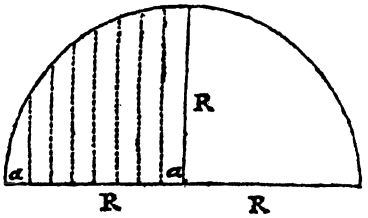
$$\begin{array}{lrrr}
 & D-a & (D-a)^2=D^2-2aD+a^2 & (D-a)^3=D^3-3aD^2+3a^2D-a^3 \\
\text{times} & a & a^2 & a^3 \\
\cline{2-4}
\text{makes} & aD-a^2 & a^2D^2-2a^3D+a^4 & a^3D^3-3a^4D^2+3a^5D-a^6
\end{array}$$

etc.

PROPOSITION 135

Corollary

Therefore the semicircle to the square of its diameter (or also the semi-ellipse to the parallelogram circumscribing the ellipse) has the same ratio as the square roots of the differences, of a series of first powers reduced by a series of second powers, to a series of terms equal to the greatest of those first powers. Therefore the complete circle (or ellipse) to that square (or parallelogram) will have twice that ratio.



For if the diameter of the circle (or ellipse) is taken to be D (of which an infinitely small part is $D/\infty = a$), and its ordinates an infinite number of lines (equally spaced) filling the semicircle (or semi-ellipse), they will be (as is known) mean proportionals

---

[52] *Rectangulorum*, or 'products', as in Proposition 120.

(or at least for the ellipse, proportional to those mean proportionals) between the segments of the diameter. Thus:

between	a	$2a$	$3a$	$4a$
and	$D-a$	$D-2a$	$D-3a$	$D-4a$
therefore	$\surd(aD-a^2)$	$\surd(2aD-4a^2)$	$(3aD-9a^2)$	$\surd(4aD-16a^2)$
or	$\surd(aD-a^2)$	$\surd(bD-b^2)$	$\surd(cD-c^2)$	$(dD-d^2)$

And therefore the sum of all, that is the semicircle (or semi-ellipse), to the same number of terms equal to $\surd D^2$ itself, thus, to the square of the diameter (or at least the diameter multiplied by the altitude) is

as $\surd(aD-a^2)+\surd(bD-b^2)+\surd(cD-c^2)+$ etc. as far as $\surd(DD-D^2)$
to $\surd D^2+\surd D^2+\surd D^2+$ etc. $=A\surd D^2=AD$

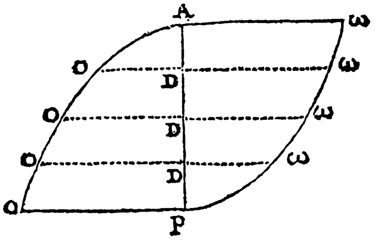
Therefore the complete circle to that same square as

$$2\surd(aD-a^2) \text{ etc. to } AD$$

PROPOSITION 136

Corollary

And hence if we take the infinite lines (the ordinates) of any half parabola set perpendicular one by one to the lines of the same in inverse position, the solid that arises, consisting of an infinite number of those rectangles (or of squares, or indeed other similar figures, equal to those rectangles) will be, to the corresponding parallelepiped of equal height (that is, whose base is equal to the square of the base of the half parabola), as the semicircle to the square of its diameter. (And indeed, the mean proportionals will be as the square roots of the ordinates of the circle or ellipse.)



Let that same parabola be APO in normal position and $PA\omega$ in inverse position. Therefore (by the nature of the parabola) the squares of the ordinates (that is, the lines DO, DO, etc. decreasing, or $D\omega$, $D\omega$, etc. increasing) will be an infinite series of first powers, thus, a, $2a, 3a$, etc. or in their place a, b, c, etc. of which the greatest may be called D (that is, the square of the base PO or $A\omega$). And therefore in inverse position they will be $D-a$, $D-2a$, $D-3a$, etc. or also $D-a$, $D-b$, $D-c$, etc.

(For if ordered from the least the increase between each is equal, or the decrease if ordered from the greatest.) And consequently the ordinates themselves (that is, the second roots of those squares) are in the former case $\surd a$, $\surd 2a$, $\surd 3a$, etc. or $\surd a$, $\surd b$, $\surd c$, etc. and in the latter $\surd(D-a)$, $\surd(D-2a)$, $\surd(D-3a)$, etc. or (in their place) $\surd(D-a)$, $\surd(D-b)$, $\surd(D-c)$, etc. Therefore setting the latter perpendicular to the former, there arise rectangles $Od\omega$. That is,

setting	$\surd(D-a)$	$\surd(D-b)$	$\surd(D-c)$	etc.
perpendicular to	$\surd a$	$\surd b$	$\surd c$	etc.
gives	$\surd(aD-a^2)$	$\surd(bD-b^2)$	$\surd(cD-c^2)$	etc.

Moreover, the sum of all the rectangles, to the rectangle $\surd D-0$ times $\surd D-0$, that is, $\surd D^2 = D$, taken the same number of times, that is, of the solid arising from that multiplication, to the said parallelepiped, is as the semicircle to the square of its diameter, from what has gone before.

And therefore also, the mean proportionals between corresponding lines OD, $D\omega$, will be as the square roots of the ordinates of the circle or ellipse. Since, indeed, the rectangles $Od\omega$ are proportional to those ordinates.

COMMENT

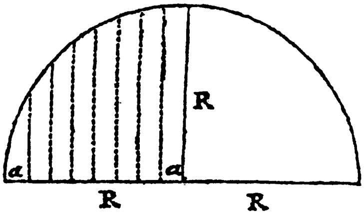
Note, however, that it is not necessary for the half parabola placed in inverse position to be exactly the same as that in normal position, for the thing succeeds no less for any two half parabolas placed in inverse position provided they are of equal altitude. In such a way, however, that if they have unequal bases, the base of the parallelepiped is not taken to be the base of either parabola squared, but equal to a rectangle of both, thus $PO \times A\omega$. Which it is sufficient to have pointed out, since the same demonstration as that preceding can also be accommodated to this, by making light changes. Whence this one also may easily be inferred.

But the figure consisting of all the mean proportionals (between OD and $D\omega$) will be elliptoid, in which, that is, the squares of the ordinates are themselves proportional to the ordinates of the ellipse, as is clear. Just as, that is, in a biquadratic parabola, the squares of the ordinates are proportional to the ordinates of the parabola. And the squares of the ordinates of the parabola are proportional to the ordinates of a triangle.

PROPOSITION 137

Corollary

In the same way, spheroids (or also elliptic conoids or pyramidoids) to a circumscribed cylinder (or prism), will have the same ratio as four times a series of first powers reduced by a series of second powers, to a series of the same number of terms equal to the greatest of the first powers. That is as 2 to 3.

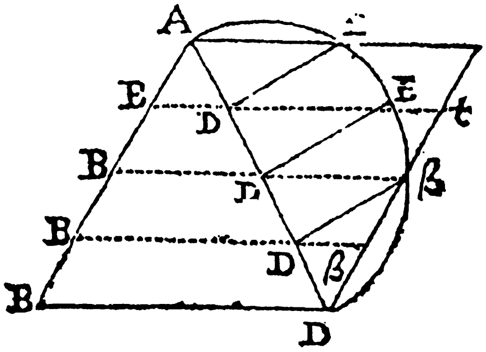


For since the lines in a circle or ellipse are twice the series $\sqrt{(aD - a^2)}$, etc. the planes in the conoid or pyramidoid will be as four times $aD - a^2$, etc. And therefore that to the circumscribed prism or cylinder as 4 to 6, or 2 to 3. By Proposition 133. Which has also been shown previously at Proposition 123.

PROPOSITION 138

Corollary

In the same way, if a parallelogram is cut by a diagonal line, and the lines of one triangle are set perpendicular to their continuations in the other, the mean proportionals will be the same number of ordinates of (either a circle or at least) an ellipse. And their squares will be the planes of a circular or elliptic (or some similar) pyramidoid or conoid.



Follows from the two preceding propositions. For the coterminous lines stand in for the segments of the diameters. And will produce either a circle or an ellipse, as may be proved from the same information as in Proposition 124.

PROPOSITION 139

Theorem

If there is proposed a series of first powers reduced by a series of third powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of equals. That is:

Differences	Squares	Cubes
$aD^2 - a^3$	$a^2D^4 - 2a^4D^2 + a^6$	$a^3D^6 - 3a^5D^4 + 3a^7D^2 - a^9$
etc. to		
$DD^2 - D^3$	$D^2D^4 - 2D^4D^2 + D^6$	$D^3D^6 - 3D^5D^4 + 3D^7D^2 - D^9$
$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}AD^3$	$\frac{1}{3} - \frac{2}{5} + \frac{1}{7} = \frac{8}{105}AD^6$	$\frac{1}{4} - \frac{3}{6} + \frac{3}{8} - \frac{1}{10} = \frac{48}{1920}AD^9$
or:		
$\frac{2}{2 \times 4}$	$\frac{2 \times 4}{3 \times 5 \times 7}$	$\frac{2 \times 4 \times 6}{4 \times 6 \times 8 \times 10}$

And so on; thus:

$$\frac{2 \times 4 \times 6 \times 8}{5 \times 7 \times 9 \times 11 \times 13}, \quad \frac{2 \times 4 \times 6 \times 8 \times 10}{6 \times 8 \times 10 \times 12 \times 14 \times 16}, \quad \frac{2 \times 4 \times 6 \times 8 \times 10 \times 12}{7 \times 9 \times 11 \times 13 \times 15 \times 17 \times 19},$$

etc.

PROPOSITION 140

Corollary

The same happens if a series of equals reduced by a series of second powers is multiplied by a series of first powers. And the squares, cubes, etc. of the former by the squares, cubes, etc. of the latter.

(Thus, if the lines of a half parabola, parallel to the diameter, are set perpendicular to the lines of a triangle. For their continuations, in the complement, are a series of second powers.)

Since, that is, $D^2 - a^2$ times a is $aD^2 - a^3$, etc.

PROPOSITION 141

Theorem

If there is proposed a series of first powers reduced by a series of fourth powers, the [sums of] squares, cubes, etc. of the differences will have known ratios to a series of equals. That is:

Differences	Squares	Cubes	
$aD^3 - a^4$	$a^2D^6 - 2a^5D^3 + a^8$	$a^3D^9 - 3a^6D^6 + 3a^9D^3 - a^{12}$	etc.
$\frac{1}{2} - \frac{1}{5} = \frac{3}{10}AD^4$	$\frac{1}{3} - \frac{2}{7} + \frac{1}{9} = \frac{1}{9}AD^8$	$\frac{1}{4} - \frac{3}{7} + \frac{3}{10} - \frac{1}{13} = \frac{81}{1820}AD^{12}$	
or:			
$\frac{3}{2 \times 5}$	$\frac{3 \times 6}{3 \times 6 \times 9}$	$\frac{3 \times 6 \times 9}{4 \times 7 \times 10 \times 13}$	
And so on, thus:	$\frac{3 \times 6 \times 9 \times 12}{5 \times 8 \times 11 \times 14 \times 17}$,	$\frac{3 \times 6 \times 9 \times 12 \times 15}{6 \times 9 \times 12 \times 15 \times 18 \times 21}$,	etc.

PROPOSITION 142

Corollary

The same holds if a series of equals reduced by a series of third powers is multiplied by a series of first powers.

(Thus, if the lines parallel to the diameter in a cubical parabola are set perpendicular to those of an inscribed triangle. For their continuations in the complement are a series of third powers. And similarly, with apropriate changes, in other propositions.)

Since, that is, $D^3 - a^3$ times a is $aD^3 - a^4$.

COMMENT

And it may be similarly considered, with appropriate changes, in any other cases whatever, where a series of this kind is composed from two or more other series multiplied by each other. As is clear.

PROPOSITION 143

Theorem

Equally, if there is proposed a series of first powers reduced by a series of fifth powers, sixth powers, etc. the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of equals.

Thus $\frac{4}{2 \times 6}$, $\frac{4 \times 8}{3 \times 7 \times 11}$, $\frac{4 \times 8 \times 12}{4 \times 8 \times 12 \times 16}$, $\frac{4 \times 8 \times 12 \times 16}{5 \times 9 \times 13 \times 17 \times 21}$, etc.

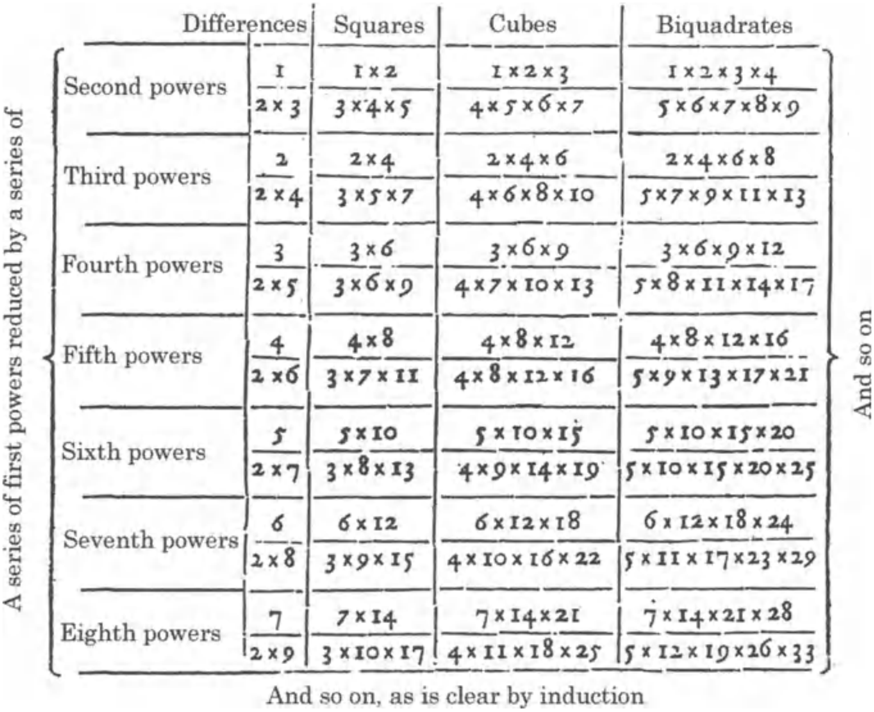
Similarly $\frac{5}{2 \times 7}$, $\frac{5 \times 10}{3 \times 8 \times 13}$, $\frac{5 \times 10 \times 15}{4 \times 9 \times 14 \times 19}$, $\frac{5 \times 10 \times 15 \times 20}{5 \times 10 \times 15 \times 20 \times 25}$, etc.

And so on, as the power of the reduced series requires. As will be clear by induction. Therefore:

PROPOSITION 144

Theorem

If there is proposed a series of first powers reduced by a series of second powers, third powers, fourth powers, etc. [the sums of] the differences themselves, and [of] their squares, cubes, biquadrates, etc. will have ratios to a series of equals, as the adjoined table indicates. Or as the numbers in the table have to 1. That is:



A series of first powers reduced by a series of

	Differences	Squares	Cubes	Biquadrates
Second powers	$\frac{1}{2\times 3}$	$\frac{1\times 2}{3\times 4\times 5}$	$\frac{1\times 2\times 3}{4\times 5\times 6\times 7}$	$\frac{1\times 2\times 3\times 4}{5\times 6\times 7\times 8\times 9}$
Third powers	$\frac{2}{2\times 4}$	$\frac{2\times 4}{3\times 5\times 7}$	$\frac{2\times 4\times 6}{4\times 6\times 8\times 10}$	$\frac{2\times 4\times 6\times 8}{5\times 7\times 9\times 11\times 13}$
Fourth powers	$\frac{3}{2\times 5}$	$\frac{3\times 6}{3\times 6\times 9}$	$\frac{3\times 6\times 9}{4\times 7\times 10\times 13}$	$\frac{3\times 6\times 9\times 12}{5\times 8\times 11\times 14\times 17}$
Fifth powers	$\frac{4}{2\times 6}$	$\frac{4\times 8}{3\times 7\times 11}$	$\frac{4\times 8\times 12}{4\times 8\times 12\times 16}$	$\frac{4\times 8\times 12\times 16}{5\times 9\times 13\times 17\times 21}$
Sixth powers	$\frac{5}{2\times 7}$	$\frac{5\times 10}{3\times 8\times 13}$	$\frac{5\times 10\times 15}{4\times 9\times 14\times 19}$	$\frac{5\times 10\times 15\times 20}{5\times 10\times 15\times 20\times 25}$
Seventh powers	$\frac{6}{2\times 8}$	$\frac{6\times 12}{3\times 9\times 15}$	$\frac{6\times 12\times 18}{4\times 10\times 16\times 22}$	$\frac{6\times 12\times 18\times 24}{5\times 11\times 17\times 23\times 29}$
Eighth powers	$\frac{7}{2\times 9}$	$\frac{7\times 14}{3\times 10\times 17}$	$\frac{7\times 14\times 21}{4\times 11\times 18\times 25}$	$\frac{7\times 14\times 21\times 28}{5\times 12\times 19\times 26\times 33}$

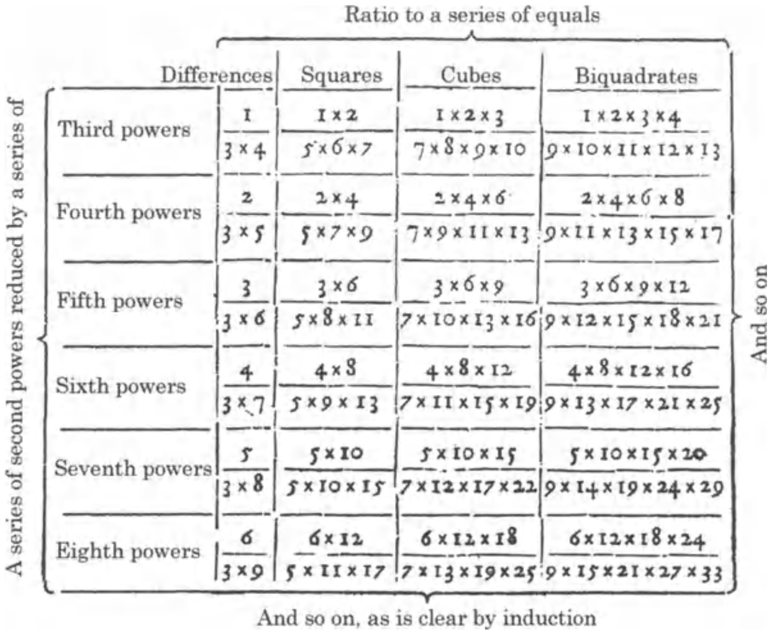
And so on

And so on, as is clear by induction

PROPOSITION 145

Theorem

Similarly, if there is proposed a series of second powers reduced by a series of third powers, fourth powers, fifth powers, sixth powers etc. [the sums of] the differences themselves, and [of] their squares, cubes, biquadrates, etc. will have ratios to a series of equals, as the adjoined table indicates. That is:



Ratio to a series of equals

A series of second powers reduced by a series of

	Differences	Squares	Cubes	Biquadrates
Third powers	$\frac{1}{3\times4}$	$\frac{1\times2}{5\times6\times7}$	$\frac{1\times2\times3}{7\times8\times9\times10}$	$\frac{1\times2\times3\times4}{9\times10\times11\times12\times13}$
Fourth powers	$\frac{2}{3\times5}$	$\frac{2\times4}{5\times7\times9}$	$\frac{2\times4\times6}{7\times9\times11\times13}$	$\frac{2\times4\times6\times8}{9\times11\times13\times15\times17}$
Fifth powers	$\frac{3}{3\times6}$	$\frac{3\times6}{5\times8\times11}$	$\frac{3\times6\times9}{7\times10\times13\times16}$	$\frac{3\times6\times9\times12}{9\times12\times15\times18\times21}$
Sixth powers	$\frac{4}{3\times7}$	$\frac{4\times8}{5\times9\times13}$	$\frac{4\times8\times12}{7\times11\times15\times19}$	$\frac{4\times8\times12\times16}{9\times13\times17\times21\times25}$
Seventh powers	$\frac{5}{3\times8}$	$\frac{5\times10}{5\times10\times15}$	$\frac{5\times10\times15}{7\times12\times17\times22}$	$\frac{5\times10\times15\times20}{9\times14\times19\times24\times29}$
Eighth powers	$\frac{6}{3\times9}$	$\frac{6\times12}{5\times11\times17}$	$\frac{6\times12\times18}{7\times13\times19\times25}$	$\frac{6\times12\times18\times24}{9\times15\times21\times27\times33}$

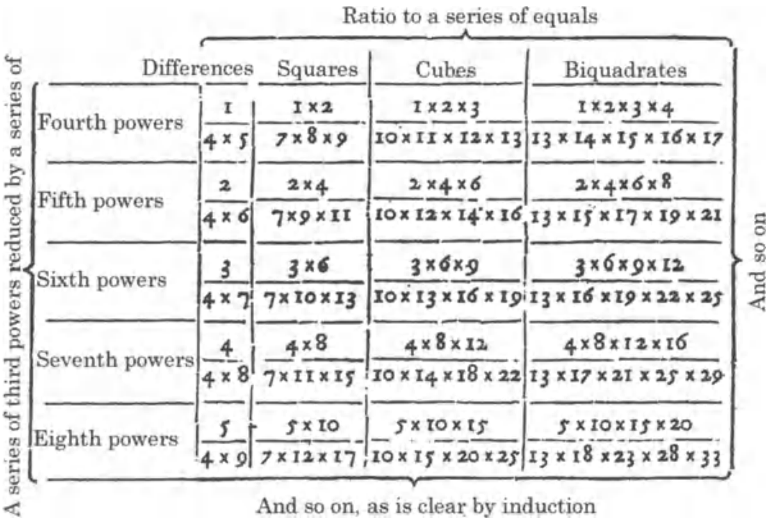
And so on

And so on, as is clear by induction

PROPOSITION 146

Theorem

In the same way, if there is proposed a series of third powers reduced by a series of fourth powers, fifth powers, sixth powers, etc. [the sums of] the differences themselves, and [of] their squares, cubes, biquadrates, etc. will have ratios to a series of equals, as the adjoined table indicates. That is:



Ratio to a series of equals

A series of third powers reduced by a series of

	Differences	Squares	Cubes	Biquadrates
Fourth powers	$\frac{1}{4\times5}$	$\frac{1\times2}{7\times8\times9}$	$\frac{1\times2\times3}{10\times11\times12\times13}$	$\frac{1\times2\times3\times4}{13\times14\times15\times16\times17}$
Fifth powers	$\frac{2}{4\times6}$	$\frac{2\times4}{7\times9\times11}$	$\frac{2\times4\times6}{10\times12\times14\times16}$	$\frac{2\times4\times6\times8}{13\times15\times17\times19\times21}$
Sixth powers	$\frac{3}{4\times7}$	$\frac{3\times6}{7\times10\times13}$	$\frac{3\times6\times9}{10\times13\times16\times19}$	$\frac{3\times6\times9\times12}{13\times16\times19\times22\times25}$
Seventh powers	$\frac{4}{4\times8}$	$\frac{4\times8}{7\times11\times15}$	$\frac{4\times8\times12}{10\times14\times18\times22}$	$\frac{4\times8\times12\times16}{13\times17\times21\times25\times29}$
Eighth powers	$\frac{5}{4\times9}$	$\frac{5\times10}{7\times12\times17}$	$\frac{5\times10\times15}{10\times15\times20\times25}$	$\frac{5\times10\times15\times20}{13\times18\times23\times28\times33}$

And so on

And so on, as is clear by induction

COMMENT

And in the same way, it will be easy either to continue these tables as far as one likes, or to compose others also for the succeeding series, thus for series of fourth powers, fifth powers, sixth powers, etc. reduced by any series of higher powers.

PROPOSITION 147

Theorem

If there is proposed a series of second roots reduced by a series of first powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of equals.

That is, as shown in Proposition 133. Thus:

Differences	Squares	Cubes
$\sqrt{aD}-\sqrt{a^2}$	$\sqrt{a^2D^2}-2\sqrt{a^3D}+\sqrt{a^4}$	$\sqrt{a^3D^3}-3\sqrt{a^4D^2}+3\sqrt{a^5D}-\sqrt{a^6}$ etc.
to		
$\sqrt{DD}-\sqrt{D^2}$	$\sqrt{D^2D^2}-2\sqrt{D^3D}+\sqrt{D^4}$	$\sqrt{D^3D^3}-3\sqrt{D^4D^2}+3\sqrt{D^5D}-\sqrt{D^6}$
$\frac{2}{3}A\sqrt{D^2}-\frac{2}{4}A\sqrt{D^2}$	$\frac{2}{4}A\sqrt{D^4}-\frac{4}{5}A\sqrt{D^4}+\frac{2}{6}A\sqrt{D^4}$	$\frac{2}{5}A\sqrt{D^6}-\frac{6}{6}A\sqrt{D^6}+\frac{6}{7}A\sqrt{D^6}-\frac{2}{8}A\sqrt{D^6}$
$\frac{2}{3}-\frac{2}{4}=\frac{2}{12}=\frac{1}{6}$	$\frac{2}{4}-\frac{4}{5}+\frac{2}{6}=\frac{4}{120}=\frac{1}{30}$	$\frac{2}{5}-\frac{6}{6}+\frac{6}{7}-\frac{2}{8}=\frac{12}{1680}=\frac{1}{140}$
$\dfrac{2}{3\times 4}=\dfrac{1}{2\times 3}$	$\dfrac{2\times 2}{4\times 5\times 6}=\dfrac{1\times 2}{3\times 4\times 5}$	$\dfrac{2\times 2\times 3}{5\times 6\times 7\times 8}=\dfrac{1\times 2\times 3}{4\times 5\times 6\times 7}$

PROPOSITION 148

Corollary

The same holds if a series of equals is reduced by a series of second roots, and multiplied by a series of second roots.

(Thus if the ordinates of a half parabola are set perpendicular to their continuations in the complement.) For $\sqrt{D}-\sqrt{a}$ times $\sqrt{a}$ makes $\sqrt{aD}-\sqrt{a^2}=\sqrt{aD}-a$ etc. And what is formed from the squares of the former $[(\sqrt{D}-\sqrt{a})^2\sqrt{a^2}]$ is equal to that formed from the squares of the latter $[(\sqrt{aD}-a)^2]$, etc.

PROPOSITION 149

Corollary

Also obvious, are the ratios arising, whether the proposed series is $a-a^2$, etc. or the series $\sqrt{a}-\sqrt{a^2}$ (or $\sqrt{a}-a$) etc.

That is, as collected in Propositions 133 and 147.

COMMENT

And therefore to the rest of the corollaries that are to be had after Proposition 133, these also (with appropriate changes), may be added without difficulty. Which it is sufficient to have indicated.

PROPOSITION 150

Theorem

If there is proposed a series of second roots reduced by a series of square roots of third powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of equals. That is:

Differences	Squares	Cubes
$\sqrt{aD^2} - \sqrt{a^3}$	$\sqrt{a^2D^4} - 2\sqrt{a^4D^2} + \sqrt{a^6}$	$\sqrt{a^3D^6} - 3\sqrt{a^5D^4} + 3\sqrt{a^7D^2} - \sqrt{a^9}$ etc.
$\frac{2}{3} - \frac{2}{5} = \frac{4}{15}$	$\frac{2}{4} - \frac{4}{6} + \frac{2}{8} = \frac{16}{192} = \frac{1}{12}$	$\frac{2}{5} - \frac{6}{7} + \frac{6}{9} - \frac{2}{11} = \frac{96}{3465}$
$\frac{4}{3 \times 5} A\sqrt{D^3}$	$\frac{4 \times 4}{4 \times 6 \times 8} A\sqrt{D^6}$	$\frac{4 \times 4 \times 6}{5 \times 7 \times 9 \times 11} A\sqrt{D^9}$

PROPOSITION 151

Corollary

The same holds if a series of equals reduced by a series of first powers is multiplied by a series of second roots.

Since $D - a$ or $\sqrt{D^2} - \sqrt{a^2}$, times $\sqrt{a}$, makes $D\sqrt{a} - a\sqrt{a}$ or $\sqrt{aD^2} - \sqrt{a^3}$.

COMMENT

And it may also be understood similarly in other cases, where the proposed series may be separated, into two or more components.

PROPOSITION 152

Theorem

If there is proposed a series of second roots reduced by a series of second powers, the [sums of] squares, cubes, biquadrates, etc. of the differences will have known ratios to a series of equals. That is:

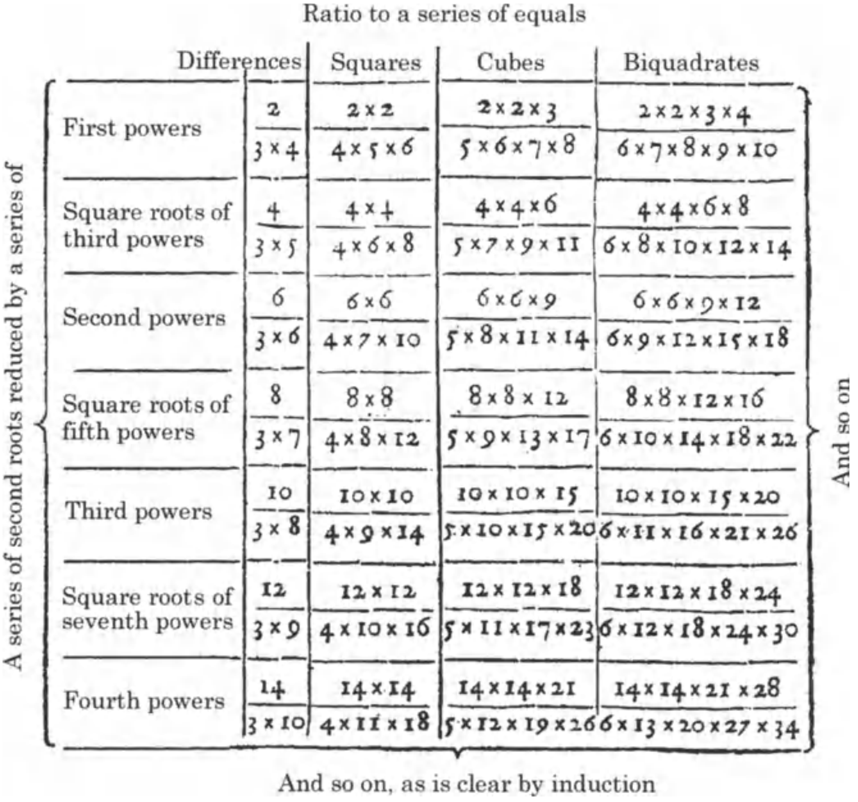
Differences $\sqrt{aD^3} - \sqrt{a^4}$	Squares $\sqrt{a^2D^6} - 2\sqrt{a^5D^3} + \sqrt{a^8}$	Cubes $\sqrt{a^3D^9} - 3\sqrt{a^6D^6} + 3\sqrt{a^9D^3} - \sqrt{a^{12}}$ etc.
$\frac{2}{3} - \frac{2}{6} = \frac{6}{18} = \frac{1}{3}$	$\frac{2}{4} - \frac{4}{7} + \frac{2}{10} = \frac{36}{280} = \frac{9}{70}$	$\frac{2}{5} - \frac{6}{8} + \frac{6}{11} - \frac{2}{14} = \frac{3224}{6160} = \frac{81}{1540}$
$\frac{6}{3 \times 6}$	$\frac{6 \times 6}{4 \times 7 \times 10}$	$\frac{6 \times 6 \times 9}{5 \times 8 \times 11 \times 14}$

And so on. And similarly for the subtraction of a series of any higher power. And therefore:

PROPOSITION 153

Theorem

If there is proposed a series of second roots reduced by a series of first powers, second powers, third powers etc. or square roots of third powers, fifth powers, etc. then [the sums of] the differences themselves, and [of] their squares, cubes, biquadrates, etc. will have ratios to a series of equals, as the adjoined table indicates. That is:



Ratio to a series of equals

A series of second roots reduced by a series of

	Differences	Squares	Cubes	Biquadrates
First powers	$\frac{2}{3 \times 4}$	$\frac{2 \times 2}{4 \times 5 \times 6}$	$\frac{2 \times 2 \times 3}{5 \times 6 \times 7 \times 8}$	$\frac{2 \times 2 \times 3 \times 4}{6 \times 7 \times 8 \times 9 \times 10}$
Square roots of third powers	$\frac{4}{3 \times 5}$	$\frac{4 \times 4}{4 \times 6 \times 8}$	$\frac{4 \times 4 \times 6}{5 \times 7 \times 9 \times 11}$	$\frac{4 \times 4 \times 6 \times 8}{6 \times 8 \times 10 \times 12 \times 14}$
Second powers	$\frac{6}{3 \times 6}$	$\frac{6 \times 6}{4 \times 7 \times 10}$	$\frac{6 \times 6 \times 9}{5 \times 8 \times 11 \times 14}$	$\frac{6 \times 6 \times 9 \times 12}{6 \times 9 \times 12 \times 15 \times 18}$
Square roots of fifth powers	$\frac{8}{3 \times 7}$	$\frac{8 \times 8}{4 \times 8 \times 12}$	$\frac{8 \times 8 \times 12}{5 \times 9 \times 13 \times 17}$	$\frac{8 \times 8 \times 12 \times 16}{6 \times 10 \times 14 \times 18 \times 22}$
Third powers	$\frac{10}{3 \times 8}$	$\frac{10 \times 10}{4 \times 9 \times 14}$	$\frac{10 \times 10 \times 15}{5 \times 10 \times 15 \times 20}$	$\frac{10 \times 10 \times 15 \times 20}{6 \times 11 \times 16 \times 21 \times 26}$
Square roots of seventh powers	$\frac{12}{3 \times 9}$	$\frac{12 \times 12}{4 \times 10 \times 16}$	$\frac{12 \times 12 \times 18}{5 \times 11 \times 17 \times 23}$	$\frac{12 \times 12 \times 18 \times 24}{6 \times 12 \times 18 \times 24 \times 30}$
Fourth powers	$\frac{14}{3 \times 10}$	$\frac{14 \times 14}{4 \times 11 \times 18}$	$\frac{14 \times 14 \times 21}{5 \times 12 \times 19 \times 26}$	$\frac{14 \times 14 \times 21 \times 28}{6 \times 13 \times 20 \times 27 \times 34}$

And so on

And so on, as is clear by induction

PROPOSITION 154

Theorem

Equally, if there is proposed a series of third roots reduced by a series of first powers, second powers, third powers etc. or cube roots of second powers, fourth powers, fifth powers, seventh powers, etc. then [the sums of] the differences themselves, and [of] their squares, cubes, biquadrates, etc. will have ratios to a series of equals, as the adjoined table indicates. That is:

Ratio of a series of equals

A series of third roots reduced by a series of

	Differences	Squares	Cubes	Biquadrates
Cube roots of second powers	$\frac{3}{4\times5}$	$\frac{3\times2}{5\times6\times7}$	$\frac{3\times2\times3}{6\times7\times8\times9}$	$\frac{3\times2\times3\times4}{7\times8\times9\times10\times11}$
First powers	$\frac{6}{4\times6}$	$\frac{6\times4}{5\times7\times9}$	$\frac{6\times4\times6}{6\times8\times10\times12}$	$\frac{6\times4\times6\times8}{7\times9\times11\times13\times15}$
Cube roots of fourth powers	$\frac{9}{4\times7}$	$\frac{9\times6}{5\times8\times11}$	$\frac{9\times6\times9}{6\times9\times12\times15}$	$\frac{9\times6\times9\times12}{7\times10\times13\times16\times19}$
Cube roots of fifth powers	$\frac{12}{4\times8}$	$\frac{12\times8}{5\times9\times13}$	$\frac{12\times8\times12}{6\times10\times14\times18}$	$\frac{12\times8\times12\times16}{7\times11\times15\times19\times23}$
Second powers	$\frac{15}{4\times9}$	$\frac{15\times10}{5\times10\times15}$	$\frac{15\times10\times15}{6\times11\times16\times21}$	$\frac{15\times10\times15\times20}{7\times12\times17\times22\times27}$
Cube roots of seventh powers	$\frac{18}{4\times10}$	$\frac{18\times12}{5\times11\times17}$	$\frac{18\times12\times18}{6\times12\times18\times24}$	$\frac{18\times12\times18\times24}{7\times13\times19\times25\times31}$
Cube roots of eighth powers	$\frac{21}{4\times11}$	$\frac{21\times14}{5\times12\times19}$	$\frac{21\times14\times21}{6\times13\times20\times27}$	$\frac{21\times14\times21\times28}{7\times14\times21\times28\times35}$
Third powers	$\frac{24}{4\times12}$	$\frac{24\times16}{5\times13\times21}$	$\frac{24\times16\times24}{6\times14\times22\times30}$	$\frac{24\times16\times24\times32}{7\times15\times23\times31\times39}$

And so on

And so on, as is clear by induction

COMMENT

And by a similar method it will not be difficult either to continue these tables as far as one likes, or even to compose others for other series, thus for series of fourth roots, fifth roots, etc. (or indeed of square roots of third roots, fifth roots, etc. or cube roots of second roots, fourth roots, etc. or of others similar), reduced by any series of higher powers.

But it is also easy to interpolate these tables (and others similarly constructed) to any length, by interposing between any horizontal sequences, as will be clear from correct consideration of the progression of the tables.

For example, in the table of Proposition 144, if a series of first powers is reduced by a series of square roots of fifth powers, this reduced series may be interposed, forming another horizontal sequence between the first and second of that table (since, that is, square roots of fifth powers, with index $\frac{5}{2}$ or $2\frac{1}{2}$, hold the mean place between second powers and third powers, with indices 2 and 3), and that sequence will be:

Differences	Squares	Cubes	Biquadrates
$\frac{1\frac{1}{2}}{2\times 3\frac{1}{2}}$	$\frac{1\frac{1}{2}\times 3}{3\times 4\frac{1}{2}\times 6}$	$\frac{1\frac{1}{2}\times 3\times 4\frac{1}{2}}{4\times 5\frac{1}{2}\times 7\times 8\frac{1}{2}}$	$\frac{1\frac{1}{2}\times 3\times 4\frac{1}{2}\times 6}{5\times 6\frac{1}{2}\times 8\times 9\frac{1}{2}\times 11}$
or $\frac{3}{2\times 7}$	$\frac{3\times 6}{3\times 9\times 12}$	$\frac{3\times 6\times 9}{4\times 11\times 14\times 17}$	$\frac{3\times 6\times 9\times 12}{5\times 13\times 16\times 19\times 22}$
or $\frac{6}{4\times 7}$	$\frac{6\times 6}{6\times 9\times 12}$	$\frac{6\times 6\times 9}{8\times 11\times 14\times 17}$	$\frac{6\times 6\times 9\times 12}{10\times 13\times 16\times 19\times 22}$

And it will not be difficult to show the same also for other tables, if the pattern of each table is observed.

And in the same manner, one may interpolate the same tables to any width, clearly by interposing others among the vertical sequences (thus, the square roots of differences, square roots of cubes, etc. or cube roots of differences, squares, biquadrates, etc. or similar), but at this point the work is not easy, if indeed it is possible.[53] Afterwards, moreover, I will try as far as I can, and indeed will show to a certain extent, that one may work out completely what I hardly dared promise except by approximation.

Meanwhile, something must be said of augmented series, lest I seem to have omitted them completely; but briefly, lest I become tedious.

PROPOSITION 155

Theorem

If there is proposed a series of equals augmented by a similar series of first powers, the [sums of] squares, cubes, biquadrates, etc. of the aggregates will

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[53] Here Wallis needs what he has needed all along, the binomial theorem for fractional indices.

have a known ratio to a series of equals. That is:

Aggregates	Squares	Cubes
$R+a$	$R^2+2aR+a^2$	$R^3+3aR^2+3a^2R+a^3$
etc. to		
$R+R$	$R^2+2RR+R^2$	$R^3+3R^2+3R^2R+R^3$
$AR+\frac{1}{2}AR$	$AR^2+\frac{2}{2}AR^2+\frac{1}{3}AR^2$	$AR^3+\frac{3}{2}AR^3+\frac{3}{3}AR^3+\frac{1}{4}AR^3$
$1+\frac{1}{2}=\frac{3}{2}$	$1+\frac{2}{2}+\frac{1}{3}=\frac{7}{3}$	$1+\frac{3}{2}+\frac{3}{3}+\frac{1}{4}=\frac{15}{4}$

Where any numerator consists of twice the preceding one increased by 1; and the denominator, of the preceding one increased by 1.

PROPOSITION 156

Corollary

Therefore, if from a trapezium (constituted from a parallelogram and a triangle, of equal base and height), there is generated a truncated conoid (or pyramidoid), (whether by turning about the axis, or otherwise), it will be to the inscribed cylinder or prism as $\frac{7}{3}$ to 1, or as 7 to 3.

That is, as the squares, of a series of equals augmented by a series of first powers, to a series of equals.

If the bases of the parallelogram and triangle are unequal, some adjustment must be introduced.

PROPOSITION 157

Corollary

If, moreover, that truncated conoid or pyramidoid is excavated by a cylinder or prism, the residue will be (to the greatest inscribed cylinder or prism) as 4 to 3.

That is, $\frac{7}{3}-\frac{3}{3}=\frac{4}{3}$ to 1.

PROPOSITION 158

Theorem

If there is proposed a series of equals augmented by a series of second powers, the [sums of] squares, cubes, biquadrates, etc. of the aggregates will have known ratios to a series of equals. (Thus, if a parallelogram is augmented by the complement of a half parabola.)

That is, for any term of the series of first powers I put a (to shorten the work), and therefore for any term of the second powers, a^2, etc. Then:

Aggregates	Squares	Cubes
$R^2 + a^2$	$R^4 + 2a^2R^2 + a^4$	$R^6 + 3a^2R^4 + 3a^4R^2 + a^6$
etc.		
$1AR^2 + \frac{1}{3}AR^2$	$1AR^4 + \frac{2}{3}AR^4 + \frac{1}{5}AR^4$	$1AR^6 + \frac{3}{3}AR^6 + \frac{3}{5}AR^6 + \frac{1}{7}AR^6$
$1 + \frac{1}{3} = \frac{4}{3}$	$1 + \frac{2}{3} + \frac{1}{5} = \frac{28}{15}$	$1 + \frac{3}{3} + \frac{3}{5} + \frac{1}{7} = \frac{288}{105}$

COMMENT

And it may be carried out by the same method if a series of equals is augmented by a series of third powers, fourth powers, etc. As is clear.

PROPOSITION 159

Theorem

If there is proposed a series of equals augmented by a series of second roots, the [sums of] squares, cubes, biquadrates, etc. of the aggregates will have known ratios to a series of equals. (Thus, if a parallelogram is augmented by a half parabola.) That is:

Aggregates	Squares	Cubes
$\sqrt{R} + \sqrt{a}$	$\sqrt{R^2} + 2\sqrt{aR} + \sqrt{a^2}$	$\sqrt{R^3} + 3\sqrt{aR^2} + 3\sqrt{a^2R} + \sqrt{a^3}$
etc.		
$A\sqrt{R} + \frac{2}{3}A\sqrt{R}$	$A\sqrt{R^2} + \frac{4}{3}A\sqrt{R^2} + \frac{2}{4}A\sqrt{R^2}$	$A\sqrt{R^3} + \frac{6}{3}A\sqrt{R} + \frac{6}{4}A\sqrt{R^3} + \frac{2}{5}A\sqrt{R^3}$
$1 + \frac{2}{3} = \frac{10}{6} = \frac{5}{3}$	$1 + \frac{2}{3} + \frac{2}{4} = \frac{68}{24} = \frac{17}{6}$	$1 + \frac{6}{3} + \frac{6}{4} + \frac{2}{5} = \frac{588}{120} = \frac{49}{10}$

PROPOSITION 160

Theorem

If a series of equals is augmented by a series of third roots, [the sums of] the aggregates and [of] their squares, cubes, etc. will have known ratios to a series of equals.

(Thus, if a parallelogram is augmented by half a cubical parabola.). That is:

Aggregates	Squares	Cubes
$\sqrt[3]{R}+\sqrt[3]{a}$	$\sqrt[3]{R^2}+2\sqrt[3]{aR}+\sqrt[3]{a^2}$	$\sqrt[3]{R^3}+3\sqrt[3]{aR^2}+3\sqrt[3]{a^2R}+\sqrt[3]{a^3}$
etc.		
$\frac{2}{3}+\frac{3}{4}=\frac{7}{4}$	$\frac{2}{3}+\frac{6}{4}+\frac{3}{5}=\frac{31}{10}$	$\frac{3}{3}+\frac{9}{4}+\frac{9}{5}+\frac{3}{6}=\frac{111}{20}$

COMMENT

And it may be carried out by the same method if there is proposed a series of equals augmented by series of fourth roots, fifth roots, etc. or also by series of square roots of cubes, supersolids, etc. or cube roots of second powers, fourth powers, etc. And the same in other cases.

PROPOSITION 161

Theorem

Equally, if a series of first powers is augmented by a series of second powers, [the sums of] the aggregates, and [of] their squares, cubes, etc. will have known ratios to a series of equals. That is:

Aggregates	Squares	Cubes
$aR+a^2$	$a^2R^2+2a^3R+a^4$	$a^3R^3+3a^4R^2+3a^5R+a^6$
etc.		
$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}AR^2$	$\frac{1}{3}+\frac{2}{4}+\frac{1}{5}=\frac{31}{30}AR^4$	$\frac{1}{4}+\frac{3}{5}+\frac{3}{6}+\frac{1}{7}=\frac{209}{140}AR^6$

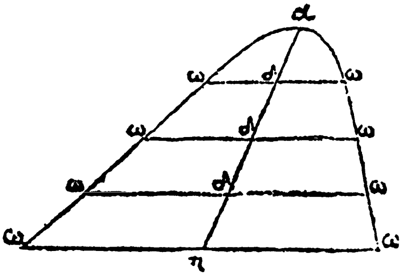
And it may be carried out by the same method if there is proposed a series of first powers (or also second powers, third powers, etc.) augmented by any other series; so it is not worth dwelling on this any longer. In all of which, the sequence of numbers, first in the numerator, then in the denominator, is clear to the eye.

PROPOSITION 162

Corollary

Therefore a hyperbolic conoid or pyramid to half the circumscribed cylinder or prism is as 5 to 6, and to the total as 5 to 12.

(It is to be understood that both the transverse diameter and the maximum intercepted diameter, are equal to the *latus rectum*, otherwise an appropriate adjustment must be introduced.)



For if the *latus rectum* of the hyperbola is taken to be l or R, the transverse diameter $t = l$, and intercepted diameter d, the squares of the ordinates will be $dl + \frac{d}{t}dl$ (by Proposition 33 of *On conic sections*)[54] or (since $t = l$) $dl + d^2$. And therefore (since l or R are fixed quantities, while d is variable, and indeed proportional to the altitude, for which may therefore be substituted a, b, c, etc.) all the squares (and also therefore the planes of the conoid of pyramidoid) will be an infinite series of first powers augmented by a series of second powers, thus $aR + a^2$, $bR + b^2$, $cR + c^2$, etc. as far as $R^2 + R^2 = 2R^2$. And therefore that series, to half a series of the same number equal to greatest (thus to AR^2), will be as 5 to 6 by what has gone before. And therefore to the complete series of equals as 5 to 12. As was proposed.

COMMENT

The same also happens if the greatest intercepted diameter is taken to be equal to the transverse diameter. As may be gathered from the following proposition.

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[54] Here for the first time Wallis used some of the algebraic formulae that he developed in *On conic sections*.

PROPOSITION 163

Corollary

If, moreover, the limitation of the preceding proposition does not apply,[55] the ratio of the conoid or pyramidoid to the circumscribed cylinder or prism, although not the same as there indicated, is nevertheless known.

For in any case, the squares of ordinates of the hyperbola are $dl + \frac{dd}{t}l$, or $dL + \frac{dd}{T}L$, or $\frac{dT+dd}{T}L$. If for the intercepted diameters, d, d, etc. there are put in turn a, b, c, etc. and it is supposed that the greatest of them is D, then the square of the greatest ordinate is $\frac{DT+DD}{T}L$. All $aT + bT + cT$ etc. (as far as DT) are equal to $\frac{1}{2}ADT$, and all $a^2+b^2+c^2$ etc. (to AD^2) are equal to $\frac{1}{3}AD^2$, the sum of which, $\frac{1}{2}ADT + \frac{1}{3}AD^2$, if multiplied by L and the product divided by T, will give $\frac{\frac{1}{2}ADT+\frac{1}{3}AD^2}{T}L$, or also $\frac{\frac{1}{2}T+\frac{1}{3}D}{T}ADL$, or thence $\frac{3T+2D}{6T}ADL$. And the ratio of this, $\frac{3T+2D}{6T}ADL$, the sum of squares of all ordinates, to $\frac{DT+D^2}{T}AL$, or $\frac{T+D}{T}ADL$, the sum of the same number of terms equal to the square of the greatest, is that of the conoid or pyramidoid to the circumscribed cylinder or prism (since the planes are proportional to those squares), that is, as $\frac{3T+2D}{6T}$ to $\frac{T+D}{T}$, or as $3T+2D$ to $6T+6D$, or as $\frac{1}{2}T+\frac{1}{3}D$ to $T+D$. Therefore:

PROPOSITION 164

Corollary

As half the transverse diameter augmented by a third of the intercepted diameter, to the sum of the transverse and intercepted diameters; or as three times the transverse together with twice the intercepted to six times both together: so is the hyperbolic conoid or pyramidoid to the circumscribed cylinder or prism (on the same base).

Clear from what has gone before.

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[55] That is, the limitation that both the transverse diameter and the maximum intercepted diameter must be equal to the *latus rectum*.

PROPOSITION 165

Corollary

In the same way, the hyperbola to the circumscribed parallelogram is as a series of square roots augmented term by term, to a series of the same number of terms equal to the square root of the greatest.

That is, if the condition of Proposition 162 holds, as $\sqrt{(aR+a^2)} + \sqrt{(bR+b^2)} + \sqrt{(cR+c^2)}$ etc. (as far as $\sqrt{(R^2+R^2)}$), to $A\sqrt{(R^2+R^2)} = A\sqrt{2R^2} = AR\sqrt{2}$. That is, as all the ordinates to the greatest taken the same number of times.

But if that condition does not hold, then it will at least be as $\sqrt{\dfrac{aT+a^2}{T}L} + \sqrt{\dfrac{bT+b^2}{T}L} + \sqrt{\dfrac{cT+c^2}{T}L}$ etc. (as far as $\sqrt{\dfrac{DT+D^2}{T}L}$), to $A\sqrt{\dfrac{DT+D^2}{T}L}$. That is, (dividing everything by $\sqrt{L}$ and multiplying by $\sqrt{T}$) as $\sqrt{(aT+a^2)} + \sqrt{(bT+b^2)} + \sqrt{(cT+c^2)}$ etc. (as far as $\sqrt{(DT+D^2)}$), to $A\sqrt{(DT+D^2)}$, as is clear from the demonstration in Proposition 163.[56]

COMMENT

And by what means the ratio of the sum of those roots, to the sum of the same number equal to the greatest, may eventually be expressed in numbers, is not so easily shown.

And therefore we here come upon the same difficulty in the quadrature of the hyperbola as we recalled several times above for the quadrature of the circle or ellipse (and various other curved figures), that is, that it must now be inquired what are the ratios for infinite series of roots of binomes, just as there for apotomes.

And indeed I was sometimes inclined to believe the thing to be quite impossible, that an infinite number of surd roots, incommensurable to each other, might be brought together in one sum that has an explicable ratio to some proposed rational quantity.

And this indeed seems to be confirmed still more strongly, since a finite series of this kind, to a series of the same number of terms equal to the greatest, has scarcely allowed any other expression of the ratio than by repetition of everything piece by piece; for rarely do two or more happen to be commensurable, that can be gathered into one sum.

For example, if the radius of a circle is taken in six equal parts, the right sines or ordinates in the quadrant standing on the ends of each of those parts will be $\sqrt{(36-0)} + \sqrt{(36-1)} + \sqrt{(36-4)} + \sqrt{(36-9)} + \sqrt{(36-16)} + \sqrt{(36-25)} + \sqrt{(36-36)}$ (by what was said at Proposition 121), or what

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[56] This is virtually the last of Wallis's geometric examples; from now on his investigations are based almost entirely on arithmetic.

it reduces to, $\sqrt{36}+\sqrt{35}+\sqrt{32}+\sqrt{27}+\sqrt{20}+\sqrt{11}+\sqrt{0}$, or as irrationals reduced to their least terms, $6+\sqrt{35}+4\sqrt{2}+3\sqrt{3}+2\sqrt{5}+\sqrt{11}+0$. The ratio, therefore, of this sum of roots to the greatest root taken the same number of times, thus $7\sqrt{(36-0)}$, or $7\sqrt{36}$, or 7×6, that is 42, can be no better expressed than

$$\frac{6+\sqrt{35}+4\sqrt{2}+3\sqrt{3}+2\sqrt{5}+\sqrt{11}+0}{42}.$$

And that is the ratio of those right sines or ordinates in the quadrant, to the same number of lines equal and parallel to the radius in the circumscribed square.

Equally, if the radius is taken in ten parts, the right sines will be $\sqrt{(100-0)}+\sqrt{(100-1)}+\sqrt{(100-4)}+\sqrt{(100-9)}+\sqrt{(100-16)}+\sqrt{(100-25)}+\sqrt{(100-36)}+\sqrt{(100-49)}+\sqrt{(100-64)}+\sqrt{(100-81)}+\sqrt{(100-100)}$. That is $\sqrt{100}+\sqrt{99}+\sqrt{96}+\sqrt{91}+\sqrt{84}+\sqrt{75}+\sqrt{64}+\sqrt{51}+\sqrt{36}+\sqrt{19}+\sqrt{0}$. Or $10+3\sqrt{11}+4\sqrt{6}+\sqrt{91}+2\sqrt{21}+5\sqrt{3}+8+\sqrt{51}+6+\sqrt{19}+0$. Which sum cannot be written otherwise more briefly than by substituting 24 for $10+8+6+0$; so the ratio of this sum to the greatest root taken the same number of times, thus to $11\sqrt{100}$ or 11×10 or 110, can be no better expressed than

$$\frac{24+3\sqrt{11}+4\sqrt{6}+\sqrt{91}+2\sqrt{21}+5\sqrt{3}+\sqrt{51}+\sqrt{19}}{110},$$

which seems still less intelligible than when the radius is taken in fewer parts, such as six.

And in the same way, as more parts of the radius are taken, so the expression for the ratio necessarily becomes more intricate; and indeed requires repetition of almost all the roots, since they happen little, indeed rarely, and only as if by chance, to be commensurable either with rational numbers or with each other. And therefore if one takes the radius in an infinite number parts, the ratio arising will appear even less expressible than here.

The same holds if, in the manner of Proposition 135, one takes the diameter of the circle in twelve parts. For then the corresponding right sines in the semicircle are $\sqrt{(0\times 12-0)}+\sqrt{(1\times 12-1)}+\sqrt{(2\times 12-4)}+\sqrt{(3\times 12-9)}+\sqrt{(4\times 12-16)}+\sqrt{(5\times 12-25)}+\sqrt{(6\times 12-36)}+\sqrt{(7\times 12-49)}+\sqrt{(8\times 12-64)}+\sqrt{(9\times 12-81)}+\sqrt{(10\times 12-100)}+\sqrt{(11\times 12-121)}+\sqrt{(12\times 12-144)}$. That is $\sqrt{(0-0)}+\sqrt{(12-1)}+\sqrt{(24-4)}+\sqrt{(36-9)}+\sqrt{(48-16)}+\sqrt{(60-25)}+\sqrt{(72-36)}+\sqrt{(84-49)}+\sqrt{(96-64)}+\sqrt{(108-81)}+\sqrt{(120-100)}+\sqrt{(132-121)}+\sqrt{(144-144)}$. That is $\sqrt{0}+\sqrt{11}+\sqrt{20}+\sqrt{27}+\sqrt{32}+\sqrt{35}+\sqrt{36}+\sqrt{35}+\sqrt{32}+\sqrt{27}+\sqrt{20}+\sqrt{11}+\sqrt{0}$. Or because of those roots taken twice, $2\sqrt{0}+2\sqrt{11}+2\sqrt{20}+2\sqrt{27}+2\sqrt{32}+2\sqrt{35}+\sqrt{36}$. Or reducing irrationals to least terms, $0+2\sqrt{11}+4\sqrt{5}+6\sqrt{3}+8\sqrt{2}+2\sqrt{35}+6$. Therefore the ratio of this sum to the same number of roots equal to the greatest, thus to $13\sqrt{36}$ or 13×6, that is to 78, can be no better expressed than as

$$\frac{0+2\surd 11+4\surd 5+6\surd 3+8\surd 2+2\surd 35+6}{78},$$

or

$$\frac{\surd 11+2\surd 5+3\surd 3+4\surd 2+\surd 35+3}{39}.$$

And this is the ratio of the sum of those right sines in the semicircle, to the same number of lines equal and parallel to the radius in the parallelogram circumscribing this semicircle. If, moreover, the greatest sine $\surd 36=6$ is assumed to be taken twice (for both quadrants at the same time) and therefore for 13 equals, are put 14 (thus for $13\times 6=78$ is put $14\times 6=84$) this will be the same ratio as was said above to arise from taking the radius in six parts.

Equally if the diameter is taken 20 parts. The right sines in the semicircle will be $\surd(0\times 20-0)+\surd(1\times 20-1)+\surd(2\times 20-4)+\surd(3\times 20-9)+\surd(4\times 20-16)+\surd(5\times 20-25)+\surd(6\times 20-36)+\surd(7\times 20-49)+\surd(8\times 20-64)+\surd(9\times 20-81)+\surd(10\times 20-100)+\surd(11\times 20-121)+\surd(12\times 20-144)+\surd(13\times 20-169)+\surd(14\times 20-196)+\surd(15\times 20-225)+\surd(16\times 20-256)+\surd(17\times 20-289)+\surd(18\times 20-324)+\surd(19\times 20-361)+\surd(20\times 20-400)$. That is $\surd 0+\surd 19+\surd 36+\surd 51+\surd 64+\surd 75+\surd 84+\surd 91+\surd 96+\surd 99+\surd 100+\surd 99+\surd 96+\surd 91+\surd 84+\surd 75+\surd 64+\surd 51+\surd 36+\surd 19+\surd 0$. Or $2\surd 0+2\surd 19+2\surd 36+2\surd 51+2\surd 64+2\surd 75+2\surd 84+2\surd 91+2\surd 96+2\surd 99+\surd 100$. (That is, the same as held above in a quadrant taking the radius in ten parts, here put twice, except that the greatest sine, in common to both quadrants, is not repeated.) Or also $0+2\surd 19+12+2\surd 51+16+10\surd 3+4\surd 21+2\surd 91+8\surd 6+6\surd 11+10$. Or finally (since $0+12+16+10=38$), $38+2\surd 19+2\surd 51+10\surd 3+4\surd 21+2\surd 91+8\surd 6+6\surd 11$. And therefore the ratio of the sum of those roots to the greatest taken the same number of times (thus, $21\times 10=210$) is

$$\frac{38+2\surd 19+2\surd 51+10\surd 3+4\surd 21+2\surd 91+8\surd 6+6\surd 11}{210}$$

or

$$\frac{19+\surd 19+\surd 51+5\surd 3+2\surd 21+\surd 91+4\surd 6+3\surd 11}{105}$$

And indeed the more parts there are taken of the radius or diameter, so much less does the ratio of all the sines, to the greatest taken the same number of times, seem expressible. Therefore if the radius or diameter is taken in infinitely many parts (which it seems must be done for our purposes) the ratio of all sines, to the radius taken the same number of times, that is, the quadrant or semicircle to the circumscribed square or parallelogram, seems wholly inexpressible, at least unless an expression of this kind is judged to be sufficient, as we showed in Propositions 121 and 135.

And thus having weighed this carefully, it nearly came about that I abandoned the investigation of the thing that, as it were, I so called for above. The one thing that gave hope was this. That is, that the same difficulty notwithstanding, in square roots, cube roots, biquadratic roots, etc. of numbers in arithmetic proportion the thing turned out not badly.

For example, if a series of second roots is continued as far as you please, thus, $\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}$, their ratio to the greatest taken the same number of times, thus, $7\sqrt{6}$, does not seem to be expressible other than as

$$\frac{\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}}{7\sqrt{6}},$$

or

$$\frac{0+1+\sqrt{2}+\sqrt{3}+2+\sqrt{5}+\sqrt{6}}{7\sqrt{6}},$$

or at least (since $0+1+2=3$) as

$$\frac{3+\sqrt{2}+\sqrt{3}+\sqrt{5}+\sqrt{6}}{7\sqrt{6}},$$

unless perhaps it pleases one to multiply both the numerator and denominator by $\sqrt{6}$ to produce the ratio $3\sqrt{6}+\sqrt{12}+\sqrt{18}+\sqrt{30}+\sqrt{36}$ to 7×6, or rather $3\sqrt{6}+2\sqrt{3}+3\sqrt{2}+\sqrt{30}+6$ to $7\times 6=42$. And similarly in other series of this kind.

But if the same series is supposed continued to infinity, it will eventually produce the ratio $\frac{2}{3}$ or 2 to 3 or 1 to $1\frac{1}{2}$, as was said in Propositions 53 and 54, the infiniteness itself indeed (which seems amazing) destroying the irrationality.

And it holds similarly for third roots, fourth roots, etc. as is clear from what was taught above in Propositions 54 and 59.

This difficulty notwithstanding, the quadrature of the simple parabola been shown both by others before this, and also by us by our method; and also the quadrature of any higher parabola (the same difficulty remaining) has been taught by us happily enough above, so clearly not all hope was lacking of eventually finding the ratio of series of universal roots (of augmented or reduced series) to a series of equals. And indeed if not in every case, at least for those so far set out; and perhaps even in those that touch on the quadrature of the circle itself or the ellipse, or also the hyperbola, something may be gained.

That what must next be inquired after may be more rightly seen, let us remember what (among other things) has been achieved so far towards the quadrature of the circle (or any ellipse).

That is, by Propositions 118 and 121, if the sequence of ratios $\frac{1}{1}$, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, $\frac{384}{945}$, etc. can be interpolated, the ratio that must be placed as intermediate between the first and second is that of a quadrant of a circle to the square of the radius, or the circle itself to the square of the diameter.

In the same way, by Propositions 133 and 135, if it were possible to interpolate this sequence of ratios $\frac{1}{1}$, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{140}$, $\frac{1}{630}$, etc., the ratio that must be placed as intermediate between the first and second is that of a semicircle to the square of the diameter.

But, above all, if it were possible to interpolate the diagonal numbers in the table in Proposition 132, that is, 1, 2, 6, 20, 70, etc., the ratio of 1 to the number intermediate between the first and second of those, is that of the circle to the square of the diameter, and the ellipse to the circumscribed parallelogram. As will be proved by the following proposition.

PROPOSITION 166

Theorem

If an infinite series of equals, first powers, second powers, third powers, etc. is multiplied term by term by itself reversed,[57] and the same also by itself directly,[58] the sums of the products of the former, to the sums of the latter, are as 1 to 1, 2, 6, 20, 70, etc., the diagonal numbers in the table in Proposition 132.

For if a series of equals (whether taken directly or reversed) is multiplied term by term by itself, it will give a series of equals, to which there belongs the ratio 1 to 1.

If, moreover, a series of first powers is thus multiplied by itself directly, it will give a series of second powers; if a series of second powers is thus multiplied it will give a series of fourth powers; if a series of third powers, it will give a series of sixth powers, etc. by Proposition 73. To which belong the ratios $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, etc. by Propositions 44 or 64.

If, moreover, a series of first powers is multiplied term by term by itself reversed (thus the series a, b, c, etc. by the series $D-a$, $D-b$, $D-c$, etc.) or in the same way a series of second powers by itself reversed (thus a^2, b^2, c^2, etc. by the series $(D-a)^2$, $(D-b)^2$, $(D-c)^2$, etc or $D^2-2aD+a^2$, $D^2-2bD+b^2$, $D^2-2cD+c^2$, etc.) or in the same way a series of third powers by itself reversed

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[57] *In seipsam inverse positam*, that is 'taken backwards' or 'reversed'. Bear in mind that Wallis's series, though it has an infinite number of terms, has a finite greatest term, and so the terms can be taken in either direction.

[58] *In seipsam directe positam*; Wallis uses 'directly' here to mean 'forward'. Earlier (in Propositions 99, 102, 103, 104, 106) he spoke of direct and reciprocal proportion; the two uses of 'direct' are linked in that the powers, or indices, of direct series go forwards $(1, 2, 3, \ldots)$ whereas the indices of reciprocal or inverse series go backwards $(-1, -2, -3, \ldots)$.

(that is a^3, b^3, c^3, etc. by $(D-a)^3$, $(D-b)^3$, $(D-c)^3$, etc.), and so on for the rest, the ratios belonging to them are $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{140}$, $\frac{1}{630}$, etc. by Propositions 133 and 134.

Therefore the ratios of the latter ratios to the former ratios are as 1 to the numbers 1, 2, 6, 20, 70, etc., that is, to the diagonal numbers in the table in Proposition 132 (as is clear from the calculations). Which was to be proved.

COMMENT

It must be noted here: in the sequence of ratios $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, etc. the denominators are arithmetic proportionals; and therefore if in the intervals there are to be interposed the same number of ratios, they will be $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, etc. by analogy with arithmetic proportionals, and the rules in Propositions 44 and 64.

But in the sequence of ratios $\frac{1}{1}$, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{140}$, $\frac{1}{630}$, etc. the denominators are 1, 6, 30, 140, 630, etc. arising from continued multiplication of the numbers $1 \times \frac{6 \times 10 \times 14 \times 18}{1 \times 2 \times 3 \times 4}$ etc. or $1 \times \frac{12 \times 20 \times 28 \times 36 etc.}{2 \times 4 \times 6 \times 8 etc.}$ (where both the numerators and denominators of the fractions are arithmetic proportionals). And therefore (by analogy with those progressions), if the number to be interposed between the first and second is called A, then the rest, to be interposed in the remaining intervals, arise from continued multiplication of the numbers $A \times \frac{16 \times 24 \times 32 etc.}{3 \times 5 \times 7 etc.}$. (And indeed, the number placed before the first is $\frac{1}{8}A$, by the same analogy. In the previous case, moreover, and therefore in the sequence soon to follow, the number before the first vanishes; that is, first to 0, then to infinity.)[59]

Finally in the sequence of ratios $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{20}$, $\frac{1}{70}$, etc. the denominators 1, 2, 6, 20, 70, arise from continued multiplication of the numbers $1 \times \frac{2 \times 6 \times 10 \times 14 etc.}{1 \times 2 \times 3 \times 4 etc.}$ or $1 \times \frac{4 \times 12 \times 20 \times 28 etc.}{2 \times 4 \times 6 \times 8 etc.}$ (where, as above, both the numerators and the denominators of the fractions are arithmetic proportionals). And therefore (by analogy with such progressions), if the number to be interposed between the first and second is called α, the rest arise from continued multiplication of the numbers $\alpha \times \frac{8 \times 16 \times 24 etc.}{3 \times 5 \times 7 etc.}$. (Moreover, $\alpha = \frac{1}{2}A$, since $\frac{1}{A}$ divided by $\frac{1}{2}$ is $\frac{2}{A} = \frac{1}{\frac{1}{2}A} = \frac{1}{\alpha}$.)

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[59] What Wallis means here is that in the sequence just discussed the multiplier before $\frac{16}{3}$ (following the same pattern) would be $\frac{8}{1}$, and therefore the term before A must be $\frac{1}{8}A$. In the previous sequence the multiplier before $\frac{6}{1}$ (following the same pattern) would be $\frac{2}{0}$ and therefore the term before 1 must be $\frac{0}{2} = 0$, and the one before that must be infinite.

PROPOSITION 167

Theorem

Therefore if an infinite series of second roots (thus $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, etc.) is multiplied term by term by itself reversed (thus by $\sqrt{(D-a)}$, $\sqrt{(D-b)}$, $\sqrt{(D-c)}$ etc.) and also by itself directly (thus the series $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, etc. by the series $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, etc.), the sum of products of the former (thus $\sqrt{(aD-a^2)}+\sqrt{(bD-b^2)}+\sqrt{(cD-c^2)}$, etc.) to the sum of the latter (thus $\sqrt{a^2}+\sqrt{b^2}+\sqrt{c^2}$ etc. or $a+b+c$ etc.) is as 1 to the intermediate number interposed between the diagonal numbers 1 and 2 in the table in Proposition 132.

Follows from what has gone before. For a series of second roots is intermediate between a series of equals and series of first powers (as is clear from what was said in Proposition 64).

Moreover, a series of second roots multiplied by itself directly (thus $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, etc. by $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, etc.) is a series of first powers (thus $\sqrt{a^2}$, $\sqrt{b^2}$, $\sqrt{c^2}$, etc. or a, b, c, etc.) to which belongs the ratio $\frac{1}{2}$ by Propositions 44 or 64. And therefore, if the ratio that belongs to a series of [second roots][60] multiplied by itself reversed (intermediate, that is, between the ratios $\frac{1}{1}$ and $\frac{1}{6}$) is said to be $\frac{1}{2\square}$, then the ratio of this $\frac{1}{2\square}$, to that $\frac{1}{2}$, that is $\frac{1}{\square}$, will be (by what has gone before) that of 1 to the number interposed between 1 and 2, in the sequence of diagonal numbers 1, 2, 6, 20, 70, etc. in the table of Proposition 132. *Which number, therefore, in what follows will be called* □. And it is half the number interposed between 1 and 6 in the sequence 1, 6, 30, 140, 630, etc.

PROPOSITION 168

Corollary

And therefore the circle to the square of its diameter is as 1 to □, that is, to the number interposed between 1 and 2 in the sequence of diagonal numbers 1, 2, 6, 20, 70, etc. in the table of Proposition 132.

For since (by Propositions 133 and 135) the semicircle to the square of its diameter is as 1 to 2□ (the number intermediate between 1 and 6 in the sequence 1, 6, 30, 140, 630, etc.) the circle (twice the semicircle) is as 1 to □ (the number intermediate between 1 and 2 in the sequence 1, 2, 6, 20, 70, etc.) by what has gone before.

And indeed the same ratio $\frac{1}{\square}$ is that which must be placed as intermediate between $\frac{1}{1}$ and $\frac{2}{3}$ in the sequence $\frac{1}{1}$, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, etc. by Propositions 118 and 121, as will also be further obvious later.

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[60] Wallis has mistakenly written ‘first powers’.

COMMENT

Since therefore (as at last we return to what we pointed out in the *Comment* to Proposition 165) the thing may be reduced to this, whether we can interpolate those ratios (noted in Proposition 118), $\frac{1}{1}$, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, etc., that is, 1 to 1, $\frac{3}{2}$, $\frac{15}{8}$, $\frac{105}{48}$, etc. (that is, to 1, $1\frac{1}{2}$, $1\frac{7}{8}$, $2\frac{9}{48}$, etc.) which arise from continued multiplication of the numbers $1 \times \dfrac{3 \times 5 \times 7 \times 9 etc.}{2 \times 4 \times 6 \times 8 etc.}$.

Or also those $\frac{1}{1}$, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{140}$, etc. (from Proposition 133), of which the denominators 1, 6, 30, 140, etc. arise from continued multiplication of the numbers $1 \times \dfrac{6 \times 10 \times 14 \times 18 etc.}{1 \times 2 \times 3 \times 4 etc.}$.

Or finally those $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{20}$, etc., that is, of 1 to 1, 2, 6, 20, etc. (the diagonal numbers in the table in Proposition 132) which arise from continued multiplication of the numbers $1 \times \frac{2}{1} \times \frac{6}{2} \times \frac{10}{3} \times \frac{14}{4}$ etc.

If, I say, we can interpolate any one sequence of these ratios, we will have the quadrature of the circle very accurately. And indeed in the first and third, the ratio to be interposed (after the first) will be $\frac{1}{\square}$, but in the second the ratio $\frac{1}{2\square}$. And therefore if the interpolation is shown in one, it may also be done without difficulty in the others.

One may therefore approach the table in Proposition 132 (as we assumed from the beginning, whence there shines greater hope of understanding the question), that we may see by what art we may interpolate it. And therefore we repeat it with spaces placed alternately (so that what were there the first, second, third sequence, etc. are here the second, fourth, sixth, etc.) and examine it a little more closely, which is to be done in various propositions following.

PROPOSITION 169

Theorem

All the numbers of the table in Proposition 132 are figurate. That is, those in the first sequence (whether vertically or horizontally) are units; those in the second, sides; those in the third, triangular numbers; those in the fourth, pyramidal numbers, and so on, thus, triangulo-triangulars, triangulo-pyramidals, pyramido-pyramidals, etc.

This is clear from inspection of the table, and by comparison (if needed) with the figurate numbers that occur in Maurolico and others. Moreover, I use those names that our master Oughtred (an exceptional mathematician) uses in his *Clavis mathematicae*, (Chapter 17, note 11).[61]

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[61] Wallis is referring here to the second and later editions of William Oughtred's *Clavis mathematicae*, those published from 1647 onwards; in the first (1631) edition, Oughtred's note on figurate numbers appears at Chapter 18, note 16. Wallis was involved in correcting the third Latin edition of the *Clavis* for publication at Oxford in 1652. See Stedall 2002, 55–87.

Moreover, what were the first, second, third sequences, etc. in that table in Proposition 132, now here repeated in the same way become the second, fourth, sixth, etc. (because of the interposed spaces to be filled, if possible, with numbers).

		Units		Sides		Triangulars		Pyramidals		Triangulo-triangulars		Triangulo-pyramidals
Units		1		1		1		1		1		1
			□									
Sides		1		2		3		4		5		6
Triangulars		1		3		6		10		15		21
Pyramidals		1		4		10		20		35		56
Triangulo-triangulars		1		5		15		35		70		126
Triangulo-pyramidals		1		6		21		56		126		252

Numbers

And so on

And so on

PROPOSITION 170

Theorem

Two sequences in the table shown, that is, units and sides, are easily interpolated (interposing as many places as one likes); in the former, obviously, by the interposition of one as many times as needed; in the latter, by the same number of arithmetic means.

Thus, one may interpose a single number everywhere; the interpolated sequences of units will be 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1; but of sides $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6, or $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4, $\frac{9}{2}$, 5, $\frac{11}{2}$, 6.

The reasoning is obvious; since the numbers in the former sequence are equals, in the latter arithmetic proportionals.

COMMENT

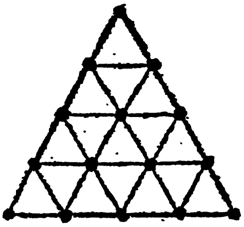
The remaining sequences are not so easily interpolated, except by first finding the true nature of each sequence, which we will investigate in the following Propositions.

PROPOSITION 171

Lemma

Let it be proposed to inquire, what is the ratio of the triangular numbers to their sides.

We will investigate that by this process:



1. If one takes the number of points that any triangular number requires, they can of necessity be displayed in the form of a triangle, and the lines may be joined as in the diagram. It is clear that the complete triangular figure is divided into as many triangles (similar both to the whole and amongst themselves) as the square of the number of the side less one (which may be demonstrated if need be from Euclid's *Elements* VI.19). And therefore if the number of the side is l, the number of small triangles will be $(l-1)^2 = l^2 - 2l + 1$.
2. Since any of these triangles has three angles, the number of these angles will be $3l^2 - 6l + 3$.
3. It must be noted that at three angular points of the whole figure, only the same number of angles adjoin (that is, one each), and therefore those three angles occupy three points, or $3P$.

4. At the remaining points along the sides, there meet three angles, any one of which therefore occupies one third of a point. Moreover, those intermediate points on the sides are $l - 2$ on each side, therefore $3l - 6$ in all (since there are three sides); and the angles adjacent to these intermediate points are $9l - 18$ (since there are three angles to each point), of which any one occupies one third of a point, or $\frac{1}{3}P$. Therefore all together occupy $\frac{9l-18}{3}P$.
5. At the remaining points left over, inside the area of the figure, there meet six angles (that is, six at each) which therefore occupy one sixth of a point. How many are those angles, thus together? The total number of angles is (as we said) $3l^2 - 6l + 3$. Now if there are subtracted 3 (taken at the corners of the whole figure) and $9l - 18$ (adjacent to the points on the sides) there remain $3l^2 - 15l + 18$, which is the number of angles meeting at points inside the area. But since any of them occupies one sixth of a point, or $\frac{1}{6}P$, they occupy together $\frac{3l^2-15l+18}{6}P$.

Finally, if all the points so found are added together, that is, $3P$ and $\frac{9l-18}{3}P$ and $\frac{3l^2-15l+18}{6}P$, their sum will be $\frac{l^2+l}{2}$ points, the number of all the points. That is, the triangular number of side l. Therefore:

PROPOSITION 172

Theorem

The side of any triangular number to the number itself is as l to $\frac{l^2+l}{2}$.

As has been shown in what has gone before.
Therefore, given a side l, there will be given a triangular number belonging to that side, thus $n = \frac{l^2+l}{2}$.
And conversely, given a triangular number, its side may be found.
That is, by solving this equation: $2n = l^2 + l$, we will have $\sqrt{(\frac{1}{4} + 2n)} - \frac{1}{2} = l$.

PROPOSITION 173

Corollary

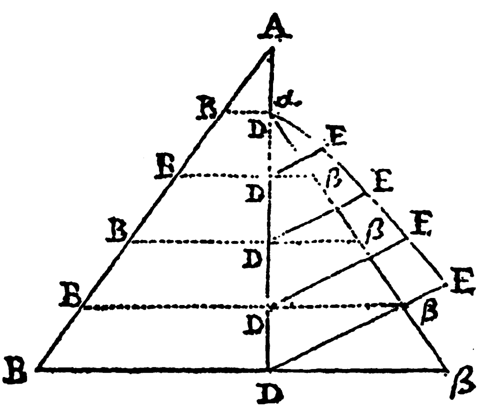
If the transverse diameter of a hyperbola is 1 and its *latus rectum* $\frac{1}{2}$, taking diameters (between the foot of the ordinate and the vertex) 1, 2, 3, 4, 5, etc., the squares of the ordinates will be 1, 3, 6, 10, etc., that is, triangular numbers whose sides are 1, 2, 3, 4, 5, etc.

This may be proved by Propositions 17 or 33 of my *On conic sections*. The figure in the following Proposition shows the hyperbola itself.

PROPOSITION 174

Corollary

In the same way, suppose that on the same line $A\alpha D$ there are two similar triangles $\alpha D\beta$, ADB, and that AD to DB and αD to $D\beta$ are as 1 to $\sqrt{\frac{1}{2}}$, and take equal steps $A\alpha = 1 = \alpha D = DD$ etc. The rectangles $BD\beta$, $BD\beta$, etc. will be to each other as 1, 3, 6, 10, etc., triangular numbers, whose sides are αD, αD, etc. But the mean proportionals between BD and $D\beta$, BD and $D\beta$ etc. are the ordinates of the hyperbola αDE whose transverse diameter is $A\alpha$ and *latus rectum* αB, or equal to it.



Clear from the calculation. For of the rectangles $BD\beta$, the first will be $\sqrt{\frac{1}{2}} \times 2\sqrt{\frac{1}{2}} = 1$. The second $2\sqrt{\frac{1}{2}} \times 3\sqrt{\frac{1}{2}} = 3$. The third $3\sqrt{\frac{1}{2}} \times 4\sqrt{\frac{1}{2}} = 6$. The fourth $4\sqrt{\frac{1}{2}} \times 5\sqrt{\frac{1}{2}} = 10$. And so on. Which are the squares of the ordinates in the hyperbola, by what has gone before. And therefore the mean proportionals $\sqrt{1}$, $\sqrt{3}$, $\sqrt{6}$, $\sqrt{10}$, etc. are the ordinates themselves.

COMMENT

If, moreover, it had been assumed that $AD = DB$ and also $\alpha D = D\beta$, then the rectangles would have been $1 \times 2 = 2$, $2 \times 3 = 6$, $3 \times 4 = 12$, $4 \times 5 = 20$, etc., twice the triangular numbers, and the mean proportionals $\sqrt{2}$, $\sqrt{6}$, $\sqrt{12}$, $\sqrt{20}$, etc. would be the ordinates of a hyperbola, in which both the *latus rectum* and transverse diameter would be 1, which, as has been said, will be clear from consideration. Which is easily accommodated to other ratios of the *latus rectum* to the transverse diameter.

PROPOSITION 175

Theorem

The sequence of triangular numbers in the previous table may be conveniently interpolated if between the sides of those numbers there are interposed as many arithmetic means as are needed, and from these are formed triangular numbers according to Proposition 172.[62]

Thus if in the sequence of triangular numbers 1, 3, 6, 10, 15, etc. a single number is to be everywhere interposed: their sides 1, 2, 3, 4, 5, etc. interpolated by arithmetic means, must be $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, etc. to which sides (by Proposition 172) correspond the triangular numbers $\frac{3}{8}$, 1, $1\frac{7}{8}$, 3, $4\frac{3}{8}$, 6, $7\frac{7}{8}$, 10, $12\frac{3}{8}$, 15, $17\frac{7}{8}$, 21, etc. Or $\frac{3}{8}$, 1, $\frac{15}{8}$, 3, $\frac{35}{8}$, 6, $\frac{63}{8}$, 10, $\frac{99}{8}$, 15, $\frac{143}{8}$, 21, etc. Or finally $\frac{3}{8}$, $\frac{8}{8}$, $\frac{15}{8}$, $\frac{24}{8}$, $\frac{35}{8}$, $\frac{48}{8}$, $\frac{63}{8}$, $\frac{80}{8}$, $\frac{99}{8}$, $\frac{120}{8}$, $\frac{143}{8}$, $\frac{168}{8}$, etc. whose differences are arithmetic proportionals.

In the same way, if two places are to be interposed in a single interval, they will produce the numbers $\frac{2}{9}$, $\frac{5}{9}$, 1, $\frac{14}{9}$, $\frac{20}{9}$, 3, $\frac{35}{9}$, $\frac{44}{9}$, 6, $\frac{65}{9}$, $\frac{77}{9}$, 10, etc. Or $\frac{2}{9}$, $\frac{5}{9}$, $\frac{9}{9}$, $\frac{14}{9}$, $\frac{20}{9}$, $\frac{27}{9}$, etc. whose differences, in the same way, are arithmetic proportionals.

PROPOSITION 176

Lemma

It is proposed to inquire what is the ratio of the pyramidal numbers to their sides.

This proposition also may be investigated by the same process as we used in Proposition 171, which anyone who wishes may try (having observed in the meantime the facts that necessarily distinguish the arrangement of pyramidal numbers from the arrangement of triangular numbers). But since it is not so easy for the reader to conceptualize the necessary placing of points in a pyramid (as not all of them can be positioned in the same plane), or the placing of the solid angles at each point, it seems more satisfactory here to show it by the method that follows. (Which indeed, except that I preferred to show another method, could have been applied also at Proposition 171.)

1. A pyramidal number is equal to a sum of triangular numbers (as is clear from what was said at Propositions 130 and 132), that is, from 1 to the triangular number with the same side as itself, inclusive. (In the same way also, triangular numbers arise from sums of sides; and sides from units; and also triangulo-triangulars from pyramidals; and so on).

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[62] Note the mixture of geometry and algebra in this theorem. Triangles and sides are geometrical concepts, while an arithmetic mean can be constructed either geometrically or algebraically. The final step in Wallis's argument, however, the construction of new triangular numbers from given sides is purely algebraic: there is no physical meaning to a triangular number based on a side of $\frac{1}{2}$ or 1 $\frac{1}{2}$ or 2 $\frac{1}{2}$ points.

2. Moreover, any triangular number of side l is $\frac{l^2+l}{2}$, by Proposition 171.
3. Therefore, having taken the sides 1, 2, 3, 4, etc. or (in their place) a, b, c, d, etc. the sum of a^2+a, b^2+b, c^2+c, d^2+d, etc. (the number of which will be equal to the greatest side) will be twice the sum of triangular numbers; and therefore half of this will be the pyramidal number, whose side is equal to the side of the greatest triangular number.
4. Therefore, the pyramidal number is half the sum of two series, continued from 1 as far as one likes, until the number of places is equal to the side of the pyramidal number sought, which may be supposed l. To which if there is added in front another place 0^2+0 (so that the series is understood to begin from 0) the number of terms will be $l+1$. And the sum of both series is already known from Propositions 2 and 20.
5. That is, the sum of a series of first powers $0+a+b+c$ etc. of which the last is l and the number of terms $l+1$, will be $\frac{l+1}{2}l$, by Proposition 2.
6. And the sum of a series of second powers $0+a^2+b^2+c^2$ etc. of which the last term is l^2 and the number of terms $l+1$, will be $\frac{l+1}{3}l^2+\frac{l+1}{6l}l^2$ or $\frac{l+1}{3}l^2+\frac{l+1}{6}l$, by Proposition 20.
7. Therefore the sum of both together (thus $\frac{l+1}{2}l+\frac{l+1}{3}l^2+\frac{l+1}{6}l$), that is, $\frac{3l^2+3l+2l^3+2l^2+l^2+l}{6}=\frac{2l^3+6l^2+4l}{6}$ is the sum of the two series, half of which sum, $\frac{l^3+3l^2+2l}{6}$ is the pyramidal number of side l. Therefore:

PROPOSITION 177

Theorem

The side of any pyramidal number to the number itself is as l to $\frac{l^3+3l^2+2l}{6}$.

As shown in what has gone before.

But, given a pyramidal number n, its side will not be known except by solving the cubic equation $6n=l^3+3l^2+2l$.

PROPOSITION 178

Theorem

The sequence of pyramidal numbers in the previous table may be conveniently interpolated if between the sides of those numbers there are interposed as many arithmetic means as are needed, and from them are formed pyramidal numbers according to the preceding Proposition.

Thus, if the sides 1, 2, 3, 4, 5, 6, etc. of pyramidal numbers 1, 4, 10, 20, 35, 56, etc. are interpolated in this way: $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6, etc. To these sides will correspond the pyramidal sequence $\frac{5}{16}$, 1, $2\frac{3}{16}$, 4, $6\frac{9}{16}$, 10, $14\frac{7}{16}$, 20, $26\frac{13}{16}$, 35, $44\frac{11}{16}$, 56, etc. or also $\frac{5}{16}$, 1, $\frac{35}{16}$, 4, $\frac{105}{16}$, 10, $\frac{231}{16}$, 20, $\frac{429}{16}$, 35, $\frac{715}{16}$, 56, etc. Or rather, $\frac{15}{48}$, 1, $\frac{105}{48}$, 4, $\frac{315}{48}$, 10, $\frac{693}{48}$, 20, $\frac{1287}{48}$, 35, $\frac{2145}{48}$, 56, etc.

PROPOSITION 179

Lemma

It is proposed to inquire what is the ratio of triangulo-triangular numbers to their sides.

This will be shown by the same method as in Proposition 176. That is:

1. A triangulo-triangular numbers is equal to the sum of all the pyramidal numbers (the sides of which are to be understood as integers, for the interpolation is not yet carried out) from 1 to the number sharing the same side, inclusive.
2. Moreover if the side is l, the pyramidal number is $\dfrac{l^3 + 3l^2 + 2l}{6}$, by Proposition 177.
3. Therefore, having taken sides 1, 2, 3, 4, etc. or (in their place) a, b, c, d, etc. the sum of $a^3 + 3a^2 + 2a$, $b^3 + 3b^2 + 2b$, $c^3 + 3c^2 + 2c$, $d^3 + 3d^2 + 2d$, etc. (the number of all of which will be equal to the greatest side, as is obvious) will be six times the sum of pyramidal numbers; and therefore a sixth of this sum will be the triangulo-triangular number whose side is the same as that of the greatest pyramid.
4. And therefore the triangulo-triangular number is one sixth of the sum of three series, continued from 1 as far as one likes until the number of terms is equal to the side of the required triangulo-triangular number, which may be called l. And therefore, if to that there is added in front another term $0^3 + 0^2 + 0$ (so the series are understood to increase from 0), then the number of terms will be $l + 1$. And the sum of each of those series is already known from Propositions 2, 20 and 40.
5. That is, the sum of twice a series of first powers, $0 + 2a + 2b + 2c$, etc. whose last term is $2l$, and with number of terms $l + 1$, will be $\dfrac{l+1}{2}2l$, by Proposition 2.
6. The sum of three times a series of second powers, $0 + 3a^2 + 3b^2 + 3c^2$, etc. whose last term is $3l^2$, and with number of terms $l + 1$, will be $\dfrac{l+1}{3}3l^2 + \dfrac{l+1}{6l}3l^2$, by Proposition 20.
7. The sum of a series of third powers, $0 + a^3 + b^3 + c^3$, etc. whose last term is l^3, and with number of terms $l + 1$, will be $\dfrac{l+1}{4}l^3 + \dfrac{l+1}{4l}l^3$ by Proposition 40.

Therefore the aggregate of these sums
(thus $\dfrac{l+1}{2}2l + \dfrac{l+1}{3}3l^2 + \dfrac{l+1}{6l}3l^2 + \dfrac{l+1}{4}l^3 + \dfrac{l+1}{4l}l^3$),

that is, $\dfrac{24l^2 + 24l + 24l^3 + 24l^2 + 12l^2 + 12l + 6l^4 + 6l^3 + 6l^3 + 6l^2}{24} =$ $\dfrac{6l^4 + 36l^3 + 66l^2 + 36l}{24}$, is the aggregate of those three series, one sixth of which, $\dfrac{l^4 + 6l^3 + 11l^2 + 6l}{24}$, is the triangulo-triangular number of side l. Therefore:

PROPOSITION 180

Theorem

The side of any triangulo-triangular number to the number itself is as l to $\dfrac{l^4 + 6l^3 + 11l^2 + 6l}{24}$.

Therefore, given side l there is given the triangulo-triangular number, thus:

$$n = \frac{l^4 + 6l^3 + 11l^2 + 6l}{24}.$$

But given a triangulo-triangular number its side will not be found except by solving this equation $24n = l^4 + 6l^3 + 11l^2 + 6l$.

PROPOSITION 181

Theorem

The sequence of triangulo-triangular numbers in the previous table may be conveniently interpolated if between the sides of those numbers there are interposed as many arithmetic means as are needed, and from them are formed triangulo-triangular numbers according to the preceding Proposition.

Thus, to the interpolated sides $\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 5, 5\frac{1}{2}$, 6, etc. correspond the triangulo-triangular numbers $\frac{35}{128}, 1, 2\frac{59}{128}, 5, 9\frac{3}{128}, 15, 23\frac{59}{128}, 35, 50\frac{35}{128}, 70, 94\frac{123}{128}$, 126, etc. or also $\frac{35}{128}$, 1, $\frac{315}{128}$, 5, $\frac{1155}{128}$, 15, $\frac{3003}{128}$, 35, $\frac{6435}{128}$, 70, $\frac{12155}{128}$, 126, etc. Or rather, $\frac{105}{384}$, 1, $\frac{945}{384}$, 5, $\frac{3465}{384}$, 15, $\frac{9009}{384}$, 35, $\frac{19305}{384}$, 70, $\frac{36465}{384}$, 126, etc.

PROPOSITION 182

Lemma

It is proposed to inquire what are the ratios of subsequent sequences of figurate numbers to their sides, that is, triangulo-pyramidal numbers, pyramido-pyramidal numbers, etc.

It is possible indeed for this to be shown by the same method that we used in Propositions 176 and 179, with the help of the Propositions noted there, that is, Propositions 2, 20 and 40 together with Proposition 43, at least if we would first pursue further the teaching of ratios of finite series of fourth, fifth, sixth (and subsequent) powers to a series of equal terms, which teaching we only briefly indicated at Proposition 43. But if anyone wants that further continuation, he may demonstrate it by another method as best pleases him, or also (unless better aids occur to him) by the help of the table itself, which we already have in hand, after which, by a way soon to be taught, we will show how to investigate the ratios of figurate numbers, to the sides of any of them, in subsequent sequences. For as at Propositions 176 and 179, from known ratios of simple finite series (thus, by Propositions 2, 20 and 40, of first powers, second powers and third powers, to series of equals) there may be investigated the ratios of this table (thus, triangular, pyramidal, triangulo-triangular numbers to their respective sides), so in turn, the latter known, the former also may be sought out, and therefore the teaching of Proposition 43 may be continued as far as one likes.

Because, moreover (as we said), that was only touched on at Proposition 43. Nor indeed is it necessary to the present purpose to proceed with it further, since from the known formulae[63] of a few sequences of this table (or the ratios of those figurate numbers to their respective sides) a method of investigating the formulae also of the subsequent sequences will begin to appear, so that I may now operate more easily by that, as is given here.

The formula for each sequence of numbers is clear from what has gone before.

Units	1
Sides	l
Triangulars	$\frac{l^2+l}{2}$
Pyramidals	$\frac{l^3+3l^2+2l}{6}$
Triangulo-triangulars	$\frac{l^4+6l^3+11l^2+6l}{24}$

It is also clear, looking more closely, that those formulae arise from continued multiplication of these quantities:

$$1\times\frac{l}{1}\times\frac{l+1}{2}\times\frac{l+2}{3}\times\frac{l+3}{4}\text{ etc.}\quad\text{or}\quad 1\times\frac{l\times(l+1)\times(l+2)\times(l+3)}{1\times2\times3\times4}$$

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[63] *Serierum characteribus*, literally ' from the properties of the sequences'. Since for Wallis the properties of each sequence had now come to be defined by an algebraic formula, *character* is translated from here onwards by 'formula'.

For

$$1 \times \frac{l+0}{1} = l$$

$$l \times \frac{l+1}{2} = \frac{l^2+l}{2}$$

$$\frac{l^2+l}{2} \times \frac{l+2}{3} = \frac{l^3+3l^2+2l}{6}$$

$$\frac{l^3+3l^2+2l}{6} \times \frac{l+3}{4} = \frac{l^4+6l^3+11l^2+6l}{24}$$

And therefore, if the multiplication of the ratio last discovered is continued further, by $\frac{l+4}{5} \times \frac{l+5}{6} \times \frac{l+6}{7}$ etc. we will have the formulae of the subsequent sequence.

$$\text{Thus} \quad \frac{l^5+10l^4+35l^3+50l^2+24l}{120}$$

$$\text{and} \quad \frac{l^6+15l^5+85l^4+225l^3+274l^2+120l}{720}$$

$$\left[\text{and} \quad \frac{l^7+21l^6+175l^5+735l^4+1624l^3+1764l^2+720l}{5040}\right]^{64}$$

And so on, as far as you please.[65]

COMMENT

In this way we say, from the continuation of the ratios or formulae of the present table, it is possible to deduce also the continuation of those ratios indicated in Proposition 43.[66] But since that will perhaps not be obvious to

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[64] This formula was included only in the edition reprinted in the *Opera mathematica* in 1695; it is included here for completeness.

[65] These formulae were first written down symbolically, almost exactly as Wallis has them here, by Thomas Harriot about fifty years earlier, except that Harriot used *nn* etc. where Wallis later wrote l^2 etc.; see British Library Add MS 6782, ff. 108, reproduced in Lohne 1979, 294. Harriot also discovered the same method of generating the numbers by successive multiplication. The formulae and the method of generating them were also known to Fermat who, however, expressed the results verbally: 'The last side multiplied by the next greater makes twice the triangle. The last side multiplied by the triangle of the next greater side makes the three times the pyramid. The last side multiplied by the pyramid of the next greater side makes four times the triangulo-triangle. And so on by the same progression ad infinitum'; Fermat to Roberval, 4 November 1636, Fermat 1891–1912, II, 84–85, see also Mahoney 1974, 230.

[66] From his formulae for figurate numbers Wallis is about to derive further results on sums of powers. It seems that Fermat was in possession of the same facts but worked the other way round: from sums of powers to formulae for figurate numbers: see Mahoney 1974, 229–233.

all, I have considered it worth a little trouble to show it in passing. Which indeed, may be introduced here without inconvenience, in a way that will perhaps not be unwelcome to some. Therefore for example:

Let it be proposed to inquire what is the ratio of a finite series of fourth powers (beginning from 0) to a series of the same number of terms equal to the greatest.

1. The formula for the triangulo-triangular numbers (by Proposition 180) is $\frac{l^4+6l^3+11l^2+6l}{24}$, and for the triangulo-pyramidals $\frac{l^5+10l^4+35l^3+50l^2+24l}{120}$, by Proposition 182.
2. Moreover, (as has often been said), a figurate number of any degree (in the present table) is the sum of all those preceding in the degree nearest to it. Therefore a triangulo-pyramidal number is a sum of triangulo-triangular numbers.
3. And therefore, taking sides 1, 2, 3, etc. or (in their place) a, b, c, etc. (of which the greatest may be called l), and forming from them triangulo-triangular numbers, their sum will be the triangulo-pyramidal number of the same side, that is $\frac{l^5+10l^4+35l^3+50l^2+24l}{120}$.
4. Moreover, the sum of the series $0+6a+6b+6c$ etc. is (by Proposition 2) $\frac{l+1}{2}6l = \frac{6l^2+6l}{2}$.
5. The sum of the series $0+11a^2+11b^2+11c^2$ etc. is (by Proposition 20) $\frac{l+1}{3}11l^2 + \frac{l+1}{6l}11l^2 = \frac{11l^3+11l^2}{3} + \frac{11l^2+11l}{6} = \frac{22l^3+33l^2+11l}{6}$.
6. The sum of the series $0+6a^3+6b^3+6c^3$ etc. is (by Proposition 40) $\frac{l+1}{4}6l^3 + \frac{l+1}{4l}6l^3 = \frac{6l^4+6l^3}{4} + \frac{6l^3+6l^2}{4} = \frac{6l^4+12l^3+6l^2}{4}$.
7. These three sums collected into one are $\frac{6l^2+6l}{4} + \frac{22l^3+33l^2+11l}{6} + \frac{6l^4+12l^3+6l^2}{4} = \frac{9l^4+40l^3+60l^2+29l}{6}$.
8. If, therefore, from twenty-four times the sum of all of them, is taken the sum of the three series,

that is, if from $(l^5+10l^4+35l^3+50l^2+24l)/5$
is taken $(9l^4+40l^3+60l^2+29l)/6$
that is, from $(6l^5+60l^4+210l^3+300l^2+144l)/30$
is taken $(45l^4+200l^3+300l^2+145l)/30$

there remains the sum of the fourth series, that is, of fourth powers

$$(6l^5+15l^4+10l^3+00l^2-1l)/30$$

That is, one thirtieth of

$$
\begin{array}{llll}
6l^5 + 6l^4 & & & \\
+\,9l^4 + 9l^3 & & & \\
& +\,1l^3 + 1l^2 & & \\
& & -\,1l^2 - 1l &
\end{array}
$$

That is $\frac{l+1}{5}l^4 + \frac{3l+3}{10}l^3 + \frac{l+1}{30}l^2 - \frac{l+1}{30}l$.

Which is therefore the sum of the series of fourth powers whose last term is l^4, with number of terms $l+1$.

Or, if for the number of terms $l+1$, there is substituted m, and therefore a series of equal terms ml^4, the series of fourth powers will be $\frac{1}{5}ml^4 + \frac{3}{10}ml^3 + \frac{1}{30}ml^2 - \frac{1}{30}ml$ (if, that is, the first term is 0, the second 1), or $\frac{ml^4}{5} + \frac{3ml^4}{10l} + \frac{ml^4}{30l^2} - \frac{ml^4}{30l^2}$.

9. Therefore a finite series of fourth powers, to a series of the same number of terms equal to the greatest, is as $\frac{1}{5} + \frac{3}{10l} + \frac{1}{30l^2} - \frac{1}{30l^2}$ to 1. Which is what was sought.[67]

And in the same way, these being known, the ratio of a series of fifth powers to a series of equals may be found with the help of the formula for the next series of the table. And thence the ratio of a series of sixth powers, with the help of the formula for the next following series in this table, and so on as far as one likes.

Moreover, that this may be better understood, it will perhaps be worth the effort to open up a little more precisely what has already been taught. For although it seems to me that I have taught it sufficiently clearly, it may be, nevertheless, that the reader less accustomed to those things might perhaps sometimes hesitate.

It should be noted, therefore, that here (as also everywhere else, where we speak of finite series) we make the number of terms $l+1$ (thus, if the first term is 0, the second is said to be 1), one more than the number of steps by which the last term is reached; that is, [than the number] of all the differences of the terms taken continually, the sum of all of which is equal to the greatest term, whether those differences are equal, as in a series of first powers (thus, 1, 1, 1, 1, etc., differences of arithmetic proportionals), or increasing, as in as series of second powers, third powers, etc. (thus, 1, 3, 5, 7, etc., differences of squares; or 1, 7, 19, 37, etc., differences of cubes, etc.), or even decreasing, as in a series of second roots, third roots, etc. (since, for example, the difference $\sqrt{3} - \sqrt{2}$ is less than $\sqrt{2} - \sqrt{1}$, and this less than $\sqrt{1} - \sqrt{0}$, etc. and so on for the rest). The number l of these differences (in any series) is one less than the number of terms, as is obvious. And the sum of all (because of the nullity

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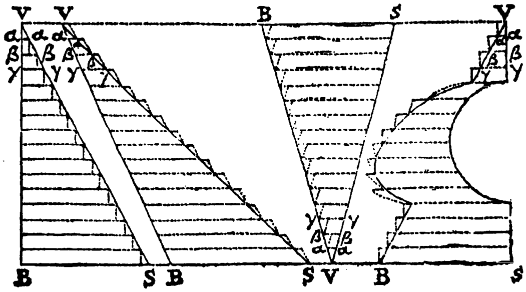
[67] When the *Arithmetica infinitorum* was reprinted in 1695 Wallis added further (lengthy) calculations for the sums of sequences of fifth and sixth powers; see Wallis 1695, I, 449–452.

of the first term, 0) is the greatest term itself. Moreover, where the number or quantity of terms is called m, the number of differences, or of steps to the greatest, will be $m - 1$.



Take, for example, a series of first powers, which is like a sum of parallelograms of equal altitude filling the figure of a circumscribed triangle; of which, if the first is called 0 (that is, of no width, although of the same altitude as the rest), the second 1, etc. and the number of all is 16, there will be 15 differences (equal to each other, that is everywhere 1), and the greatest term therefore 15. And therefore since in each parallelogram the common altitude is $\frac{1}{16}VB$ (see figure 1) and the continual increase in width $\frac{1}{16}BS$, all the altitudes taken at once, that is, the altitude of the inscribed figure, are $VB = \frac{16}{16}VB$, but at the same time all the increases in width, that is, the base of the inscribed figure, are not BS, but $\frac{15}{16}BS$, or $BS - \frac{1}{16}BS$. If, moreover, one proceeds yet one step further, adjoining under the base one further parallelogram, we will indeed have the width of BS precisely, but the altitude now becomes augmented, that is $VB + \frac{1}{16}VB$. But if (in figure 2) the figure is taken to be circumscribed by parallelograms, then first the altitudes of all the parallelograms taken together, that is, the altitude of the circumscribed figure, are VB (which is now to be imagined perpendicular to the base), and then all the parts of the width taken together, that is, the base of the circumscribed figure, are BS precisely. But now the series begins not from 0, but 1; but if this series is continued one step further above the vertex (so as to begin from 0) the altitude thus increased will now be $VB + \frac{1}{16}VB$, as is obvious. And therefore the inscribed figure continued one step below the base and the circumscribed figure continued one step above the vertex amount to the same thing.

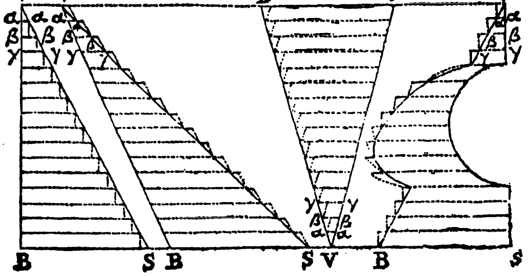
And the total in this triangle (and by the same reasoning in other figures, unless the steps are unequal) is sufficiently evident. For (besides that it is clear enough from what has already been said) if in the triangle there are taken any number of lines parallel to the base (in which count we wish to include the base itself, and the point of the vertex) and the same number of parallelograms adjacent to them, then if all those are assumed to lie under their lines, the lowest of them will be under the base; if above, the highest will be above the vertex. If, moreover, we suppose that those lines lie neither at the top nor the bottom of the parallelograms but pass through the middle of them, then both the highest and lowest of them will be part in, part out, of the triangle. Therefore whatever place we suppose those lines to have with respect to the parallelograms, the figure constituted from the parallelograms (so long as it begins from 0) will either have its base a little less or its altitude a little greater than has that true triangle.

And this excess or defect, as long as one is dealing with finite series, must be wholly taken account of. Where, however, one is dealing with infinite series, it may be safely neglected. For since the more terms there are assumed, the smaller becomes the difference of either the base or the altitude, if one proceeds to infinity it vanishes; indeed $\frac{1}{\infty}$ (an infinitely small part) may be taken for nothing (at least, observing some limitations of which we will soon speak). Thus, for example, if a triangle with altitude A, base B, is inscribed with parallelograms, in each of which the altitude is $\frac{1}{\infty}A$, and the increase of width is $\frac{1}{\infty}B$, the inscribed altitude will be $\infty \times \frac{1}{\infty}A$, and the base not B but $B - \frac{1}{\infty}B$. For the number of altitudes is ∞ and of differences $\infty - 1$. But if the figure so inscribed is continued one step below the base, or the circumscribed one step above the vertex, the base will be $\infty \times \frac{1}{\infty}B = B$, the altitude $A + \frac{1}{\infty}A$; indeed the number of increments is ∞ and of altitudes $\infty + 1$. Where therefore one deals with finite series, by the altitude and base must be understood the altitude and base of the adscribed figure (whether inscribed or circumscribed) not, however, of that to which it is adscribed; but in an infinite series it is all the same whether one understands the former or the latter, since the difference is infinitely small, and therefore vanishing or zero. For $\infty, \infty + 1, \infty - 1$, amount to the same thing. And just as when a polygon with infinitely many sides is taken for a circle, it is all the same whether it is understood inscribed or circumscribed (that is, whether the radius is supposed equal to a line from the centre to a vertex, or to one taken from the centre to the middle of a side, the difference of which, because of the infinite number of sides, is infinitely small), so in our adscription it is all the same (because of the infinitely small difference) whether the altitude or base of the inscribed or circumscribed figure is taken for the true one. And indeed as in an inscribed or circumscribed polygon with infinitely many sides, the sides are supposed equal to each other, that is, their right sines and tangents are equal both to each other and to the arcs themselves, so here also both the bases and altitudes of the figures consisting of inscribed or circumscribed parallelograms must be supposed equal both among themselves and to that of the adscribed figure; that is, if one wishes to speak precisely, not to differ except by an infinitely small part.

In the same way, in the figure in Proposition 5, in the figure inscribed by similar sectors of spirals, if the number of sectors is finite, it will be a finite series, whose first term is 0, but the last is the last of the inscribed sectors (whose radius is one part less than that of the last circumscribing sector), and the arcs of all those sectors taken together equal half the arc of the coterminous circle, that is, coterminous with the figure consisting of sectors, not coterminous with the true spiral. And if the number of sectors (just as was supposed there) is supposed infinite, the arcs of all the sectors taken together to this point will be equal to half the arc of the coterminous circle, that is, coterminous with the figure consisting of these infinitely many sectors; which [arc] however is either itself identical with that coterminous with the true spiral, or certainly less than it by an infinitely small part of itself (that is, nothing). But if instead of inscribed sectors there are taken circumscribed, the arc of the coterminous circle will be increased by one part of itself (whether the number of sectors is finite or infinite) so that half of it will be equal to that to be had by taking all the arcs of the sectors together, and therefore it must be supposed to have begun one step before the beginning of the true spiral, so that the arc of the first sector is 0. For in arithmetic proportionals, unless the first term is 0, the sum of all will not be equal to half the last multiplied by the number of terms.

Moreover, what has been shown in these figures, may be understood (with appropriate changes) of any others, that is, the number of terms (if begun from 0) will be one more than the number of differences, or small parts, from which the greatest term is constituted (whether those differences are equal or unequal); and therefore if in the adscribed figure (whether inscribed or circumscribed) the base is taken equal to the base of the proposed figure, to which it is adscribed, (whether by continuing the inscribed one step below the base, or the circumscribed one step above the vertex), the altitude of the former will be one part greater than the altitude of the latter (whether that part is finite or infinite). Where, moreover, the number of parts of the altitude of the latter is assumed infinite, it will be in the former $\infty + 1$; or if the altitude of the latter is A, that of the former will be $A + \frac{1}{\infty}A$. If in the former it is A (as we usually put it), it will be in the latter $A - \frac{1}{\infty}A$, which however (at infinity) amounts to the same thing on account of the infinitely small difference.

But when we say that an infinitely small part may be taken as nothing, this must be received with caution, for this does not hold everywhere, but sometimes offers occasion to lapse. Since from an infinitely small part multiplied infinitely there sometimes arises a sufficiently large quantity, namely, that of which that part was a divisor, although infinitely small. For $\frac{1}{\infty} \times \infty = 1$ and $\frac{1}{\infty}A \times \infty = A$.



We have shown an example in the *Comment* to Proposition 13. If (in figures 1 and 2) one were to conclude, because the sides of the infinite number of parallelograms (constituting line *VB*) and the sides of the trapezia (completing *VS*) taken piece by piece do not differ from each other except by an infinitely small part (since both the latter and the former are infinitely small, indeed $\frac{1}{\infty}$ of the lines *VB, VS*), that this therefore is to be discounted, and that the sides of the parallelograms and trapezia are to be said to be equal; and that therefore (since from the addition of equals to equals, the sums are equal) the infinite number of the former is equal to the infinite number of the latter, that is, all of *VS* is equal to all of *VB*. This is clearly a paradox (into which, nevertheless, one is quite inclined to fall unless one takes care). For although the differences piece by piece are infinitely small (that is, $\frac{1}{\infty} VS - \frac{1}{\infty} VB$), nevertheless the sum of all (an infinite number), has a sufficiently noteworthy magnitude, that is, $SV - VB$.

And meanwhile in the same parallelograms and trapezia (if we look at the area) not only do they have infinitely small differences taken piece by piece, but also the sum of the former and the sum of the latter (that is, an infinite number of parallelograms taken together and an infinite number of trapezia) differ from each other only by an infinitely small part, which does not hold for their sides.

The reason for the distinction is this: since where one deals with the comparison of sides [of parallelograms and trapezia], taking any two respectively, although the difference is less as the number of all is greater, yet it is always the same ratio by which each difference is diminished as the number of differences is increased; and therefore the sum of the differences, to be divided, is not diminished. But where one is dealing with areas, not only are the differences of any two (trapezia and parallelograms) taken respectively diminished, but also the sum of all of them; and indeed the more differences there are, the less is the sum of them, until at length not only does each [parallelogram] differ infinitely little from each [trapezium] (which it would not be sufficient to have demonstrated) but so also do all [the parallelograms] from all [the trapezia] taken together, as is clear from the demonstrations. And this I have considered worth the trouble of noting somewhat more fully, because in this place I have noticed some are inclined to fall.

Lest, moreover, anyone here suspects this danger, that while we have the accurate altitude of any figure, we also have the same increased by an infinitely small part of itself, this one thing may sufficiently restore their security, that, other things being equal, the increase of altitude of any figure (whether plane or solid) increases the area or size only in the same ratio. And therefore where the increase in altitude is only some infinitely small part of itself, the increase of the whole figure will also be only in the same ratio, that is, by some infinitely small part of itself, or $\frac{1}{\infty}$ of the whole figure; because (since there are taken so many at a time but not infinitely many) the space will be less than any assigned quantity, and may therefore be taken as nothing.

But finally it may be asked why I choose the inscribed figure rather than the circumscribed, therefore beginning almost everywhere from 0 rather than 1? Particularly since the circumscribed figure (not continued one step above the vertex so that it begins from 0, but rather from 1) has precisely both the same base and altitude as that to which it is circumscribed, whether in series of first, or second, or any subsequent powers, and whether a finite or infinite series?

I say it is indeed possible for what we have dealt with to be done by either method, that is, by inscribed or circumscribed figures (which we also pointed out above at Proposition 43 which gave the opportunity for all this *Comment*, indeed the greater part of it could not be conveyed more quickly since it depends on the Proposition immediately preceding this). Therefore, for example, a series of first powers may be denoted indifferently by 1, 2, 3, etc. or by 0, 1, 2, etc. for the first term 0 adds nothing to the sum of the rest. And indeed I already at one time set out my lemmas by both methods, although either was sufficient for our demonstration, so I did not think the reader should be burdened with both, especially since I was mostly looking at infinite series, and have scarcely made use of finite series other than in lemmas to have at hand for theorems of infinite series.

And meanwhile circumscribed figures, if the thing is weighed carefully, are no more like the figures by which they are circumscribed, than are inscribed. For example, the inscribed agrees with the given figure as to altitude and width at the vertex but differs as to the base (that is, the width at the lowest point); the same inscribed figure continued one step below the base (or the circumscribed so continued above the vertex) agrees with the given figure as to base, and width at the vertex, but differs as to altitude. But the circumscribed (not continued) indeed agrees with the given figure as to base and altitude but not as to width at the vertex, which in one is 0, in the other 1.

Since, therefore, to this extent circumscribed and inscribed figures behave indifferently as far as our business is concerned, I prefer our series to begin with 0 rather than 1, partly because although an inscribed figure seems to be better suited, nevertheless both can be adjusted (as has already been said), whether it is supposed continued above the vertex or below the base; partly because in this way (since the lowest term is 0) the sum of the extremes is the same as the greatest term; but especially so that I can, without going a long way round in words, understand under the name of a series of first powers not only 0, 1, 2, 3, etc. but also 0, 2, 4, 6, etc. or 0, 3, 6, 9, etc. or 0, 4, 8, 12, etc. and similarly others beginning from 0, whatever the second term; and under the name of a series of second powers not only 0, 1, 4, 9, etc. but also 0, 2, 8, 18, etc. or 0, 3, 12, 27, etc. and similarly others. And the same in subsequent series.

If anyone however prefers to begin their series from 1, they may set out the results in this manner.

Series of first powers:

$$\frac{0+1}{1+1}=\frac{1}{2}$$
$$\frac{0+1+2}{2+2+2}=\frac{1}{2}$$
$$\frac{0+1+2+3}{3+3+3+3}=\frac{1}{2}$$
$$\frac{0+1+2+3+4}{4+4+4+4+4}=\frac{1}{2}$$
$$\frac{0+1+2+3+4+5}{5+5+5+5+5+5}=\frac{1}{2}$$

etc.

$$\frac{1}{2}=\frac{1}{2}$$
$$\frac{1+2}{3+3}=\frac{1}{2}$$
$$\frac{1+2+3}{4+4+4}=\frac{1}{2}$$
$$\frac{1+2+3+4}{5+5+5+5}=\frac{1}{2}$$
$$\frac{1+2+3+4+5}{6+6+6+6+6}=\frac{1}{2}$$

etc.

Or also:

$$\frac{0+1}{2+2}=\frac{1}{2}-\frac{1}{4}$$
$$\frac{0+1+2}{3+3+3}=\frac{1}{2}-\frac{1}{6}$$
$$\frac{0+1+2+3}{4+4+4+4}=\frac{1}{2}-\frac{1}{8}$$

$$\frac{0+1+2+3+4}{5+5+5+5+5}=\frac{1}{2}-\frac{1}{10}$$

etc.

$$\frac{1+2}{2+2}=\frac{1}{2}+\frac{1}{4}$$
$$\frac{1+2+3}{3+3+3}=\frac{1}{2}+\frac{1}{6}$$
$$\frac{1+2+3+4}{4+4+4+4}=\frac{1}{2}+\frac{1}{8}$$

$$\frac{1+2+3+4+5}{5+5+5+5+5}=\frac{1}{2}+\frac{1}{10}$$

etc.

Series of second powers:

$$\frac{0+1}{1+1}=\frac{1}{3}+\frac{1}{6}$$
$$\frac{0+1+4}{4+4+4}=\frac{1}{3}+\frac{1}{12}$$
$$\frac{0+1+4+9}{9+9+9+9}=\frac{1}{3}+\frac{1}{18}$$
$$\frac{0+1+4+9+16}{16+16+16+16+16}=\frac{1}{3}+\frac{1}{24}$$

etc.

$$\frac{1}{4}=\frac{1}{3}-\frac{1}{12}$$
$$\frac{1+4}{9+9}=\frac{1}{3}-\frac{1}{18}$$
$$\frac{1+4+9}{16+16+16}=\frac{1}{3}-\frac{1}{24}$$
$$\frac{1+4+9+16}{25+25+25+25}=\frac{1}{3}-\frac{1}{30}$$

etc.

Or also:

$$\frac{0+1}{4+4}=\tfrac{1}{3}-\tfrac{5}{24}$$
$$\frac{0+1+4}{9+9+9}=\tfrac{1}{3}-\tfrac{12}{81}$$
$$\frac{0+1+4+9}{16+16+16+16}=\tfrac{1}{3}-\tfrac{22}{192}$$
$$\frac{0+1+4+9+16}{25+25+25+25+25}=\tfrac{1}{3}-\tfrac{35}{375}$$

etc.

$$\frac{1+4}{4+4}=\tfrac{1}{3}+\tfrac{7}{24}$$
$$\frac{1+4+9}{9+9+9}=\tfrac{1}{3}+\tfrac{15}{81}$$
$$\frac{1+4+9+16}{16+16+16+16}=\tfrac{1}{3}+\tfrac{26}{192}$$
$$\frac{1+4+9+16+25}{25+25+25+25+25}=\tfrac{1}{3}+\tfrac{40}{375}$$

etc.

Series of third powers:

$$\frac{0+1}{1+1}=\tfrac{1}{4}+\tfrac{1}{4}$$
$$\frac{0+1+8}{8+8+8}=\tfrac{1}{4}+\tfrac{1}{8}$$
$$\frac{0+1+8+27}{27+27+27+27}=\tfrac{1}{4}+\tfrac{1}{12}$$
$$\frac{0+1+8+27+64}{64+64+64+64+64}=\tfrac{1}{4}+\tfrac{1}{16}$$

etc.

$$\tfrac{1}{8}=\tfrac{1}{4}-\tfrac{1}{8}$$
$$\frac{1+8}{27+27}=\tfrac{1}{4}-\tfrac{1}{12}$$
$$\frac{1+8+27}{64+64+64}=\tfrac{1}{4}-\tfrac{1}{16}$$
$$\frac{1+8+27+64}{125+125+125+125}=\tfrac{1}{4}-\tfrac{1}{20}$$

etc.

And similarly in the subsequent series, which I have abstained from listing lest I over extend myself. The reader may, if he wishes, with no great work either turn these arguments into theorems, or generate other similar ones for the subsequent series, if he has paid attention to what I have already taught.

But there is also yet another way of setting out the series up to here (if the reader is attracted by diversity), which will also sometimes be no less useful. If, that is, the series is begun neither from 0 (as in the inscribed figure) nor from 1 (as in the circumscribed figure) but from an intermediate quantity, thus $\frac{1}{2}$ (which therefore represents a figure intermediate between an inscribed and circumscribed, or greater than an inscribed and less than a circumscribed), to give, for example, a series of first powers $\frac{1}{2}+1\frac{1}{2}+2\frac{1}{2}+3\frac{1}{2}$ etc. or (which derives from that) $1+3+5+7$ etc. In which case the argument for first powers is to be set out thus.

$$\frac{\frac{1}{2}}{1}=\frac{1}{2}$$

$$\frac{\frac{1}{2}+1\frac{1}{2}}{2+2}=\frac{1}{2}$$

$$\frac{\frac{1}{2}+1\frac{1}{2}+2\frac{1}{2}}{3+3+3+3}=\frac{1}{2}$$

$$\frac{\frac{1}{2}+1\frac{1}{2}+2\frac{1}{2}+3\frac{1}{2}}{4+4+4+4+4}=\frac{1}{2}$$

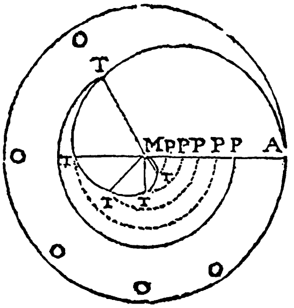
$$\frac{1}{2}=\frac{1}{2}$$

$$\frac{1+3}{4+4}=\frac{1}{2}$$

$$\frac{1+3+5}{6+6+6}=\frac{1}{2}$$

$$\frac{1+3+5+7}{8+8+8+8}=\frac{1}{2}$$

And indeed this method best of all suits Propositions 15 and 16 where we compare the figure contained in the spiral with that in a parabola. For if in the spiral figure, having taken any number of lines *MT* making angles successively equal to each other, there are supposed sectors inscribed in each space, their arcs will be as 0, 1, 2, 3, etc., but if circumscribed, as 1, 2, 3, 4, etc. But if they are applied so that the arcs of the sectors are bisected by the spiral, they will be as $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, etc. or as 1, 3, 5, 7, etc. (that is, as differences of square numbers).



And therefore if those arcs, (whether finite or infinite in number, although infinite in the same way that words are facts: I have considered this curiosity should there be omitted) are supposed taken in line and continued in turn so as to make the same number of segments of the diameter of a parabola (placed continuously), whence the intercepted diameters come out to be 1, 4, 9, 16, etc. (for $1+3=4, 1+3+5=9$, etc.) to which correspond ordinates (as square roots of the diameters) which will be to each other as 1, 2, 3, 4, etc., that is, as those lines *MT, MT,* etc. themselves passing through the ends of similar sectors in the true spiral.

And by this method one may compare a figure consisting not only of an infinite number of sectors (which we did there), but also of a finite number, contained in the spiral, with the figure made from the same number of parallelograms contained in the parabola. Which indeed (without a new figure) may be sufficiently understood. If those arcs of sectors are denoted $1a, 3a, 5a$, etc. and the true radii (proportional to those) $2r, 6r, 10r$, etc. the sectors will be $1ar, 9ar, 25ar$, etc. (that is, half the product of the radius and arc respectively). Having taken continual segments along the diameter of a parabola in the same way, $1a, 3a, 5a$, etc., and therefore intercepted diameters $1a, 4a (= 1a + 3a), 9a (= 4a + 5a)$, etc. and ordinates that correspond to those diameters (as their square roots), $2r, 4r, 6r$, etc. (which divide spaces not of equal altitude,[68] but whose altitudes are in arithmetic proportion, as 1, 3, 5, etc., as is obvious), the parallelograms inscribed in those spaces will be $1a \times 0r, 3a \times 2r, 5a \times 4r$, etc. or $0ar, 6ar, 20ar$, etc. or circumscribed will be $1a \times 2r, 3a \times 4r, 5a \times 6r$, etc., or $2ar, 12ar, 30ar$, etc. Moreover, those intermediate, part inscribed, part circumscribed (which, that is, have a width,[69] that is the arithmetic mean between the two ordinates bounding the space) or (which amounts to the same thing) inscribed trapezia, will be $1a \times 1r, 3a \times 3r, 5a \times 5r$, etc., or $1ar, 9ar, 25ar$, etc. equal one by one to the proposed sectors; of which the arcs, that is, are equal to the altitudes of the parallelograms (that is, to the segments of the diameter of the parabola), but the radii twice the widths of the parallelograms; or if the radii of the sectors are equal to the widths of the parallelograms, the parallelograms will be twice the sectors.

But it is time I put an end to this extended *Comment*; since why I have omitted anything here was explained above.

PROPOSITION 183

Theorem

The side of any figurate number, in any sequence of the given table (of Proposition 132) continued as far as one likes, will have a known ratio to its figurate number.

That is, as indicated in the preceding Proposition.

PROPOSITION 184

Theorem

And therefore it will not be difficult to interpolate the subsequent sequences in the given table continued as far as one likes.

That is, having found the proper formula of each by Proposition 182, the interpolation may be done as in Propositions 175, 178 and 181.

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[68] The 'altitude' of each space is the length of the segment along the diameter of the parabola.

[69] The 'width' of each parallelogram (actually a rectangle) is the length of the ordinate that bounds it.

			Units		Sides		Triangulars		Pyramidals		Triangulo-triangulars	
Numbers		∞	1		$\frac{1}{2}$		$\frac{3}{8}$		$\frac{15}{48}$		$\frac{105}{384}$	
	Units	1	1	1	1	1	1	1	1	1	1	1
			1	□	$1\frac{1}{2}$		$1\frac{7}{8}$		$2\frac{9}{48}$		$2\frac{177}{384}$	
	Sides	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	l
			1		$2\frac{1}{2}$		$4\frac{3}{8}$		$6\frac{27}{48}$		$9\frac{9}{384}$	
	Triangulars	$\frac{3}{8}$	1	$1\frac{7}{8}$	3	$4\frac{3}{8}$	6	$7\frac{7}{8}$	10	$12\frac{3}{8}$	15	$\frac{l^2+l}{2}$
			1		$3\frac{1}{2}$		$7\frac{7}{8}$		$14\frac{21}{48}$		$23\frac{177}{384}$	
	Pyramidals	$\frac{15}{48}$	1	$2\frac{9}{48}$	4	$6\frac{27}{48}$	10	$14\frac{21}{48}$	20	$26\frac{39}{48}$	35	$\frac{l^3+3l^2+2l}{6}$
			1		$4\frac{1}{2}$		$12\frac{3}{8}$		$26\frac{39}{48}$		$50\frac{105}{384}$	
	Triangulo-triangulars	$\frac{105}{384}$	1	$2\frac{177}{384}$	5	$9\frac{9}{384}$	15	$23\frac{177}{384}$	35	$50\frac{105}{384}$	70	$\frac{l^4+6l^3+11l^2+6l}{24}$

And so on

The same table, as was promised, will be here shown thus interpolated.

Or otherwise more expediently: after the interpolation of sequences both horizontally and vertically as far as one likes has been begun by Propositions 170 and 175, etc., one may continue it further as far as one likes solely by the summation of numbers already found; for not only the numbers in the table in Proposition 132 (where we pointed out the same thing), but also those interposed by interpolation, arise from the summation of two others, one above, one to the left (not indeed adjacent as in Proposition 132, because of the place now interpolated) but taken after one interpolated place. As will be clear from inspection.

What has already been said, moreover, in various preceding propositions about the interpolation of one place in each space may also be easily accommodated, with appropriate changes, to two or three or more interposed places.

COMMENT

It must be noted here that it is possible to accomplish all this work of interpolation so far shown (even without finding the correct formula for any sequence) with the help of the reminders to be had in the *Comments* to Propositions 126 and 154. That is, by first interpolating the vertical sequences and then repeating the interpolations in the same way in the horizontal sequences. But while it is not injudicious to investigate the formula for each distinct sequence, and the reader may not perhaps be ungrateful, it may please to him to proceed by another method rather than that used.

Since this has arisen, moreover, it is clear from the interpolation carried out between each sequence, whether vertically of horizontally, in the table of Proposition 132, that new sequences have already emerged amongst them, not yet complete, however, but with gaps. And indeed that place (signified by the symbol □) whose completion I wish for the most, remains as yet empty. If, moreover, it was given to fill any one of those empty places, then the rest could be filled without difficulty, as will be clear from Proposition 188.

But since the table in Proposition 132 is now to be had interpolated by new sequences, in order that the interposed sequences have their appropriate titles according to the scheme of that table, this following Proposition is to be noted.

PROPOSITION 185

Theorem

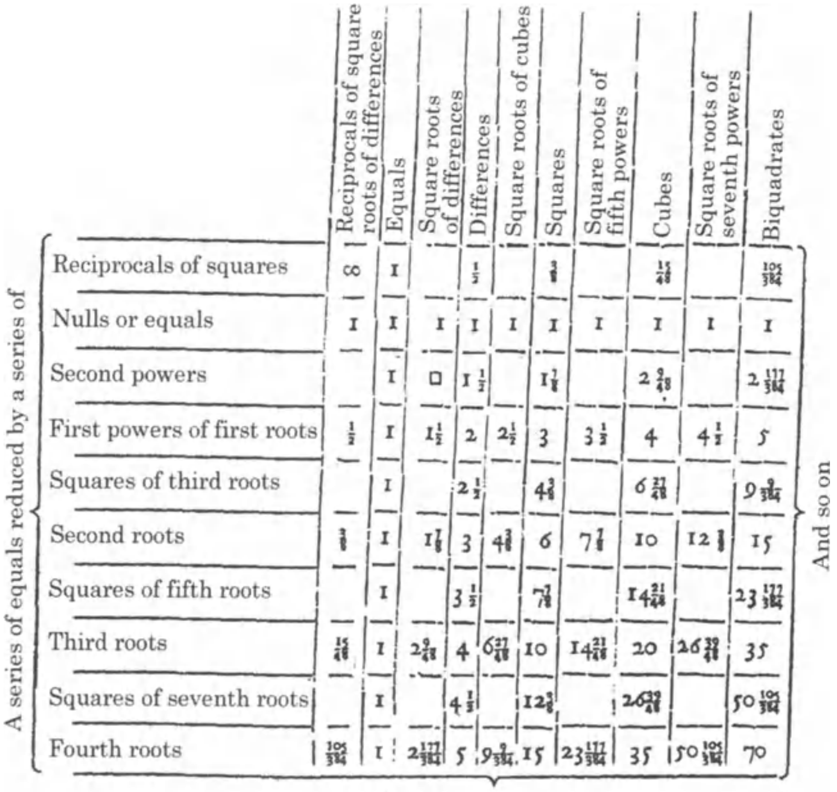
If a new sequence is interposed among the sequences of the table in Proposition 132, in order that it may be given its correct title, the indices of the powers positioned there must be noted. And only those powers are to be interposed whose indices hold the correct relationship to the original indices.

Thus, since the powers found at the head of that table have indices 0, 1, 2, 3, 4, etc., by the interpolation of one place now done everywhere the indices of the powers positioned there will be $-\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4$, etc.

In the same way, since the indices of powers positioned down the margin of that table are $\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc. or $\frac{2}{0}, \frac{2}{2}, \frac{2}{4}, \frac{2}{6}, \frac{2}{8}$, etc., the indices of the powers now positioned will be $\frac{2}{-1}, \frac{2}{0}, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6}, \frac{2}{7}, \frac{2}{8}$, etc. (or for $\frac{2}{-1}$ you may substitute $-\frac{2}{1}$ or $\frac{-2}{1}$ or -2, which amounts to the same thing).

Therefore the table now interpolated may be had in this way.

If an infinite series of equal terms is reduced by a similar series of first roots, or second roots, or third roots, etc. [the sums of] the differences, and [of] their squares, cubes, etc. will be in the same ratio to a corresponding series of equals as 1 to the numbers in the following table.



A series of equals reduced by a series of

	Reciprocals of square roots of differences	Equals	Square roots of differences	Differences	Square roots of cubes	Squares	Square roots of fifth powers	Cubes	Square roots of seventh powers	Biquadrates
Reciprocals of squares	∞	1		$\frac{1}{2}$		$\frac{3}{8}$		$\frac{15}{48}$		$\frac{105}{384}$
Nulls or equals	1	1	1	1	1	1	1	1	1	1
Second powers		1	□	$1\frac{1}{2}$		$1\frac{7}{8}$		$2\frac{9}{48}$		$2\frac{177}{384}$
First powers of first roots	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
Squares of third roots		1		$2\frac{1}{2}$		$4\frac{3}{8}$		$6\frac{27}{48}$		$9\frac{9}{384}$
Second roots	$\frac{3}{8}$	1	$1\frac{7}{8}$	3	$4\frac{3}{8}$	6	$7\frac{7}{8}$	10	$12\frac{3}{8}$	15
Squares of fifth roots		1		$3\frac{1}{2}$		$7\frac{7}{8}$		$14\frac{21}{48}$		$23\frac{177}{384}$
Third roots	$\frac{15}{48}$	1	$2\frac{9}{48}$	4	$6\frac{27}{48}$	10	$14\frac{21}{48}$	20	$26\frac{39}{48}$	35
Squares of seventh roots		1		$4\frac{1}{2}$		$12\frac{3}{8}$		$26\frac{39}{48}$		$50\frac{105}{384}$
Fourth roots	$\frac{105}{384}$	1	$2\frac{177}{384}$	5	$9\frac{9}{384}$	15	$23\frac{177}{384}$	35	$50\frac{105}{384}$	70

And so on

And so on

COMMENT

And here now one may note another series of those we mentioned in the *Comments* to Propositions 165 and 168, namely one that I had taught before at Propositions 118 and 121, unexpectedly arises also in the present table, that is, in the third sequence across.

PROPOSITION 186

Theorem

This is clear, that an infinite series of universal roots, to a series of the same number of equals, may have a rational ratio.

That is, by the preceding Propositions we will have:

Square roots of differences	Differences	Square roots of cubes
$\sqrt{(\sqrt{R}-\sqrt{a})}$	$\sqrt{R}-\sqrt{a}$	$\sqrt{(\sqrt{R^3}-3\sqrt{R^2a}+3\sqrt{Ra^2}-\sqrt{a^3})}$
$\sqrt{(\sqrt{R}-\sqrt{b})}$	$\sqrt{R}-\sqrt{b}$	$\sqrt{(\sqrt{R^3}-3\sqrt{R^2b}+3\sqrt{Rb^2}-\sqrt{b^3})}$
$\sqrt{(\sqrt{R}-\sqrt{c})}$	$\sqrt{R}-\sqrt{c}$	$\sqrt{(\sqrt{R^3}-3\sqrt{R^2c}+3\sqrt{Rc^2}-\sqrt{c^3})}$
etc. to		
$\sqrt{(\sqrt{R}-\sqrt{R})}$	$\sqrt{R}-\sqrt{R}$	$\sqrt{(\sqrt{R^3}-3\sqrt{R^3}+3\sqrt{R^3}-\sqrt{R^3})}$
$\frac{8}{15}\sqrt[4]{R}$	$\frac{1}{3}A\sqrt{R}$	$\frac{8}{35}\sqrt[4]{R^3}$
$\sqrt{(\sqrt[3]{R}-\sqrt[3]{a})}$	$\sqrt[3]{R}-\sqrt[3]{a}$	$\sqrt{(\sqrt[3]{R^3}-3\sqrt[3]{R^2a}+3\sqrt[3]{Ra^2}-\sqrt[3]{a^3})}$
$\sqrt{(\sqrt[3]{R}-\sqrt[3]{b})}$	$\sqrt[3]{R}-\sqrt[3]{b}$	$\sqrt{(\sqrt[3]{R^3}-3\sqrt[3]{R^2b}+3\sqrt[3]{Rb^2}-\sqrt[3]{b^3})}$
$\sqrt{(\sqrt[3]{R}-\sqrt[3]{c})}$	$\sqrt[3]{R}-\sqrt[3]{c}$	$\sqrt{(\sqrt[3]{R^3}-3\sqrt[3]{R^2c}+3\sqrt[3]{Rc^2}-\sqrt[3]{c^3})}$
etc. to		
$\sqrt{(\sqrt[3]{R}-\sqrt[3]{R})}$	$\sqrt[3]{R}-\sqrt[3]{R}$	$\sqrt{(\sqrt[3]{R^3}-3\sqrt[3]{R^2R}+3\sqrt[3]{R^3}-\sqrt[3]{RR^2})}$
$\frac{48}{105}A\sqrt[6]{R}$	$\frac{1}{4}A\sqrt[3]{R}$	$\frac{48}{315}\sqrt[6]{R^3}=\frac{48}{315}A\sqrt{R}$
or $\frac{16}{35}A\sqrt[6]{R}$		$\frac{16}{105}\sqrt[6]{R^3}=\frac{16}{105}A\sqrt{R}$

And by the same method for any other sequence of this table for which the interpolation has been completed.

Therefore nothing is lacking for perfecting the same in the remaining sequences (and in particular for the quadrature of the circle), except that there should be discovered a method of filling the empty places, or (which comes down to the same thing) that there should be found the correct formulae for those sequences. And indeed although it is not obvious how to find the formulae for the interpolated sequences, nevertheless one may come to know from the following Proposition what ratios they have to each other, so that if by any art we may find one of them, at once the rest are also found.

PROPOSITION 187

Theorem

In the table of Proposition 184: just as taking any number 1 of the second sequence (that is, the first of the even sequences), the formulae for the remaining terms (from the even sequences) arise from continued multiplication of the numbers $1 \times \frac{l}{1} \times \frac{l+1}{2} \times \frac{l+2}{3} \times \frac{l+3}{4}$ etc. (as was said in Proposition 182)[70] or (which amounts to the same thing) $1 \times \frac{2l}{2} \times \frac{2l+2}{4} \times \frac{2l+4}{6} \times \frac{2l+6}{8}$ etc., so taking any letter A of the first (of the odd) sequences,[71] the formulae for the remaining terms from the odd sequences arise from continued multiplication of numbers $A \times \frac{2l-1}{1} \times \frac{2l+1}{3} \times \frac{2l+3}{5} \times \frac{2l+5}{7}$ etc. And therefore if one of these becomes known, the rest also immediately follow.

For here one claims an analogy with arithmetic progressions, which are seen in both the numerators and denominators. And the induction confirmed this for all the places that are filled, so that there may be no doubt but that the same may also be considered in the empty places.

And therefore from the formulae for the odd sequences, if one term becomes known the rest also follow.

PROPOSITION 188

Theorem

In the sequences in the table of Proposition 184, if the first terms A are labelled $\alpha, A, \beta, B, \gamma, C, \delta, D, \varepsilon, E$, etc.[72] and the second terms (that is, the

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[70] Wallis mistakenly has Proposition 178 here.

[71] Here Wallis takes A to represent any term of the first sequence, thus $\infty, 1, \frac{1}{2}, \dots$.

[72] Here Wallis uses A in two distinct ways: (i) to represent in a general way the first terms of the even sequences (see note 5), but also (ii) to denote in particular the first term of the second sequence.

first of the evens) 1, then all the rest (both even and odd) arise from continued multiplication of the following numbers. Thus:

	Odd	Even
In the first	$\alpha \times \frac{0 \times 2 \times 4 \times 6 \times 8 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the second	$A \times \frac{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the third	$\beta \times \frac{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{3 \times 5 \times 7 \times 9 \times 11 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the fourth	$B \times \frac{3 \times 5 \times 7 \times 9 \times 11 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{4 \times 6 \times 8 \times 10 \times 12 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the fifth	$\gamma \times \frac{4 \times 6 \times 8 \times 10 \times 12 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{5 \times 7 \times 9 \times 11 \times 13 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the sixth	$C \times \frac{5 \times 7 \times 9 \times 11 \times 13 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{6 \times 8 \times 10 \times 12 \times 14 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the seventh	$\delta \times \frac{6 \times 8 \times 10 \times 12 \times 14 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{7 \times 9 \times 11 \times 13 \times 15 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$
In the eighth	$D \times \frac{7 \times 9 \times 11 \times 13 \times 15 \text{ etc.}}{1 \times 3 \times 5 \times 7 \times 9 \text{ etc.}}$	$1 \times \frac{8 \times 10 \times 12 \times 14 \times 16 \text{ etc.}}{2 \times 4 \times 6 \times 8 \times 10 \text{ etc.}}$

And so on.

Proved from what has gone before. Or also (as before) by analogy with an arithmetic progression. And indeed, it was confirmed by induction in all the filled places, so that there may be no doubt but that the same may be considered in the empty places.

If, moreover, anyone hesitates at the odd places of the first sequence (which I assert arise from continued multiplication of the numbers $\alpha \times \frac{0 \times 2 \times 4 \times 6 \times 8 etc.}{1 \times 3 \times 5 \times 7 \times 9 etc.}$), that is, lest the figure 0 which is seen there completely destroy the whole continued multiplication, however large it becomes, and make all the terms of the sequence vanish into 0 or nothing, it must be understood how this danger has been guarded against, because the term A in this sequence is ∞, or infinity, (just as we showed above in the *Comment* to Proposition 166), and therefore unless 0 follows (to diminish the force of that ∞) all the terms of the sequence would have turned out to be ∞, or infinity. But both of them together conveniently remedy this danger. For although $\infty \times 0$ does not definitely designate any number (and therefore nothing can thence be concluded with certainty about the rest of the quantities), it can nevertheless come up in place of virtually all numbers in turn. For any number divided by ∞ will give a quotient 0 and conversely. Thus $1/\infty = 0$, $1/0 = \infty$; $2/\infty = 0$, $2/0 = \infty$; $3/\infty = 0$, $3/0 = \infty$. And so on, for any others. And therefore (since *divisor* multiplied by *quotient* must restore the number *divided*), it must be that $\infty \times 0 = 1$, or $\infty \times 0 = 2$, or $\infty \times 0 = 3$; and so on for any other numbers.

PROPOSITION 189

Theorem

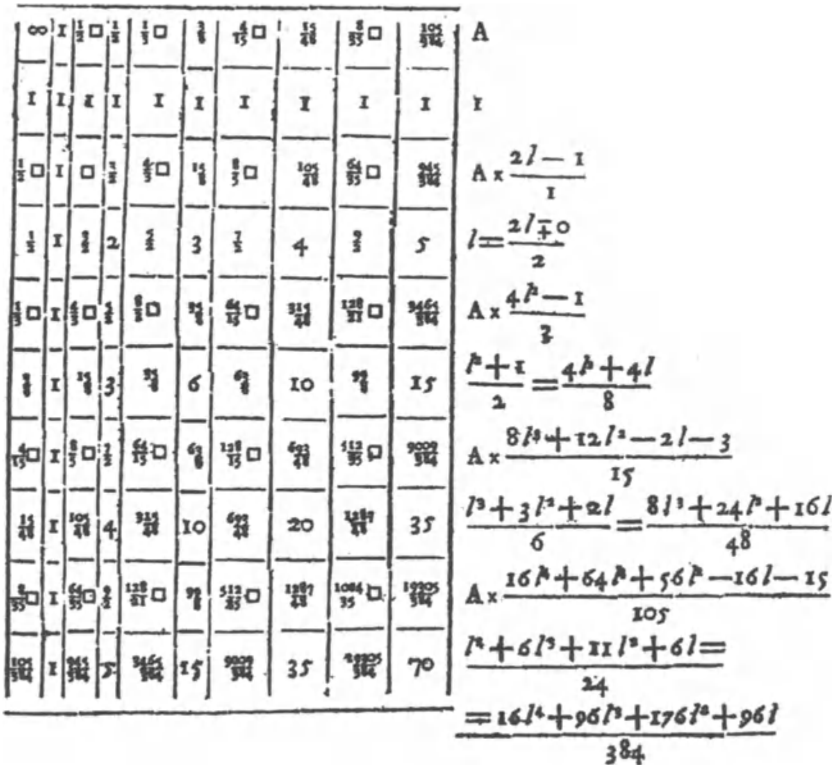
Here it follows that if in the empty spaces of the table of Proposition 184 any one is filled with a known number, then all the rest will also be known.

For example, if the number designated by this symbol □ is assumed known, all the rest also become known; which, that is, will have the ratio to that quantity as is indicated here below.

The whole process is shown by the preceding propositions.

It must also be noted here, moreover, that any intermediate number is the sum of two others, one taken from above, the other carried to the right (not to the next place but after one interposed).

We may also adjoin the formula for any sequence (as far as it is known from Propositions 182 and 187), so that the reader may better see how far we have taken the thing.



∞	1	$\frac{1}{2}\square$	$\frac{1}{2}$	$\frac{1}{3}\square$	$\frac{3}{8}$	$\frac{4}{15}\square$	$\frac{15}{48}$	$\frac{8}{35}\square$	$\frac{105}{384}$	A
1	1	1	1	1	1	1	1	1	1	1
$\frac{1}{2}\square$	1	$\square$	$\frac{3}{2}$	$\frac{4}{3}\square$	$\frac{15}{8}$	$\frac{8}{3}\square$	$\frac{105}{48}$	$\frac{64}{35}\square$	$\frac{945}{384}$	$A \times \frac{2l-1}{1}$
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$l = \frac{2l \mp 0}{2}$
$\frac{1}{3}\square$	1	$\frac{4}{3}\square$	$\frac{5}{2}$	$\frac{8}{3}\square$	$\frac{35}{8}$	$\frac{64}{15}\square$	$\frac{315}{48}$	$\frac{128}{35}\square$	$\frac{3465}{384}$	$A \times \frac{4l^2-1}{3}$
$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	$\frac{l^2+l}{2} = \frac{4l^2+4l}{8}$
$\frac{4}{15}\square$	1	$\frac{8}{5}\square$	$\frac{7}{2}$	$\frac{64}{15}\square$	$\frac{63}{8}$	$\frac{128}{15}\square$	$\frac{693}{48}$	$\frac{512}{35}\square$	$\frac{9009}{384}$	$A \times \frac{8l^3+12l^2-2l-3}{15}$
$\frac{15}{48}$	1	$\frac{105}{48}$	4	$\frac{315}{48}$	10	$\frac{693}{48}$	20	$\frac{1287}{48}$	35	$\frac{l^3+3l^2+2l}{6} = \frac{8l^3+24l^2+16l}{48}$
$\frac{8}{35}\square$	1	$\frac{64}{35}\square$	$\frac{9}{2}$	$\frac{128}{35}\square$	$\frac{99}{8}$	$\frac{512}{35}\square$	$\frac{1287}{48}$	$\frac{1024}{35}\square$	$\frac{19305}{384}$	$A \times \frac{16l^4+64l^3+56l^2-16l-15}{105}$
$\frac{105}{384}$	1	$\frac{945}{384}$	5	$\frac{3465}{384}$	15	$\frac{9009}{384}$	35	$\frac{19305}{384}$	70	$\frac{l^4+6l^3+11l^2+6l}{24} = \frac{16l^4+96l^3+176l^2+96l}{384}$

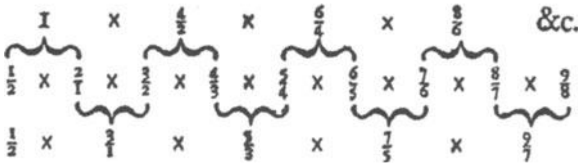
COMMENT

And indeed until now we seem to have carried the thing through happily enough. But here, at last, I am at a loss for words. For I do not see in what manner I may produce either the quantity □, or the formula for the sequence A.[73] (Nor therefore how to attain completely the formulae for the odd sequences, though their ratios to each other are known, nor the odd places in the odd sequences, although the ratios of these to each other are also known.) For although if the lateral numbers are integers, thus, 1, 2, 3, 4, etc. the first terms of their sequences may be written down, thus, 1, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{15}{48}$, etc. it is nevertheless not easy to understand in what manner the ratio of these numbers to their respective lateral numbers may be expressed by any one equation; or whence also to the remaining lateral numbers (in the odd places) $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc. there may be fitted the first term of their sequences. For although here no small hope seemed to shine, nevertheless, this slippery Proteus whom we have in hand, both here and above, frequently escapes and disappoints hope. In what manner, moreover, having also been constrained here he might have shown his face, it will perhaps not be unwelcome to have put forward. Namely:

PROPOSITION 190

Theorem

In the fourth (or second even) sequence, numbers taken alternately (in even places) 1, 2, 3, 4, 5, etc. arise from continued multiplication of the numbers, or fractions, $1 \times \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4}$ etc. or $1 \times \frac{4}{2} \times \frac{6}{3} \times \frac{8}{4} \times \frac{10}{5}$, etc.; and (in odd places) $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, etc. from the multiplication $\frac{1}{2} \times \frac{3}{1} \times \frac{5}{3} \times \frac{7}{5} \times \frac{9}{7}$ etc. So because of the interpolation of one number in each space (so that from both odd and even places intermingled there arises a single sequence), any of the ratios by which the first term of all, whether of the evens or odds, is continually multiplied, must be separated into two ratios (as, for example, the ratio 1 to 2 is composed from the ratios 1 to $1\frac{1}{2}$ and $1\frac{1}{2}$ to 2) in this way:



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[73] The sequence of first terms.

In the sixth (or third even) sequence, if the same is to be done, any ratio must be separated in the same way into two ratios, but both of those are composed of two ratios, in this way:

$$1 \times \frac{6}{2} \times \frac{8}{4} \times \frac{10}{6} \quad \&c.$$

$$\overbrace{\frac{1\times3}{2\times4} \times \frac{2\times4}{1\times3}} \times \overbrace{\frac{3\times5}{2\times4} \times \frac{4\times6}{3\times5}} \times \overbrace{\frac{5\times7}{4\times6} \times \frac{6\times8}{5\times7}} \times \overbrace{\frac{7\times9}{6\times8} \times \frac{8\times10}{7\times9}} \quad \&c.$$

$$\frac{3}{8} \times \frac{5}{1} \times \frac{7}{3} \times \frac{9}{5} \quad \&c.$$

That is, by first separating any ratio into four ratios (thus $\frac{6}{2} = \frac{3\times4\times5\times6}{2\times3\times4\times5}$ and $\frac{7}{3} = \frac{4\times5\times6\times7}{3\times4\times5\times6}$, and so on for the rest), and then distributing them alternately into two classes.

In the eighth (or fourth even) sequence, any ratio must be separated in the same way into two ratios, but both of those are composed of three ratios, thus:

$$1 \times \frac{8}{2} \times \frac{10}{4} \quad \&c.$$

$$\overbrace{\frac{1\times3\times5}{2\times4\times6} \times \frac{2\times4\times6}{1\times3\times5}} \times \overbrace{\frac{3\times5\times7}{2\times4\times6} \times \frac{4\times6\times8}{3\times5\times7}} \times \overbrace{\frac{5\times7\times9}{4\times6\times8} \times \frac{6\times8\times10}{5\times7\times9}} \quad \&c.$$

$$\frac{15}{48} \times \frac{7}{1} \times \frac{9}{3} \quad \&c.$$

That is, by first separating any ratio into six ratios (thus $\frac{8}{2} = \frac{3\times4\times5\times6\times7\times8}{2\times3\times4\times5\times6\times7}$ etc. and $\frac{9}{3} = \frac{4\times5\times6\times7\times8\times9}{3\times4\times5\times6\times7\times8}$ etc.) which must then be distributed alternately into two classes.

And similarly in the tenth sequence, twelfth sequence, etc. any ratio must be separated into eight ratios, ten ratios, etc. which must then be distributed alternately into two classes.

(Moreover, in the second (or first even) sequence, there is no need for the separation of the ratios, but since all are the same ratio of equality, or $\frac{1}{1}$, that same ratio is also everywhere interposed, for $\frac{1}{1} \times \frac{1}{1} = \frac{1}{1}$.)

But if we attempt this in the odd sequences, that is, so that any ratio is separated (proceeding evenly) into two ratios, the thing does not come out so happily.

So (for example), since (by analogy with the rest) the ratios of the fifth sequence must be separated into three ratios, of the seventh into five ratios, etc. (always an odd number), an equal partitioning of them into two classes, as needed for the required interpolation, cannot be done.

This whole Proposition (by inspection of the table) is sufficiently clear in itself to those who are attentive.

The thing will perhaps appear somewhat more clearly if I separate some of the ratios of the sequences into two, three, four ratios, etc. (as each sequence requires). That is:

In the third sequence

$$\frac{2}{1}=\frac{2}{1}$$
$$\frac{3}{2}=\frac{3}{2}$$
$$\frac{4}{3}=\frac{4}{3}$$
$$\frac{5}{4}=\frac{5}{4}$$
$$\frac{6}{5}=\frac{6}{5}$$
$$\frac{7}{6}=\frac{7}{6}$$
$$\frac{8}{7}=\frac{8}{7}$$
$$\frac{9}{8}=\frac{9}{8}$$

In the fourth sequence

$$\frac{3}{1}=\frac{2\times3}{1\times2}$$
$$\frac{4}{2}=\frac{3\times4}{2\times3}$$
$$\frac{5}{3}=\frac{4\times5}{3\times4}$$
$$\frac{6}{4}=\frac{5\times6}{4\times5}$$
$$\frac{7}{5}=\frac{6\times7}{5\times6}$$
$$\frac{8}{6}=\frac{7\times8}{6\times7}$$
$$\frac{9}{7}=\frac{8\times9}{7\times8}$$
$$\frac{10}{8}=\frac{9\times10}{8\times9}$$

In the fifth sequence

$$\frac{4}{1}=\frac{2\times3\times4}{1\times2\times3}$$
$$\frac{5}{2}=\frac{3\times4\times5}{2\times3\times4}$$
$$\frac{6}{3}=\frac{4\times5\times6}{3\times4\times5}$$
$$\frac{7}{4}=\frac{5\times6\times7}{4\times5\times6}$$
$$\frac{8}{5}=\frac{6\times7\times8}{5\times6\times7}$$
$$\frac{9}{6}=\frac{7\times8\times9}{6\times7\times8}$$
$$\frac{10}{7}=\frac{8\times9\times10}{7\times8\times9}$$
$$\frac{11}{8}=\frac{9\times10\times11}{8\times9\times10}$$

In the sixth sequence

$$\frac{5}{1}=\frac{2\times3\times4\times5}{1\times2\times3\times4}$$
$$\frac{6}{2}=\frac{3\times4\times5\times6}{2\times3\times4\times5}$$
$$\frac{7}{3}=\frac{4\times5\times6\times7}{3\times4\times5\times6}$$
$$\frac{8}{4}=\frac{5\times6\times7\times8}{4\times5\times6\times7}$$
$$\frac{9}{5}=\frac{6\times7\times8\times9}{5\times6\times7\times8}$$
$$\frac{10}{6}=\frac{7\times8\times9\times10}{6\times7\times8\times9}$$
$$\frac{11}{7}=\frac{8\times9\times10\times11}{7\times8\times9\times10}$$
$$\frac{12}{8}=\frac{9\times10\times11\times12}{8\times9\times10\times11}$$

In the seventh sequence

$$\frac{6}{1}=\frac{2\times3\times4\times5\times6}{1\times2\times3\times4\times5}$$
$$\frac{7}{2}=\frac{3\times4\times5\times6\times7}{2\times3\times4\times5\times6}$$
$$\frac{8}{3}=\frac{4\times5\times6\times7\times8}{3\times4\times5\times6\times7}$$
$$\frac{9}{4}=\frac{5\times6\times7\times8\times9}{4\times5\times6\times7\times8}$$
$$\frac{10}{5}=\frac{6\times7\times8\times9\times10}{5\times6\times7\times8\times9}$$
$$\frac{11}{6}=\frac{7\times8\times9\times10\times11}{6\times7\times8\times9\times10}$$
$$\frac{12}{7}=\frac{8\times9\times10\times11\times12}{7\times8\times9\times10\times11}$$
$$\frac{13}{8}=\frac{9\times10\times11\times12\times13}{8\times9\times10\times11\times12}$$

In these sequences (however far continued) and all subsequent sequences, it must be noted that ratios from any even sequences are separated into an even number of others, which may therefore be conveniently distributed (as was said) into two classes; ratios from any odd sequences, however, are separated into an odd number of others, which therefore can not be so distributed.

COMMENT

If, moreover, anyone thinks a cure might sufficiently conveniently be applied to this problem by separating the ratios of the fifth sequence, seventh sequence, etc. (not into three ratios, five ratios, etc. but) into six ratios, ten ratios, etc. (that is twice three, twice five, etc.) in such a way that the ratios (now even in number) may be distributed into two classes: the thing cannot on any account succeed as wished. For this indeed amounts to the same thing as if the ratios of the fourth sequence, sixth sequence, eighth sequence, etc. are separated (not into two ratios, four ratios, six ratios, etc. but) into four ratios, eight ratios, twelve ratios, etc. and after that distributed alternately into two classes. Which indeed if it were done, would not produce the ratios sought (which we showed above) but others somewhat different from those, as will be clear to the experienced.

And indeed I am inclined to believe (what from the beginning I suspected) that this ratio we seek is such that it cannot be forced out in numbers according to any method of notation so far accepted, not even by surds (of the kind implied by Van Schooten in connection with the roots of certain cubic equations, in his *Appendix* to the treatise *On a complete description of conic sections*,[74] or in the thinking of Viète, Descartes and others) so that it seems necessary to introduce another method of explaining a ratio of this kind, than by true numbers or even by the accepted means of surds.

And indeed this, whether opinion or conjecture, seems to be confirmed here, since if we have the appropriate formula, for any even sequence (in the table of Proposition 184) so also we might have obtained a formula of this kind for any odd sequence; then, just as for the formulae for the even sequences we have taught how to investigate the ratio of finite series of first powers, second powers, third powers, fourth powers, etc., to a series of the same number of terms equal to the greatest of those (in the *Comment* to Proposition 182), so by formulae of the same kind for odd sequences, it would seem there could be investigated similarly the ratio of finite series of second roots, third roots, etc. to a series of the same number of terms equal to the greatest of these: why this is not to be hoped for, moreover, we showed in the *Comment* to Proposition 165.

And therefore what arithmeticians usually do in other work, must also be done here; that is, where some impossibility is arrived at, which indeed

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[74] Frans van Schooten, *De organica conicarum sectionum in plano descriptione tractatus*, Leiden 1646. The *Appendix* gives Cardano's formula for the solution of $z^3 = * - pz + q$ as $\sqrt[3]{+\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} - \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$. This was perhaps the first time that Wallis saw Cardano's formula in Cartesian notation; he himself had arrived at the same result in 1647 or 1648 but using Oughtred's notation; see Wallis to Collins, 12 April 1673, in Rigaud 1841, II, 564–566. For Wallis's self-confessed ignorance of the arithmetic of surds in his early years see also Wallis to Collins, 6 May 1673, *ibid.* II, 578. Wallis's copy of Van Schooten's 1646 *Tractatus* with his annotations on the flyleaf is now Bodleian Library Savile Bb.10.

must be assumed to be done, but nevertheless cannot actually be done, they consider some method of representing what is assumed to be done, though it may not be done in reality.

And this indeed happens in all operations of arithmetic involving resolution,[75] for example, in subtraction: if it is proposed that a larger number must be taken from a smaller, thus 3 from 2 or 2 from 1, since this can not be shown in reality, there are considered negative numbers, by means of which a supposed subtraction of this kind may be expressed, thus $2 - 3$, or $1 - 2$, or -1.

In division, if it is proposed that a number must be divided by another which is not a divisor,[76] thus 3 by 2, since this can not be shown in reality, there is invented a method of indicating a supposed division of this kind, in this form: $\frac{3}{2}$ or $1\frac{1}{2}$.

In the extraction of roots, if there is proposed a number that is not in its nature truly a power, for example, if there is sought the square root of 12, since that root cannot be expressed as any integer or fractional number, there is invented a method of indicating any supposed root of this kind in this form: $\sqrt{}12$ or $2\sqrt{}3$.

Equally, in a geometric progression, thus, 3, 6, 12, etc. if there is sought a new term to be interposed between 3 and 6, it is said to be $3\sqrt{}2$, or $\sqrt{}18$, or $\sqrt{}(3 \times 6)$, or better (since it amounts to the same thing), $\sqrt{}(2 \times 9)$, which is the same as to say more explicitly, *the mean term between 3 and 6 in the progression 3, 6, 12, etc. or between 2 and 9 in the progression 2, 9, $40\frac{1}{2}$, etc.* Thus if between 3 and 6 there are to be interposed two geometric means, the first will be $\sqrt[3]{3 \times 3 \times 6}$ or $\sqrt[3]{}54$ or rather $3\sqrt[3]{}2$ (that is, 3 times the cube root of the common multiplier 2), and so on in other cases.

If, moreover, a geometric progression, which is assumed to be formed by continued multiplication of the first term by any numbers equal to each other (thus, 3, 6, 12, 24, etc. from the continued multiplication $3 \times 2 \times 2 \times 2$ etc.) does not always have rational intermediate terms, it is no wonder if that does not happen in a progression formed by continued multiplication of the first term by any succeeding unequal numbers, whether increasing or decreasing (thus 1, 2, 6, 24, etc. from the continued multiplication $1 \times 2 \times 3 \times 4$ etc., or 1, $\frac{3}{2}$, $\frac{15}{8}$, $\frac{105}{48}$, etc. from the continued multiplication $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6}$ etc.

As much, moreover, holds here; since it is not possible to designate that quantity (□) by a true number (not even by the usual said radicals, or surds), there may be sought some method of expressing it in some way. Therefore, as $\sqrt{}(3 \times 6)$ signifies *the mean term between 3 and 6 in a regular geometric progression 3, 6, 12,* etc. (from the continued multiplication $3 \times 2 \times 2$ etc.) so $\mathfrak{m}(1|\frac{3}{2})$ signifies *the mean term between 1 and $\frac{3}{2}$ in a decreasing hypergeometric*

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[75] *Resolution* is here used as the opposite of *composition*, thus of subtraction as opposed to addition, division as opposed to multiplication, or extraction of roots as opposed to composition of powers.

[76] *Non metitur*, literally 'by which it is not measured'.

progression (from the continued multiplication $1 \times \frac{3}{2} \times \frac{5}{4}$ etc.) which will be: $\square = \mathfrak{m}(1|\frac{3}{2})$. And therefore *the circle, to the square of its diameter, is as* 1 *to* $\mathfrak{m}(1|\frac{3}{2})$. Which indeed is the true Quadrature of the Circle expressed in numbers, as far as the nature of those numbers may be shown.

And indeed, just as in a regular geometric progression, 3, 12, 48, etc., anyone who says the term intermediate between 3 and 12 is $\sqrt{}(3 \times 12)$ may not be said to have set the thing out satisfactorily, since that term may be more explicitly said to be 6 (for $\sqrt{}(3 \times 12) = \sqrt{}36 = 6$). But anyone who assigns between 3 and 6 (in the progression 3, 6, 12, etc.) the intermediate term $\sqrt{}(3 \times 6)$ (or rather $\sqrt{}18$, or $3\sqrt{}2$) may be said to have set it out sufficiently, since it is not possible to assign a true number. Thus if in the progression 1, $\frac{15}{8}$, $\frac{954}{384}$, etc. anyone says that $\mathfrak{m}(1|\frac{15}{8})$ is the intermediate term between the terms 1 and $\frac{15}{8}$, he has not taught the thing sufficiently explicitly, for he could have said $\frac{3}{2}$. But anyone who assigns $\mathfrak{m}(1|\frac{3}{2})$ as the intermediate term between 1 and $\frac{3}{2}$ must be said to have set the thing out sufficiently, since this term cannot be expressed in true numbers; therefore it suffices if it is indicated in some way.

And, further, although $\sqrt{}(3 \times 6)$ (in the progression 3, 6, 12, etc.) or $\sqrt{}18$ or $3\sqrt{}2$ cannot be expressed accurately in true numbers, it may, nevertheless, be signified as closely as one wishes (thus greater than 4.24 but less than 4.25; or greater than 4.2426 but less than 4.2427; or greater than 4.242639 but less than 4.242640, and so on); so also the number $\square = \mathfrak{m}(1|\frac{3}{2})$ may be signified as closely as one wishes in true numbers, though not exactly, thus, greater than 1.27 but less than 1.28; greater than 1.2732 but less than 1.2733; greater than 1.273239 but less than 1.273240, and so on, as may be put together either from our table (which will be shown in the following Proposition) or also in various other ways.

Therefore I see no reason why the ratio of the circle to its circumscribed square (or also the ellipse to the circumscribed parallelogram), that is, 1 to $\square = \mathfrak{m}(1|\frac{3}{2})$, or $\square = 1\mathfrak{m}\frac{3}{2}$ (that is, 1 to the term intermediate between 1 and $\frac{3}{2}$ in the progression 1, $\frac{3}{2}$, $\frac{15}{8}$, etc.) may not be said to be just as systematically explained as the ratio of the side of a square to its diagonal, that is as 1 to $1\sqrt{}2$, or, or to $\sqrt{}(1 \times 2)$ (that is, as 1 to the intermediate term between 1 and 2 in the progression 1, 2, 4, etc.), except that this notation $\sqrt{}2$ or $\sqrt{}(1 \times 2)$ is already accepted (thought is was at one time new) while ours is now introduced for the first time because of the new kind of progression now for the first time (as I believe) discovered. Moreover, just as the notation for surd numbers (thus, $\sqrt{}2$, etc.) introduced into arithmetic the method of adding, subtracting, multiplying, dividing etc. for surd roots, so it will not be difficult to apply operations of this kind to this our new method of notation, which however is not the purpose of the present work. Meanwhile I am not ignorant that for perfecting this notation more accurately, there must be adjoined distinct symbols, thus $\mathfrak{m}^2, \mathfrak{m}^3, \mathfrak{m}^4$, etc. as will indicate either a single mean, or the first of two, or three, etc., just as is also usually done for the sign $\sqrt{}$, thus $\sqrt{}^2, \sqrt{}^3, \sqrt{}^4$, etc. to signify a square root, cube root, fourth root, etc., that is, either a single mean proportional or the first of two, three, etc. In the same way, other distinct signs must be added which indicate, in the continued multiplications (of the given interpolated sequences), whether they increase by ones, or by twos, threes, etc. But all this, and whatever similar problems, must await more exact inquiry into these progressions, if mathematicians are of the opinion this should be admitted into arithmetic (and why less should be done, I do not see). It is sufficient for the purpose of the present work that we wish to indicate it in some way and to supply in plain words what is lacking in symbols. If, moreover, this method of notation thought out by me is less pleasing to mathematicians, I would as happily allow it to be changed to a way that they show more appropriate.

Howsoever this may be, I must indeed acknowledge that I am still unable to supply formulae of this kind for the odd sequences as for the even sequences in the table; nor for the odd places in the odd sequences (though I have now shown the ratios of those to each other) according to any method of notation (that I know yet accepted). And although in those above, often by fortune and by breaking paths never, as far as I know, trodden before, I have discovered some of the hoped for conclusions, I could scarcely, however, (for the reasons already shown) have dared to hope that likewise here also everything would come out as wished. If, by chance, anyone else from here on treading in my footsteps arrives at length at what it was not given to me to arrive at (for I would not wish to proclaim to the skilled the limits of all other methods in the same way as for mine), and discover more useful methods of expressing those same quantities, I would certainly not bear any ill will. In the meantime I believe it will be by no means unwelcome to mathematicians that I have offered some new light, not (as I judge it) wholly to be disparaged, on the obscurity of problems concerning the quadrature of the circle, and to have expressed that in numbers as far as the nature of numbers allows.

What we have already found, moreover, it may also be pleasing to set out in some following Propositions, in a form a little changed. And first indeed it may be signified as closely as one wishes by whole numbers, and afterwards also by straight lines.

PROPOSITION 191

Problem

It is proposed to inquire, what is the value of the term $\square$ (in the table of Proposition 189), as closely as one wishes using whole numbers.

That the thing may come out more easily, the terms of the progression (the same produced again) $\frac{1}{2}\square$, 1, $\square$, $\frac{3}{2}$, $\frac{4}{3}\square$, $\frac{3\times5}{2\times4}$, $\frac{4\times6}{3\times5}\square$, $\frac{3\times5\times7}{2\times4\times6}$, etc. may be called $\alpha, a, \beta, b, \gamma, c, \delta, d$, etc.

Moreover, $1:2=\alpha:\beta$, and $2:3=a:b$, and $3:4=\beta:\gamma$, and $4:5=b:c$, and $5:6=\gamma:\delta$, and $6:7=c:d$.

That is, $\frac{\beta}{\alpha}=\frac{2}{1}, \frac{b}{a}=\frac{3}{2}, \frac{\gamma}{\beta}=\frac{4}{3}, \frac{c}{b}=\frac{5}{4}, \frac{\delta}{\gamma}=\frac{6}{5}, \frac{d}{c}=\frac{7}{6}$, etc.
Therefore (since the multiplying ratios continually decrease) we will have

$$\frac{\beta}{a} \text{ is } \begin{cases} \text{the lesser of both[77]} \quad \frac{a}{\alpha}\times\frac{\beta}{a}=\frac{\beta}{\alpha}=\frac{2}{1}, \quad \text{therefore less than} \quad \sqrt{\tfrac{2}{1}}=\sqrt{1\tfrac{1}{1}} \\ \text{the greater of both} \quad \frac{\beta}{a}\times\frac{b}{\beta}=\frac{b}{a}=\frac{3}{2}, \quad \text{therefore greater than} \quad \sqrt{\tfrac{3}{2}}=\sqrt{1\tfrac{1}{2}} \end{cases}$$

$$\text{and therefore } \beta = a\times\frac{\beta}{a}=\Box \text{ is } \begin{cases} \text{less than} \quad 1\sqrt{2}=1\sqrt{1\tfrac{1}{1}} \\ \text{greater than} \quad 1\sqrt{\tfrac{3}{2}}=1\sqrt{1\tfrac{1}{2}} \end{cases}$$

In the same way

$$\frac{\gamma}{b} \text{ is } \begin{cases} \text{the lesser of both} \quad \frac{b}{\beta}\times\frac{\gamma}{b}=\frac{\gamma}{\beta}=\frac{4}{3}, \quad \text{therefore less than} \quad \sqrt{\tfrac{4}{3}}=\sqrt{1\tfrac{1}{3}} \\ \text{the greater of both} \quad \frac{\gamma}{b}\times\frac{c}{\gamma}=\frac{c}{b}=\frac{5}{4}, \quad \text{therefore greater than} \quad \sqrt{\tfrac{5}{4}}=\sqrt{1\tfrac{1}{4}} \end{cases}$$

$$\text{and therefore } \gamma = b\times\frac{\gamma}{b}=\frac{4}{3}\Box \text{ is } \begin{cases} \text{less than} \quad \frac{3}{2}\times\sqrt{1\tfrac{1}{3}} \\ \text{greater than} \quad \frac{3}{2}\times\sqrt{1\tfrac{1}{4}} \end{cases}$$

that is, $\Box$ is less than $\frac{3\times 3}{2\times 4}\times\sqrt{1\tfrac{1}{3}}$, greater than $\frac{3\times 3}{2\times 4}\times\sqrt{1\tfrac{1}{4}}$

And (by the same reasoning)

$$\delta = c\times\frac{\delta}{c}=\frac{4\times 6}{3\times 5}\Box \text{ is } \begin{cases} \text{less than} \quad \frac{3\times 5}{2\times 4}\times\sqrt{1\tfrac{1}{5}} \\ \text{greater than} \quad \frac{3\times 5}{2\times 4}\times\sqrt{1\tfrac{1}{6}} \end{cases}$$

that is, $\Box$ is less than $\frac{3\times 3\times 5\times 5}{2\times 4\times 4\times 6}\times\sqrt{1\tfrac{1}{5}}$, greater than $\frac{3\times 3\times 5\times 5}{2\times 4\times 4\times 6}\times\sqrt{1\tfrac{1}{6}}$

And (continuing the work in this way according to the rules of the table) it will be found that

$$\Box \text{ is } \begin{cases} \text{less than} \quad \frac{3\times 3\times 5\times 5\times 7\times 7\times 9\times 9\times 11\times 11\times 13\times 13}{2\times 4\times 4\times 6\times 6\times 8\times 8\times 10\times 10\times 12\times 12\times 14}\times\sqrt{1\tfrac{1}{13}} \\ \text{greater than} \quad \frac{3\times 3\times 5\times 5\times 7\times 7\times 9\times 9\times 11\times 11\times 13\times 13}{2\times 4\times 4\times 6\times 6\times 8\times 8\times 10\times 10\times 12\times 12\times 14}\times\sqrt{1\tfrac{1}{14}} \end{cases}$$

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[77] Wallis's argument here is that β/a is the smaller of the two quantities a/α and β/a (because of the decreasing ratio), and is therefore less than the square root of their product. Wallis does not make himself entirely clear, and Christiaan Huygens was puzzled by this part of the argument, and failed to understand why Wallis went on to take a square root; see Huygens to Wallis, [11]/21 July 1656, Beeley and Scriba 2003, 189–192.

And so on as far as one likes. In such a way, that is, that the numerator of the fraction arises from continually multiplying odd numbers 3, 5, 7, etc. placed twice, but the denominator from continually multiplying even numbers 2, 4, 6, etc. also placed twice, except the first and last, which are put only once; and finally all that ratio, or fraction, thus formed, is multiplied by the square root of 1 increased by some fraction of itself, namely that which has as its denominator the last of the odd numbers in the continued multiplication, if we seek a number too large, or of the evens, if we seek a number too small.

And by this method it may be done as far as one likes until the difference between the greater and the smaller becomes less than any assigned quantity (which, therefore, if one supposes the operation continued infinitely, will at last disappear). Which indeed, in case it is needed, will be demonstrated here.

Thus, as has already been said of the numbers in the continued multiplication, the greatest of the evens (that is, the final factor of the denominator) may be called z, and therefore the greatest of the odds (that is, the final factor of the numerator) will be $z-1$ (that is, the other less one). Therefore (since the same multiplier is combined with both) the number too large to the number too small will be as $\sqrt{1\frac{1}{z-1}}$ to $\sqrt{1\frac{1}{z}}$, that is, as the final surd number in the former to the final surd number in the latter), that is, as $\sqrt{\frac{z}{z-1}}$ to $\sqrt{\frac{z+1}{z}}$, that is, as $\sqrt{\frac{z^2}{z-1}}$ to $\sqrt{(z+1)}$ that is as $\sqrt{z^2} = z$ to $\sqrt{(z^2-1)}$. Moreover it may happen (by increasing the quantity z as needed) that the difference between the roots $\sqrt{z^2}$ and $\sqrt{(z^2-1)}$, that is, $z - \sqrt{(z^2-1)}$, becomes less than any assigned quantity (as is known, and was also said elsewhere by me at Proposition 39 of *On conic sections*). And therefore the number too large exceeds the number too small by a fraction less than any assigned quantity.[78] Which was to be proved.

Since, moreover, as is clear from what has been said, by increasing the number z infinitely, the number too large exceeds the number too small by a fraction less than any assigned quantity, the differences between them (and therefore of either from the true quantity) will be infinitely small, that is, nothing.

Further, since the number z is thus increased infinitely, that fractional part of 1 adjoined to it will be infinitely small; it will be $\sqrt{1\frac{1}{z}}$ or $\sqrt{1\frac{1}{z-1}}$, which amounts to the same thing therefore as $\sqrt{1}$ or 1 (on account of the vanishing infinitely small part), which by multiplication changes nothing. We say that the fraction $\frac{3\times 3\times 5\times 5\times 7\times 7etc.}{2\times 4\times 4\times 6\times 6\times 8etc.}$ or $\frac{9\times 25\times 49\times 81etc.}{8\times 24\times 48\times 80etc.}$ continued infinitely is itself precisely the required number □, and the ratio of 1 to this is that of the circle to the square of its diameter. Or (if this is more pleasing), as the denominator of that fraction is to the numerator, so we may say is the circle to the square of its diameter. And, as the numerator is to the denominator, so is the square to the circle. That is, as the product of the continued multiplication 9 × 25 × 49 × 81 etc. (squares of

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[78] Wallis's proof has interesting elements of a later limit argument, but is incomplete. His argument that $z - \sqrt{(z^2-1)}$ can be made less than any assigned quantity is correct; he has ignored, however, the fact that this quantity is multiplied by a fraction that increases with each new pair of multipliers. The convergence of the fraction therefore depends on the properties of not one but two infinite processes.

odd numbers) to the product of 8 × 24 × 48 × 80 etc. (the same squares decreased by one), continued infinitely.

Moreover, if some more curious person inquires how far that continued multiplication must be continued until at last that given difference, or less than that, is arrived at, or so that the number too large exceeds the number too small, by however small a part of itself (or not even that), that will be investigated by this method.

Let the greater quantity be called m, the smaller n, and let their difference, that part however small, thus $\frac{a}{b}m = m - n$, and let it be inquired how far the work must be continued, that is, what will be the number z, the greatest (simple) multiplier that produces that difference (or even less than it).

Since therefore $m - n = \frac{a}{b}m$, we will have $n = m - \frac{a}{b}m$, and $m : n = m : m - \frac{a}{b}m = \frac{b}{b}m$: $\frac{b-a}{b}m = b : b - a = z : \sqrt{(z^2 - 1)}$ (by the method demonstrated). Therefore $b\sqrt{(z^2 - 1)} = bz - az$. And (squaring everywhere) $b^2z^2 - b^2 = b^2z^2 + a^2z^2 - 2abz^2$. And then (deleting b^2z^2 everywhere and transposing the rest) $2abz^2 - a^2z^2 = b^2$. And finally (dividing everywhere) $z^2 = \frac{b^2}{2ab - a^2}$. Therefore the square root of this number (if it is an even number), or at least (if it is either a fraction or a surd or an odd number) the even number next greater than that root, will be the greatest of the multipliers that arrives at the assigned difference or certainly less than that. Which was to be investigated.

The same another way

After this our description of that quantity □, we may also add another, which I have received from that most noble person and very skilled geometer, Lord William Viscount and Baronet Brouncker.

Since I showed him some of my progressions, and indicated by what rule they proceeded, meanwhile asking him to indicate in what form he thought that quantity might usefully be described. That Noble Gentleman, having thought it over himself, judged by a method of infinites of his own that the same quantity could be most conveniently described in this form:

$$\square = 1\cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{2\cfrac{81}{2}}}}} \text{ etc.}$$

That is, if to one there is added a fraction that has a denominator continually broken, by the rule that the numerators of each small fraction are 1, 9, 25, etc., squares of odd numbers 1, 3, 5, etc., but the denominators everywhere 2 with an adjoined fraction, and thus infinitely. Adding this at the same time, that, wherever at length it pleases one to stop, instead of the final 2 with the fraction afterwards cut off, there may be put (according to the place where one requires it to stand) any of 3, 5, 7, 9, etc. (arithmetically proportional from 1, in whole numbers); that is, if it is required to stand in the first place, 3, if in the second, 5, if in the third, 7, and so on, putting the number that defines the place, doubled and increased by one. And, in the same manner, if it is required to stand at an odd place of the fraction it will produce a number too large; but if even, too small. And the longer it is carried on, the more nearly it approaches in either case the true number.

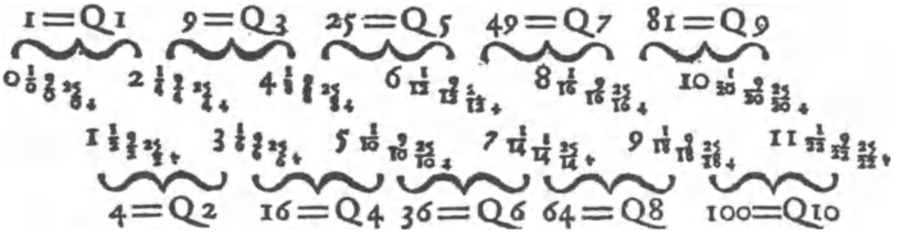
1	too small
$1\frac{1}{3}$	too large
$1\cfrac{1}{2\cfrac{9}{5}}$	too small
$1\cfrac{1}{2\cfrac{9}{2\cfrac{25}{7}}}$	too large
$1\cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{9}}}}$	too small

And he has described in the same form the remaining numbers sought in our table, and interpolated others of our progressions, similar to those in the table shown. But to open up all the process of his method would take longer than can be spared here. I hoped, moreover, that at some time the thing itself would be publicly shown by him in an orderly way.

COMMENT

But since I see that persuading the Noble Gentleman that he himself wishes to undertake it is going to be more difficult, I will endeavour to show the thing according to his thinking, as closely as I can and briefly.

The Noble Gentleman noticed that two consecutive odd numbers, if multiplied together, form a product which is the square of the intermediate even number minus one (thus, $1 \times 3 = 4 - 1 = 2^2 - 1$, $3 \times 5 = 15 = 16 - 1 = 4^2 - 1$, etc.) And similarly two consecutive evens form a product which is one less than the square of the intermediate odd number (thus, $0 \times 2 = 0 = 1 - 1 = 1^2 - 1, 2 \times 4 = 8 = 9 - 1 = 3^2 - 1, 4 \times 6 = 24 = 25 - 1 = 5^2 - 1$, etc.) He asked, therefore, by what ratio the factors must be increased to form a product, not those squares minus one, but equal to the squares themselves. He found this could be done, moreover, if both factors are increased by a fraction that has a denominator continually broken, infinitely so, in the form we have shown above. That is, the numerators of the part fractions are squares of odd numbers, but the denominators are everywhere twice an integer, increased by a fraction, and so infinitely. In this form, continued as far as one likes.[79]



Moreover, those factors thus constituted, continued as far as one likes (though not infinitely) form a product either less than the required square, if the number of fractions adjoined to the integer is even, or greater, if odd; so that, however, the longer this is carried on, the more nearly it approaches the required square, which is confirmed by this demonstration.

Let the first whole number of any required [pair of] factors be F, and the next $F+2$. The number between, therefore, (to be squared) is $F+1$. The product of the former, F^2+2F, is less than the square of the latter, F^2+2F+1.

Now one fraction is adjoined to each factor. Therefore the factors

$F+\dfrac{1}{2F}$ and $F+2+\dfrac{1}{2F+4}$ form a product

$\dfrac{4F^4+16F^3+20F^2+8F+9}{4F^2+8F}$, which will be greater than the square

$$F^2+2F+1=\frac{4F^4+16F^3+20F^2+8F}{4F^2+8F}.$$

Then two more fractions are adjoined; the resulting factors

$F+\dfrac{1}{2F+\frac{9}{2F}}$ and $F+2+\dfrac{1}{2F+4+\frac{9}{2F+4}}$ form a product

$$\frac{4F^3+11F}{4F^2+9}\times\frac{4F^3+24F^2+59F+54}{4F^2+16F+25}=$$

$$\frac{16F^6+96F^5+280F^4+480F^3+649F^2+594F}{16F^4+64F^3+136F^2+144F+225}$$

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[79] The first fraction, beginning with zero, oscillates between zero and infinity, but multiplied by the next fraction, beginning with 2, it is supposed to make 1.

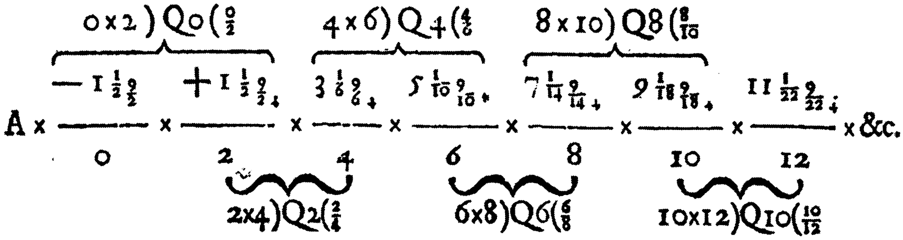
which is less than the square $F^2 + 2F + 1 =$

$$\frac{16F^6 + 96F^5 + 280F^4 + 480F^3 + 649F^2 + 594F + 225}{16F^4 + 64F^3 + 136F^2 + 144F + 225}.$$

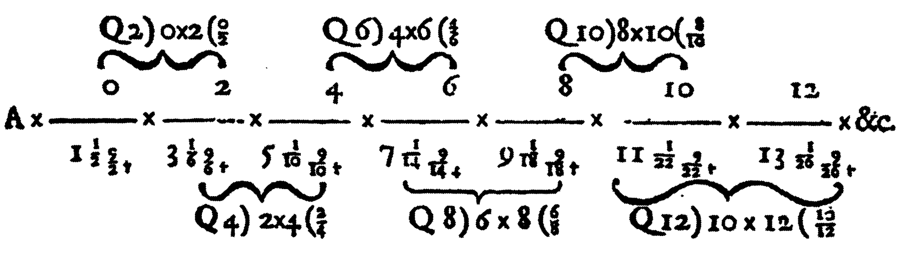
And thus it may be done as far as one likes; it will form a product which will be (in turn) now greater than, now less than, the given square (the difference, however, continually decreasing, as is clear), which was to be proved.

These having been found, moreover, they may be so adapted to our sequences, that thence the desired terms in the table become known, described according to this method of notation.

For example, (putting A for the first term of any sequence), the first sequence in our table is composed (as was shown above) in the odd places by the continued multiplication $A \times \frac{0}{1} \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}$ etc. or $A \times \frac{0}{2} \times \frac{4}{6} \times \frac{8}{10} \times \frac{12}{14}$ etc. And in the even places $1 \times \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8}$ etc. or $1 \times \frac{2}{4} \times \frac{4}{6} \times \frac{8}{10} \times \frac{14}{16}$ etc. That is,



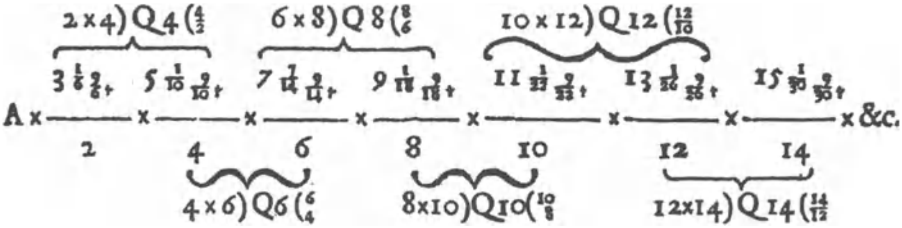
Or also (which clearly comes down to the same thing) in this form



Either way, the ratios from which the numbers to be put in either odd or even places are composed, may be separated into two ratios (as is clear), from which may be constituted the numbers to be put in each place continually; that is, the numbers in first the even then the odd, reduced to one sequence by common interpolation.

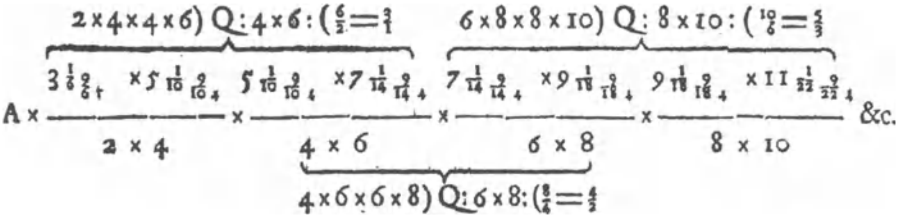
Similarly, in the third sequence, in which the numbers occupying odd places are composed by the continued multiplication $A \times \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7}$ etc.

or $A \times \frac{4}{2} \times \frac{8}{6} \times \frac{12}{10} \times \frac{16}{14}$ etc., and in the even places by $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \frac{9}{8}$ etc. or $1 \times \frac{6}{4} \times \frac{10}{6} \times \frac{14}{10} \times \frac{18}{16}$ etc. That is,

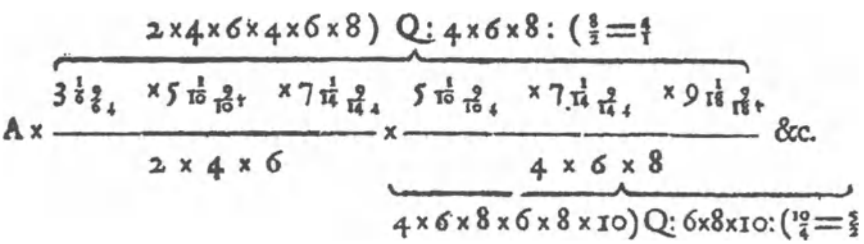


And these are indeed abundantly sufficient for completing all the numbers of our table, since it has already been shown above at Proposition 188, that any one of the desired numbers being found, the rest immediately become known. If anyone, however, desires that just as we resolved our ratios in the first and third sequences, so also to have the ratios of the remaining sequences resolved in the same way, it will indeed be possible that that also should be done, so that, however, in the fourth sequence (that is, the second after the sequence of units), there will be a separation of any ratios produced into two ratios; in the fifth, into three, and so on.

That is, in the fourth in this form:

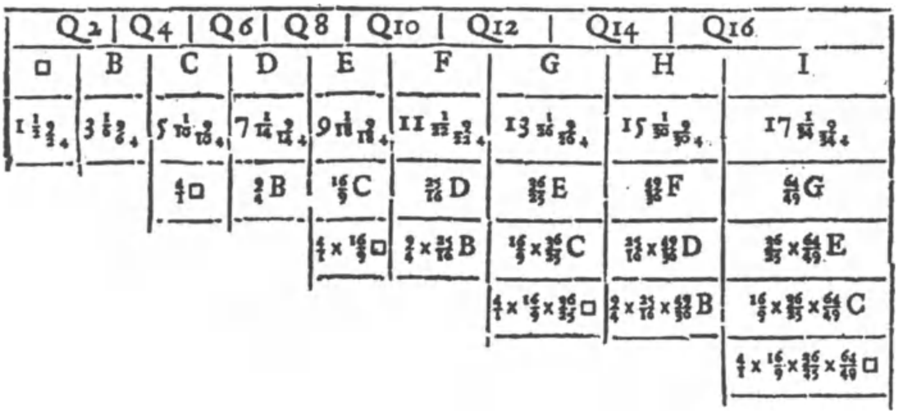


In the fifth sequence, in this form:



And similarly in the following sequences. That is, each ratio of the fourth, fifth, sixth sequence etc. is composed from two, three, four ratios etc. of the third sequence.

Since, moreover, for each of those sequences of ratios now discovered (from which the numbers of our table may be constructed by continued multiplication), it might seem convenient that they can be shown in various forms, and (just as needed) transformed from one to another, so that (standing in for other fractions) they may better support the operations of arithmetic: it is also possible to do this conveniently by what was given at the beginning of this *Comment*, that is, by the resolution (as was there taught) of squares of even numbers into the factors[80] there indicated, each being written (for convenience) in symbols, in this way:



Q2	Q4	Q6	Q8	Q10	Q12	Q14	Q16	
□	B	C	D	E	F	G	H	I
$1\frac{1}{2}\frac{9}{2}{}_4$	$3\frac{1}{6}\frac{9}{6}{}_4$	$5\frac{1}{10}\frac{9}{10}{}_4$	$7\frac{1}{14}\frac{9}{14}{}_4$	$9\frac{1}{18}\frac{9}{18}{}_4$	$11\frac{1}{22}\frac{9}{22}{}_4$	$13\frac{1}{26}\frac{9}{26}{}_4$	$15\frac{1}{30}\frac{9}{30}{}_4$	$17\frac{1}{34}\frac{9}{34}{}_4$
		$\frac{4}{1}\square$	$\frac{9}{4}B$	$\frac{16}{9}C$	$\frac{25}{16}D$	$\frac{36}{25}E$	$\frac{49}{36}F$	$\frac{64}{49}G$
				$\frac{4}{1}\times\frac{16}{9}\square$	$\frac{9}{4}\times\frac{25}{16}B$	$\frac{16}{9}\times\frac{36}{25}C$	$\frac{25}{16}\times\frac{49}{36}D$	$\frac{36}{25}\times\frac{64}{49}E$
						$\frac{4}{1}\times\frac{16}{9}\times\frac{36}{25}\square$	$\frac{9}{4}\times\frac{25}{16}\times\frac{49}{36}B$	$\frac{16}{9}\times\frac{36}{25}\times\frac{64}{49}C$
								$\frac{4}{1}\times\frac{16}{9}\times\frac{36}{25}\times\frac{64}{49}\square$

For since $\square B = 2^2 = 4$, and $BC = 4^2 = 16$,
we will have $4 : 16 = \square B : BC = \square : C = \frac{16}{4}\square = 4\square$;
and $\frac{1}{4}\square = C$
In the same way, $\frac{4}{\square} = B$ and $\frac{4}{B} = \square$.
And the same in the other places.

Therefore with the ratios recently found, there are for the sequences:

$$\text{First } A \times \frac{0}{\begin{array}{l}1\frac{1}{\;\;9}\\ 2\frac{}{2+}\end{array}} \times \frac{2}{\begin{array}{l}3\frac{1}{\;\;9}\\ 6\frac{}{6+}\end{array}} \times \frac{4}{\begin{array}{l}5\frac{1}{\;\;9}\\ 10\frac{}{10+}\end{array}} \times \frac{6}{\begin{array}{l}7\frac{1}{\;\;9}\\ 14\frac{}{14+}\end{array}} \times \frac{8}{\begin{array}{l}9\frac{1}{\;\;9}\\ 18\frac{}{18+}\end{array}} \times \frac{10}{\begin{array}{l}11\frac{1}{\;\;9}\\ 22\frac{}{22+}\end{array}}$$

$$\times \frac{12}{\begin{array}{l}13\frac{1}{\;\;9}\\ 26\frac{}{26+}\end{array}} \times \text{ etc.}$$

$$\text{that is } A \times \frac{0}{\square} \times \frac{2}{B} \times \frac{4}{C} \times \frac{6}{D} \times \frac{8}{E} \times \frac{10}{F} \times \frac{12}{G} \times \text{ etc.}$$

$$\text{Second } A \times \frac{2}{2} \times \frac{4}{4} \times \frac{6}{6} \times \frac{8}{8} \times \frac{10}{10} \times \frac{12}{12} \times \text{ etc.}$$

---

[80] *Rectangulorum aequalium latera*, literally 'sides of equal rectangles'.

$$\text{Third } A\times\frac{B}{2}\times\frac{C}{4}\times\frac{D}{6}\times\frac{E}{8}\times\frac{F}{10}\times\frac{G}{12}\times\frac{H}{14}\times\text{etc.}$$

$$\text{Fourth } A\times\frac{BC}{2\times4}\times\frac{CD}{4\times6}\times\frac{DE}{6\times8}\times\frac{EF}{8\times10}\times\frac{FG}{10\times12}\times\frac{GH}{12\times14}\times\frac{HI}{14\times16}\times\text{etc.}$$

$$\text{Fifth } A\times\frac{BCD}{2\times4\times6}\times\frac{CDE}{4\times6\times8}\times\frac{DEF}{6\times8\times10}\times\frac{EFG}{8\times10\times12}\times\frac{FGH}{10\times12\times14}\times\frac{GHI}{12\times14\times16}\times\text{etc.}$$

And thus in the rest (where, however, it is to be understood that A is not everywhere the same quantity, but in the first sequence $A=\infty$, or better $\infty\square$, in the second $A=1=\infty\square\times\frac{0}{\square}$, in the third $A=\frac{1}{2}\square=1\times\frac{2}{B}$, in the fourth $A=\frac{1}{2}=\frac{1}{2}\square\times\frac{1}{\square}$, in the fifth $A=\frac{1}{3}\square=\frac{1}{2}\times\frac{6}{D}=\frac{1}{2}\times\frac{1}{6}C=\frac{1}{2}\times\frac{4}{6}\square$. And so on. For the first vertical sequence is the same as the first across (as is clear above). They may thus be shown, merely by an equally powerful demonstration, so that only one of these infinite numbers is needed in each expression, and often indeed not one. For example, in the fifth sequence, the second ratio $\frac{CDE}{4\times6\times8}=\frac{36E}{4\times6\times8}=\frac{64C}{4\times6\times8}=\frac{4\square}{3}=\frac{16}{3B}=\frac{C}{3}=\frac{12}{D}=\frac{3E}{16}$ etc. and therefore by multiplying this ratio written in any of these ways, by the second term of that sequence, 1, (found in our table) the third term is to be had:

$$1\times\frac{4\square}{3}=\frac{16}{3B}=\frac{C}{3}=\frac{12}{D}=\frac{3E}{16}\text{ etc.}$$

And this third term multiplied by the next ratio:

$$\frac{DEF}{6\times8\times10}=\frac{64F}{6\times8\times10}=\frac{100D}{6\times8\times10}=\frac{900B}{1920}=\frac{15}{8\square}=\frac{15B}{32}=\frac{15}{2C}=\frac{5D}{24}=\frac{40}{3E}\text{ etc.}$$

will give the fourth term of the same sequence, $\frac{5}{2}=2\frac{1}{2}$, that is, the same as the table shows. (But the same term is also equally shown by multiplication of the second term, 1, by the ratio composed of the second and third.) And in this way one may show each term of our table, having made use sometimes of some one, but more often not even one, of those infinite numbers, the first of which indeed, that denoted $\square$, we introduced into our table.

Therefore the ratio of the circle to the square of its diameter (as already said) $\frac{1}{4}pd:d^2=1:\square=\frac{4}{B}=\frac{C}{4}=\frac{9}{D}=\frac{9E}{64}=\frac{225}{16F}=\frac{25G}{256}$etc. And similarly (since the ratio of the circumference to the diameter is four times the ratio of the circle to the square, because, that is, $\frac{1}{4}pd : d^2 = \frac{1}{4}p : d$), the ratio of the circumference to the diameter, $p : d = \square : 4$, and the diameter to the circumference, $d : p = \square : 4 =$

$$1 : \frac{4}{\square} = B = \frac{16}{C} = \frac{4D}{9} = \frac{256}{9E} = \frac{64F}{225} = \frac{1024}{25G} \text{ etc.}$$

That is, 1 to □ is the ratio of the circle to the square of its diameter

and 1 to B is the ratio of the diameter to the circumference.

It remains that I should show a reason (not so much from necessity as for the sake of clarity) why in assigning the value of the quantity □, as I said above, for the final denominator of the continually broken fraction (taking it to stand wherever one likes), there is to be put not 2 but rather 3, 5, 7, 9, etc. as the place where it is to stand requires. The reason indeed is this.

Since it may be assumed (as already taught) that $\square \times B = 2^2 = 4$, and that $\square = 1\frac{1}{2\frac{9}{2+}}$ and $B = 3\frac{1}{6\frac{9}{6+}}$, then if we divide $4 = 2^2$ by B (the next factor) it will give □. If the quantity B is taken incomplete, it will produce not the quantity □ itself but another which will be either greater or less, according as the imperfect value taken for B is less or greater than the exact value of B. That is, if for the divisor B is taken 3, having done the division it will give $1\frac{1}{3}$ for □; if for the divisor we take $3\frac{1}{6}$, it will give $1\frac{1}{2\frac{9}{5}}$; if $3\frac{1}{6\frac{9}{6}}$, it will give $1\frac{1}{2\frac{9}{2\frac{25}{7}}}$. And so on, as is clear from the calculation itself.

And so it will be for $B = 3\frac{1}{6+}$, $C = 5\frac{1}{10+}$, $D = 7\frac{1}{14+}$, etc. That is, for B the last denominator will be one of these: 5, 7, 9, 11, etc. (namely, the one that the place where it stands requires) because 3 (the whole number with which the description of the quantity B begins) continually increases in arithmetic progression in twos. And similarly in C, one of these: 7, 9, 11, 13, etc. And in D one of these: 9, 11, 13, 15, etc. (which, that is, in the former from 5, and in the latter from 7, continually increase in arithmetic progression in twos). And similarly in those that follow, which the calculation itself will indicate.

And generally, in any of those quantities to be described (in whatever place at length one would wish to stop) for the last denominator there may be taken twice the number that denotes the place of the fraction, increased by that whole number that begins the description.

If anyone asks, moreover, why in this process (in designating the last denominator) we make the division by the second factor rather than the first, the reason is, that thus the thing proceeds more conveniently. For as those denominators now go forward from the initial whole number increasing arithmetically; if the division were done by the first factor, the denominators would go backwards from the initial whole number, decreasing (which would confuse the description more), as trying it will show. And therefore by that rule, if (for example) the quantity F is written as $10^2 = 100$ divided by E, the denominators thus produced will be 9, 7, 5, 3, 1, −1, etc. If, however, by the first method, as $12^2 = 144$ divided by G, they would have been 13, 15, 17, 19, 21, 23, 25, etc., that is, the former decreasing from 11, the latter increasing; and, moreover, wherever the latter method gives quantities altogether too large, the former will likewise give them too large, and conversely. Whence it is also clear, that the first method of writing the final denominator is not only less confusing, but is also more accurate than the second. For since the excess and defect are always determined by the final fraction (by the addition of which, a quantity which was too large becomes too small, and conversely), where the denominator is greater (keeping the same numerator) the fraction is less, and therefore either the excess or defect is less, than if the denominator were [smaller].[81] Therefore putting in place denominators continually increasing will decrease the error, and those continually decreasing will increase it. Which indeed is true as far as you like, so that not just in our correction which proceeds by continual increase of denominators, anyone may find to their advantage (or rather, disadvantage on account of the said reason), that it may be taken so far that the increased denominator is greater than the general denominator, (namely, that which is equal to twice the whole number at the beginning) for, until that is arrived at, changing from the general denominator to the increasing denominator does not diminish, but increases the adjoined fraction, therefore also the error.

There seems to remain yet one more thing, that is, that I show by what rule continually broken fractions of this kind may be conveniently reduced to ordinary fractions.

Moreover, while it may be done by a method known to everyone, beginning from the end, and going back until one eventually arrives at the beginning, all the same it seems desirable that it may by done by starting from the beginning and proceeding as far as one likes. Therefore, we will now show how this may be done.

Therefore, let any continually broken fraction of this kind be written thus:

$$\cfrac{a}{\alpha\cfrac{b}{\beta\cfrac{c}{\gamma\cfrac{d}{\delta\cfrac{e}{\varepsilon}}}}}\ \text{etc.}$$

Therefore it may be agreed that the reduction may be set up by the accepted method, in this way:

$$\frac{a}{\alpha} = \frac{a}{\alpha}$$

$$\cfrac{a}{\alpha\cfrac{b}{\beta}} = \frac{a\beta}{b+\alpha\beta}$$

---

[81] Wallis has mistakenly written 'greater' (*major*) here.

$$\cfrac{a}{\alpha\cfrac{b}{\beta\cfrac{c}{\gamma}}} = \frac{ac + a\beta\gamma}{\alpha c + b\gamma + \alpha\beta\gamma}$$

$$\cfrac{a}{\alpha\cfrac{b}{\beta\cfrac{c}{\gamma\cfrac{d}{\delta}}}} = \frac{a\beta d + ac\delta + a\beta\gamma\delta}{bd + \alpha\beta d + \alpha c\delta + b\gamma\delta + \alpha\beta\gamma\delta}$$

etc.

And so on, as needed. Whence we may put together this rule, with the help of which we may begin the reduction from the beginning, continuing as far as we like:

$$\begin{array}{ccccc} P^{[82]} & Q^{[83]} & P & Q & Q \\ \hline N_3 & \times\,\dfrac{N_1}{N_2} & +\,D_3 & \times\,\dfrac{N_2}{D_2} & =\dfrac{N_3}{D_3} \end{array}$$

That is, if (of three consecutive fractions) the numerator of the third given, is multiplied by the numerator of the first just sought out, and the denominator of the third given, by the numerator of the second just sought out, the sum will be the numerator of the third sought. And similarly, if the numerator of the third given, is multiplied by the denominator of the first just sought out, and the denominator of the third given, by the denominator of the second just sought out, the sum will be the denominator sought.

An example may make the thing clear.

Let the fraction to be reduced be: $\cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{2\cfrac{81}{11}}}}}$

The work may be set up thus. Having found the second fraction by the usual method, the third and those following may be had thus.

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[82] *Propositus*, or proposed, or given.

[83] *Quaesitus*, or sought out.

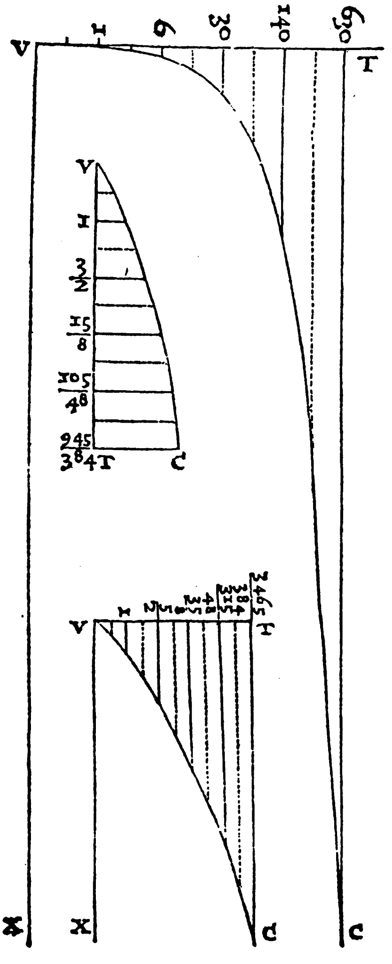
$\frac{25}{2}$	$\times 1 = 25$ $\times 2 = 4$	29
$\frac{25}{2}$	$\times 2 = 50$ $\times 13 = 26$	76
$\frac{49}{2}$	$\times 2 = 98$ $\times 29 = 58$	156
$\frac{49}{2}$	$\times 13 = 637$ $\times 76 = 152$	789
$\frac{81}{11}$	$\times 29 = 2349$ $\times 156 = 1716$	4065
$\frac{81}{11}$	$\times 76 = 6156$ $\times 789 = 8679$	14835

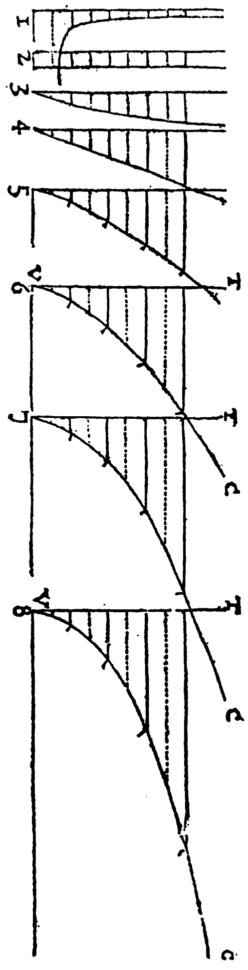
P	Q
$\frac{1}{2}$	$\frac{1}{2}$
$\frac{9}{2}$	$\frac{2}{13} = \cfrac{1}{2\frac{9}{2}}$
$\frac{25}{2}$	$\frac{29}{76} = \cfrac{1}{2\cfrac{9}{2\cfrac{25}{2}}}$
$\frac{49}{2}$	$\frac{156}{789} = \cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{2}}}}$
$\frac{81}{11}$	$\frac{4065}{14835} = \cfrac{1}{2\cfrac{9}{2\cfrac{25}{2\cfrac{49}{2\cfrac{81}{11}}}}}$

And so on as far as needed. The reason for the operations is clear from what has already been said.

If anyone still wonders, moreover, how it comes about that these continually broken fractions are alternately now greater than, now less than the required quantity, (according to whether it pleases anyone to stop here or there), he may briefly be given the reason for it. Since it is certain that an integer without any adjoined fraction is too small, the first fraction adjoining that integer increases the quantity, but the smaller it is itself, the less it increases it. Here, therefore, the first fraction increases the quantity, and indeed as far as this, that now what was too small becomes too large. And keeping the same numerator of this fraction, if the denominator is increased (which comes about by the adjoining of a second fraction) the first fraction, and therefore also the whole quantity, is decreased by adjoining a second. And this decrease will be smaller (and therefore the total quantity greater) as the denominator of this second fraction (keeping the same numerator) is increased, which comes about by adjoining a third fraction. Therefore the third fraction decreases the second, and therefore increases the first, and thus also, therefore, the whole quantity. And similarly in the subsequent steps. Thus the fourth adjoined fraction decreases the third, that is, increases the second, decreases the first and therefore the whole quantity. The fifth decreases the fourth, and therefore increases the third, decreases the second, and increases the first and therefore the whole quantity. Therefore adjoining fractions in odd places increases, in even places decreases, the quantity. Which is to be understood not only of this, but of any other fractions thus continually broken (in the denominators).

And thus far I have shown the thinking of the Noble Gentleman as briefly and clearly as I could. And what else I thought could be said about his method, I have indicated briefly.





PROPOSITION 192

Theorem

Suppose there is a smooth curve VC (not moving jerkily from here to there) with axis VX, and with tangent to the vertex VT, from which to the curve there are taken lines parallel to the axis and equally spaced from each other, of which the second, fourth, sixth, eighth, etc. (in even places) are as 1, 6, 30, 140, etc. (which numbers arise from continued multiplication of these: $1 \times \frac{6}{1} \times \frac{10}{2} \times \frac{14}{3} \times \frac{18}{4}$ etc.) Then as the second is to the third (that is, as 1 to the number interposed between 1 and 6), so will be the semicircle to the square of its diameter.

Follows from Propositions 139 and 135.

PROPOSITION 193

Theorem

Suppose there is proposed a smooth curve *VC*, to which there runs to the vertex a line *VT*, from which to the curve there are taken any number of parallel lines equally spaced from each other, of which the second, fourth, sixth, eighth, tenth, etc. (in even places) are as $1, \frac{3}{2}, \frac{15}{8}, \frac{105}{48}, \frac{945}{384}$, etc. (which numbers arise from continued multiplication of these: $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \frac{9}{8}$ etc.) Then as the second is to the third (that is, as 1 to the number that must be interposed between 1 and $\frac{3}{2}$), so will be the circle to the square of its diameter. (And also as the second is to the fifth, so will be three times the circle to four times its square, etc.)

Follows from Propositions 118, 121 and 185.

PROPOSITION 194

Theorem

Suppose there is a smooth curve *VC*, with axis *VX*, and tangent to the vertex *VT*, and of the lines taken from there to the curve (parallel to the axis and equally spaced) the second, fourth, sixth, eight, tenth, etc. (in even places) are as $1, \frac{5}{2}, \frac{35}{8}, \frac{315}{48}, \frac{3465}{384}$, etc. (which numbers arise from continued multiplication of these: $1 \times \frac{5}{2} \times \frac{7}{4} \times \frac{9}{6} \times \frac{11}{8}$ etc.) Then as the second is to the third (that is, as 1 to the number that must be interposed between 1 and $\frac{5}{2}$) so will be the circle to $\frac{4}{3}$ of the circumscribed square (or the square of its diameter), or three times the circle to four times the circumscribed square. (And as also as the second is to the fifth, so will be three times the circle to eight times its square, etc.)

Clear from what has gone before and from the table in Proposition 189.

COMMENT

And indeed innumerable propositions of this kind may be deduced from the same table (in Proposition 189), certainly having formed some or other curves of this kind according to the course of the table.

And what kind of curves all these will be it will not be very easy to judge. All the same, one may observe certain things here. That is, in the sixth sequence the (triangular) numbers 1, 3, 6, 10, etc. (which arise by continued multiplication of these: $1 \times \frac{6}{2} \times \frac{8}{4} \times \frac{10}{6} \times \frac{12}{8}$ etc.) are as the squares of the ordinates in a hyperbola, as was said at Proposition 173.

In the fourth sequence, the (arithmetically proportional) numbers 1, 2, 3, 4, etc. (which arise from continued multiplication of these: $1 \times \frac{4}{2} \times \frac{6}{4} \times \frac{8}{6}$ etc.) are as the squares of the ordinates of a parabola, or as lines in a triangle, as is obvious.

In the second sequence, the (equal) numbers 1, 1, 1, 1, etc. (which arise from continued multiplication of these: $1 \times \frac{2}{2} \times \frac{4}{4} \times \frac{6}{6}$ etc.) are as the squares of lines (or also as the lines themselves) in a parallelogram, as is obvious. Therefore in the second and fourth sequence, in reality there arise straight lines for the curves, namely, in the latter the side of a triangle, in the former the side of parallelogram.

In the sixth, eighth, tenth, etc. (taking alternate sequences), there will arise yet more complex curves, but their formulae are no less accurately designated in the said table than the known formulae of the parabola, hyperbola and ellipse.

Moreover, in the remaining interposed sequences, the first, third, fifth, etc. (in odd places) there arise the same smooth and regular curves (thus of the kind Descartes would understand as *geometric*),[84] although their formulae are more difficult to set out, as they are intermediate between the known formulae of sequences placed in even positions, according to what we showed of the form of the progressions in Proposition 187.

What the exhibited curves, indicated by each sequence of the table (whether taken in even or odd places), present for inspection, the adjoined figure shows, which exhibits those depicted curves one by one, taking the correct measure of each on (as it is said) the same scale.

Meanwhile it must be noted (what the inspection itself also indicates) that the convexity of VC (turning against the line VT), which in the last of the curves is greatest, gradually decreases in the previous ones (if we reckon it backwards), until in the fourth place the curve passes to a line (which is intermediate between concave and convex), thence in the third to concave, and in the second to a parallel, and finally in the first is recurved (that is, it continually approaches a line by those steps by which the rest recede). The

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[84] Descartes' definition of a geometric curve is *not* that it should be smooth and regular, but that it can be described by a single equation in the coordinates. Wallis has failed to find such an equation for the odd curves, but goes on to argue (see below) that such formulae or equations must exist.

same line VT, which in the fifth and following is a tangent, in the fourth cuts the line, in the third is a diameter of the curve, in the second a parallel, and finally in the first an asymptote.

What are the properties of all these curves, moreover, and by what methods they may most conveniently be described, the more curious may not for the present ask of me (for I am already tired and weary enough from the varied and difficult journey), nor that I should further handle the quadrature of the hyperbola (which was done above). And indeed it is possible that it may be pleasing to some that I offer this business while remaining silent on it, since it allows them, the method having been already demonstrated, to be diverted by the same mysteries.

And all are such as Descartes would have by the name of *geometric,* which without doubt is the case, since it is already sufficiently established for the even places, their formulae having now been discovered, and therefore also for odd places (though their formulae are not so conveniently written) it cannot be thought otherwise. What kind of equations they are that belong to each, will be clear from the regularity itself. For since to the fourth sequence there belongs a linear equation; to the sixth, a quadratic; to the eighth, a cubic; etc. (that is, the highest power is linear, a square, a cube, etc.) so to the interpolated sequences, equations must pertain that are intermediate to these (thus, to fifth powers, that which is intermediate between quadratic and linear; and it may be judged in the same way for the rest). But that equations of this kind may be satisfactorily written in the accepted way, is perhaps to be doubted.

To me, all the same, it is sufficient (and indeed repays the taking up of this labour) to have pursued the thing this far, and treading a new path to have uncovered the same by other ways; indeed what might lead me was not therefore easy to foretell at the beginning, but that pertaining to the quadrature of curves (or at least some of them), and other more difficult problems of this kind, seemed to direct the course correctly. Nor indeed was our hope disappointed. For although for the circle, its ratio to the square (which I do not deny I also looked to from the beginning) did not appear so plainly as we wished, as in various other curves, to be explained in some accepted way of notation (but by some meanderings it led me, and at length stopped at something unsayable); the reward for this labour, however, is to have indicated that [quadrature] as far as the nature of numbers allows, so that nothing more remains than that it should be agreed between mathematicians by what notation (whether mine or another yet to be considered for decision) they wish to indicate that unsayable ratio. And in other curves thus no less, everything has come out as wished (and indeed often beyond what was hoped for), in that I have shown innumerable quadratures of curves, some quite unknown until now, as well as some indeed known before, but now taught by a new and easier method. And in innumerable other intricate mathematical problems (thus of pyramids, conoids and spheres, of spiral lines and the spaces contained in them, of parabolas, and others in passing), I have either been the first to complete them, or have much elucidated. In the same way, I have reduced figures continued to infinity, both plane and solid, to known and finite measure (not one only, which was already done by Torricelli, which seemed amazing enough, but many).

Indeed, it would have been easy (by the method one preferred) to have inferred innumerable other propositions in passing (which no one skilled in these things can doubt) since that doctrine that I hand over is sufficiently fertile in its consequences. And indeed, in the first parts of this treatise I have more copiously inserted consequences of this kind, particularly so that I might indicate what this doctrine offered. But in what followed I did that more sparingly, partly because now our method and its usefulness was clear from what had gone before up to then, so that now anyone could show it by his own effort; partly also lest the number of propositions (which now seemed to swell) and therefore the whole treatise should grow to an exceedingly heavy bulk. Therefore much has been indicated lightly in passing, which if the more diligent wanted to follow up, would require, rather, a whole inquiry to each part.

There remains this: we beseech the skilled in these things, that what we thought worth showing, they will think worth openly receiving, and whatever it hides, worth imparting more properly by themselves to the wider mathematical community.

PRAISE BE TO GOD

Glossary

(The number in brackets after each definition indicates the Proposition where the term first appears.)

Arithmetic proportionals: Quantities that increase or decrease by regular addition of a fixed quantity. (1)

Binomes and apotomes: A *binome* is a quantity of the form $\sqrt{a} + \sqrt{b}$, and an *apotome* of the form $\sqrt{a} - \sqrt{b}$. (*Comment* following 127)

Circle: In Wallis's text a *circle* is always a plane figure with area. The bounding line is the *circumference.* (6)

Conoid: A solid formed by rotation of a curve around an axis of symmetry (a diameter) or an ordinate; a *parabolic conoid* is generated by the rotation of a parabola. (4)

Diameter: One of the principal axes of a conic (and for a right conic an axis of symmetry). If such an axis is aligned with the x-axis, with the vertex of the curve at the origin, then the length of the diameter as far as a given point is given by the x-coordinate (see also *ordinate*). (14)

The diameter is also sometimes called the *intercepted diameter.* (88)

Figure: A plane figure with an area. (5)

Index: The number denoting a power, thus the *index* of x^n is n. (64)

Latus rectum: The total length of the ordinates passing through a focus of a conic. For a parabola with equation $y^2 = kx$ the *latus rectum* is $2k$. (15)

Ordinate: In modern notation, the length of an ordinate of a curve at any point is given by the y-coordinate (see also *diameter*). (14)

Parabola: A curve whose equation in its simplest form is $y^n = kx$. For the common (or simple) parabola $n = 2$, while for *cubical, biquadratic* or *supersolid* parabolas, $n = 3$, 4 or 5, respectively. The latter are known as *higher* parabolas. (4; Comment following 38)

A *right* parabola has ordinates at right angles to its diameter, while an *inclined* parabola (cut from an inclined cone) has ordinates at some other angle to the diameter. (4)

A *truncated* parabola is cut off by a line $x = d$. (14)

Proportionals: Quantities of the form a, ar, ar^2, $ar^3, \ldots$. (Strictly these are *geometric proportionals*; see also *arithmetic proportionals* above.)

The *mean proportional* between x and y is $\sqrt{xy}$ (since y: $\sqrt{xy} = \sqrt{xy}$: x). (75)

The *third proportional* of two (ordered) quantities x and y is y^2/x. (84)

Pyramid: A solid with polygonal cross-sections parallel to the base; in a *parabolic pyramid* any cross-section through the vertex is bounded by parabolic curves. (4)

Right conics: Conics in which the ordinates are at right angles to the diameter. (4)

Segment: A portion of length of a line or curve. (11)

Sequences and series: A *sequence* is a finite or infinite list of terms, while a *series* is now generally understood as a sequence of partial sums. Wallis uses the single Latin word *series* both for a list of terms and as a collective noun to denote a set of such terms, usually summed. I have used *sequence* where Wallis describes individual terms generated according to some rule, but *series* where it is clear that he means all the terms taken collectively. My use of *series* in the text is thus not strictly in keeping with modern mathematical conventions, but nor was Wallis's. (1)

Sines, right and versed: The *right sine* of an arc subtending an angle 2θ at the centre of a circle is half the length of the chord connecting its ends, that is, $r \sin\theta$. The *versed sine* is the distance between the centre of the arc and the chord connecting its ends, that is, $r(1 - \cos\theta)$. (*Comment* following 38)

Universal root: A root of two or more quantities added together. (*Comment* following 165)

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Index

Sources and Studies in the History of Mathematics and Physical Sciences

Continued from page ii

T. Hawkins
Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics 1869–1926

A. Hermann/K. von Meyenn/V.F. Weisskopf (Eds.)
Wolfgang Pauli: Scientific Correspondence I: 1919–1929

C.C. Heyde/E. Seneta
I.J. Bienaymé: Statistical Theory Anticipated

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Ibn Al-Haytham's *Completion of the Conics*

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Length, Widths, Surfaces: A Portrait of Old Babylonian Alegbra and Its Kin

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The Arabic Version of Euclid's *Optics*, Volumes I and II

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Naṣīr al-Dīn al-Ṭūsī's *Memoir on Astronomy*
(al-Tadhkira fī ^{c}ilm al-hay'a)

B.A. Rosenfeld
A History of Non-Euclidean Geometry

Sources and Studies in the History of Mathematics and Physical Sciences

Continued from the previous page

J. Sesiano
Books IV to VII of Diophantus' ***Arithmetica*****: In the Arabic Translation Attributed to Qustā ibn Lūqā**

L.E. Sigler
Fibonacci's ***Liber Abaci*****: A Translation into Modern English of Leonardo Pisano's Book of Calculation**

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The Arithmetic of Infinitesimals: John Wallis 1656

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