

SOME PROBLEMS OF DIOPHANTINE APPROXIMATION.

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II.

The trigonometrical series associated with the elliptic \mathfrak{J} -functions.

2. 0. — Introduction.

2. 00. The series

$$2 \sum_1^{\infty} q^{(n-\frac{1}{2})^2}, \quad 1 + 2 \sum_1^{\infty} q^{n^2}, \quad 1 + 2 \sum_1^{\infty} (-1)^n q^{n^2},$$

where $q = e^{\pi i \tau}$, are convergent when the imaginary part of τ is positive, and represent the elliptic \mathfrak{J} -functions

$$\mathfrak{J}_2(0, \tau), \quad \mathfrak{J}_3(0, \tau), \quad \mathfrak{J}_4(0, \tau).^1$$

When τ is a real number x , the series become oscillating trigonometrical series which, if we neglect the factor 2 and the first terms of the second and third series, may be written in the forms

¹ The notation is that of TANNERY and MOLK's *Théorie des fonctions elliptiques*. We shall refer to this book as *T. and M.*

$$\sum e^{(n-\frac{1}{2})^2 \pi i x}, \quad \sum e^{n^2 \pi i x}, \quad \sum (-1)^n e^{n^2 \pi i x}.$$

These series, the real trigonometrical series formed by taking their real or imaginary parts, and the series derived from them by the introduction of convergence factors, possess many remarkable and interesting properties. It was the desire to elucidate these properties which originally suggested the researches whose results are contained in this series of papers, and it is to their study that the present paper is devoted.¹

2. 01. We shall write

$$(2. 011) \quad s_n^2 = \sum_{\nu < n} e^{(\nu-\frac{1}{2})^2 \pi i x}, \quad s_n^3 = \sum_{\nu < n} e^{\nu^2 \pi i x}, \quad s_n^4 = \sum_{\nu < n} (-1)^\nu e^{\nu^2 \pi i x}.$$

It is obvious that, if s_n is any one of s_n^2 , s_n^3 , s_n^4 , then

$$(2. 012) \quad s_n = O(n).$$

Our object is to obtain more precise information about s_n ; and we shall begin by a few remarks about the case in which x is rational. In this case s_n is always of one or other of the forms

$$O(1), \quad An + O(1),$$

where A is a constant. It is not difficult to discriminate between the different cases; it will be sufficient to consider the simplest of the three sums, viz. s_n^3 .

We suppose, as plainly we may do without loss of generality, that x is positive. Then x is of one or other of the forms

$$\frac{2\lambda + 1}{2\mu}, \quad \frac{2\lambda}{4\mu + 1}, \quad \frac{2\lambda + 1}{2\mu + 1}, \quad \frac{2\lambda}{4\mu + 3},$$

according as the denominator of $\xi = \frac{1}{2}x$ is congruent to 0, 1, 2, or 3 to modulus 4.

¹ Some of the properties in question are stated shortly in our paper 'Some problems of Diophantine Approximation' published in the *Proceedings of the fifth International Congress of Mathematicians*, Cambridge, 1912.

Now it is easy to verify that

$$\sum_0^{s-1} e^{2\nu^2 \pi i r/s}$$

is of the forms

$$(\pm 1 \pm i) \sqrt{s}, \pm \sqrt{s}, 0, \pm i \sqrt{s}$$

according as $s \equiv 0, 1, 2, 3 \pmod{4}$; and from this it follows immediately that s_n^2 is of the forms

$$(\pm 1 \pm i) An + O(1),$$

$$\pm An + O(1),$$

$$O(1),$$

$$\pm iAn + O(1),$$

in these four cases. Thus, for example, the series

$$\sum \cos(\nu^2 \pi x)$$

oscillates finitely if x is of the form $(2\lambda + 1)/(2\mu + 1)$ or $2\lambda/(4\mu + 3)$, and diverges if x is of the form $(2\lambda + 1)/2\mu$ or $2\lambda/(4\mu + 1)$.¹

2. I. — O and o Theorems.

2. 10. We pass to the far more difficult and interesting problems which arise when x is irrational. The most important and general result which we have proved in this connexion is that

$$(2. 101) \quad s_n = o(n)$$

for any irrational x . This result may be established by purely elementary reasoning which can be extended so as to show that such series as

¹ This result (or rather the analogous result for the sine series) is stated by BROWWICH, *Infinite Series*, p. 485, Ex. 10. We have been unable to find any complete discussion of the question, but the necessary materials will be found in DIRICHLET-DEDEKIND, *Vorlesungen über Zahlentheorie*, pp. 285 *et seq.* See also RIEMANN, *Werke*, p. 249; GENOCCHI, *Atti di Torino*, vol. 10, p. 985.

$$(2. 102) \quad \sum e^{n^2 \pi i x}, \sum e^{n^4 \pi i x}, \dots$$

also possess the same property. We do not propose to include this proof in the present paper. Although elementary, it is by no means particularly easy; and it will find a more natural place in a paper dealing with the higher series (2. 102). In the present paper we shall establish the equation (2. 101) by arguments of a more transcendental, though really simpler, character, which depend ultimately on the formulae for the linear transformation of the \mathcal{F} -functions, and will be found to give much more precise results for particular classes of values of x .

2. 11. It is very easy to see that, as a rule, the equation (2. 101) must be very far from expressing the utmost that can be asserted about s_n .

It follows from the well known theorem of RIESZ-FISCHER that the series

$$(2. 111) \quad \sum \frac{\cos n^2 \pi x}{n^{\frac{1}{2} + \delta}}, \quad \sum \frac{\sin n^2 \pi x}{n^{\frac{1}{2} + \delta}} \quad (\delta > 0)$$

are FOURIER'S series. Hence, by a theorem of W. H. YOUNG¹, it follows that they become convergent almost everywhere after the introduction of a convergence factor $n^{-\delta'}$ ($\delta' > 0$). As δ and δ' are both arbitrarily small, the series themselves must converge almost everywhere. Hence the equation

$$(2. 112) \quad s_n^2 = o\left(n^{\frac{1}{2} + \delta}\right)$$

must hold for almost all values of x . It is evident that the same argument may be applied to s_n^2 and s_n^4 , and to the analogous sums associated with such series as (2. 102).

If, instead of the series (2. 111), we consider the series

$$(2. 113) \quad \sum \frac{\cos n^2 \pi x}{n^{\frac{1}{2}} (\log n)^{\frac{1}{2} + \delta}}, \quad \sum \frac{\sin n^2 \pi x}{n^{\frac{1}{2}} (\log n)^{\frac{1}{2} + \delta}},$$

and use, instead of YOUNG'S theorem, the more precise theorem that any FOURIER'S series becomes convergent almost everywhere after the introduction of a convergence factor $1/\log n$,² we find that we can replace (2. 112) by the more precise equation

$$(2. 114) \quad s_n^2 = o\left\{n^{\frac{1}{2}} (\log n)^{\frac{3}{2} + \delta}\right\};$$

¹ *Comptes Rendus*, 23 Dec. 1912.

² HARDY, *Proc. Lond. Math. Soc.*, vol. 12, p. 370. The theorem was also discovered independently by M. RIESZ.

and it is evident that we can obtain still more precise equations by the use of repeated logarithmic factors. These we need not state explicitly, for none of them are as precise as those which we shall obtain later in the paper. These latter results have, moreover, a considerable advantage over those enunciated here, in that the exceptional set of measure zero, for which our equations may possibly cease to hold, will be precisely defined instead of being, as here, entirely unspecified. The main interest of the argument sketched here lies in the fact that it can be extended to series such as (2. 102).¹

2. 120. We proceed now to the analysis on which the principal results of the paper depend. These are contained, first in the equation (2. 101), and secondly in the equation

$$(2. 1201) \quad s_n = O(\sqrt{n}),$$

which we shall prove for extensive classes of values of x .

In Chap. 3 of his *Calcul des Résidus*, LINDELÖF gives an extremely elegant proof of the formula

$$(2. 1202) \quad \sum_0^{q-1} e^{n^2 \pi i p/q} = \sqrt{\frac{i q}{p}} \sum_0^{p-1} e^{-n^2 \pi i q/p},$$

where p and q are positive integers of which one is even and the other odd.² Our first object will be to obtain, by an appropriate modification of LINDELÖF's argument, analogous, though naturally rather less simple, formulae, applicable to the series $\sum e^{n^2 \pi i x}$, where x is irrational, and to the other series which we are considering.

We shall, however, consider sums of a more general form than those of which we have spoken hitherto, viz. the sums

$$(2. 1203) \quad \begin{cases} s_n^2(x, \theta) = \sum_{\nu \leq n} e^{(\nu - \frac{1}{2})^2 \pi i x} \cos(2\nu - 1)\pi\theta, \\ s_n^3(x, \theta) = \sum_{\nu \leq n} e^{\nu^2 \pi i x} \cos 2\nu\pi\theta, \\ s_n^4(x, \theta) = \sum_{\nu \leq n} (-1)^\nu e^{\nu^2 \pi i x} \cos 2\nu\pi\theta. \end{cases}$$

¹ The argument may even be extended to series of the type $\sum e^{\lambda n^2 i x}$, where λn is not necessarily a multiple of π ; but for this we require a whole series of theorems concerning DIRICHLET'S series.

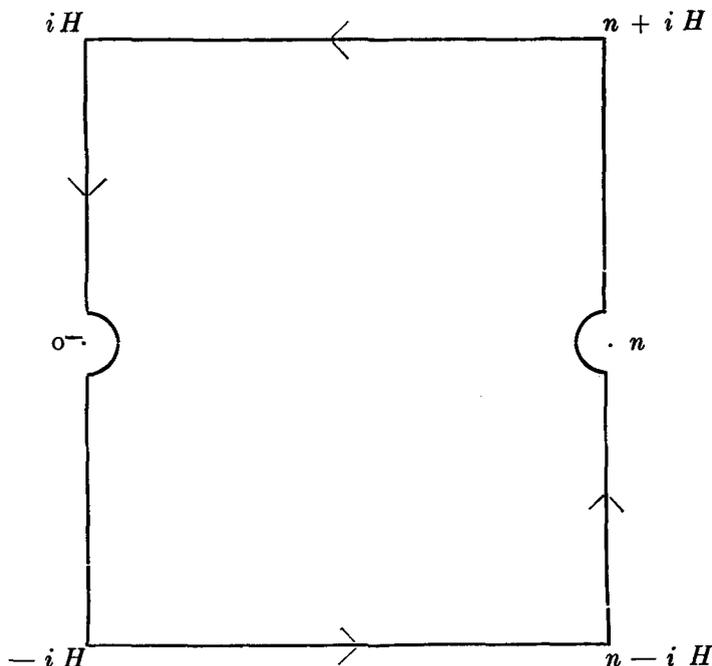
² The formula is due to GENOCCHI and SCHAAR. See LINDELÖF, *l. c.*, p. 75, for references to the history of the formula.

Here x and θ are positive and less than 1, x is irrational, and n is not necessarily an integer. These sums are related to the functions $\mathcal{G}_2(\nu, \tau), \dots$ as s_n^2, \dots are related to $\mathcal{G}_2(0, \tau), \dots$

2. 121. We consider the complex integral

$$\int e^{z^2 \pi i x} \cos 2z\pi\theta \pi \cot \pi z \, dz$$

taken round the contour C shown in the figure. We suppose that the points $0, n$ are in the first instance avoided, as in the figure, by small semicircles of



radius ρ , and that ρ is then made to tend to zero. An obvious application of CAUCHY'S Theorem gives the result

$$(2. 1211) \quad \sum_0^n e^{v^2 \pi i x} \cos 2v\pi\theta = \frac{1}{2i} P \oint_C e^{z^2 \pi i x} \cos 2z\pi\theta \cot \pi z \, dz,$$

where P is the sign of CAUCHY'S principal value, and the dashes affixed to the sign of summation imply that the terms for which $\nu = 0$ and $\nu = n$ are to be divided by 2.

We shall find it convenient to divide the contour C into two parts C_1 and C_2 , its upper and lower halves, and to consider the integrals along C_1 and C_2 ,

separately. When we attempt to do this a difficulty arises from the fact that, owing to the poles of the subject of integration at $z=0$ and $z=n$, the two integrals are not separately convergent. This difficulty is, however, trivial and may be avoided by means of a convention.

Suppose that $f(x)$ is a real or complex function of a real variable x which, near $x=\alpha$, is of the form

$$\frac{C}{x-\alpha} + \varphi(x),$$

where $\varphi(x)$ is a function which possesses an absolutely convergent integral across $x=\alpha$; and suppose that, except at $x=\alpha$, $f(x)$ is continuous in the interval (a, A) , where $a < \alpha < A$. Then CAUCHY'S principal value

$$P \int_a^A f(x) dx$$

exists; but $f(x)$ has no integral in any established sense from a to α or from α to A . We shall, however, write

$$P \int_a^a f(x) dx = \lim_{\varepsilon \rightarrow 0} \left\{ \int_a^{a-\varepsilon} f(x) dx - C \log \varepsilon \right\},$$

$$P \int_a^A f(x) dx = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{a+\varepsilon}^A f(x) dx + C \log \varepsilon \right\},$$

and it is clear that, with these conventions, we have

$$P \int_a^a f(x) dx + P \int_a^A f(x) dx = P \int_a^A f(x) dx.$$

It is clear, moreover, that a similar convention may be applied to complex integrals such as those which we are considering; thus

$$P \int_0^{iH} e^{z^2 \pi i} \cos 2z \pi \theta \pi \cot \pi z dz$$

(taken along the line $0, iH$) is to be interpreted as meaning

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{i\varepsilon}^{iH} e^{\nu^2 \pi i x} \cos 2z\pi\theta \pi \cot \pi z \, dz + \log \varepsilon \right).$$

We may now write (2. 1211) in the form

$$(2. 1212) \quad \sum_0^n e^{\nu^2 \pi i x} \cos 2\nu\pi\theta = \frac{1}{2i} \left(P \int_{C_1} - P \int_{C_2} \right) e^{\nu^2 \pi i x} \cos 2z\pi\theta \cot \pi z \, dz,$$

where now C_1 and C_2 are each supposed to be described starting from 0 . In the first of these two integrals we write

$$\cot \pi z = i + \frac{2i}{e^{2s\pi i} - 1},$$

and in the second

$$\cot \pi z = -i - \frac{2i}{e^{-2s\pi i} - 1}.$$

The two constant terms in these expressions give rise to integrals which may be taken along the real axis from 0 to n , instead of along C_2 and C_1 ; uniting and transposing these terms we obtain

$$(2. 1213) \quad \sum_0^n e^{\nu^2 \pi i x} \cos 2\nu\pi\theta - \int_0^n e^{\nu^2 \pi i x} \cos 2z\pi\theta \, dz = I_1 + I_2,$$

where

$$I_1 = P \int_{C_1} \frac{e^{\nu^2 \pi i x} \cos 2z\pi\theta}{e^{-2s\pi i} - 1} \, dz,$$

$$I_2 = P \int_{C_2} \frac{e^{\nu^2 \pi i x} \cos 2z\pi\theta}{e^{2s\pi i} - 1} \, dz.$$

We now write

$$\frac{1}{e^{-2s\pi i} - 1} = e^{2s\pi i} + e^{4s\pi i} + \dots + e^{2(k-1)s\pi i} + \frac{e^{2ks\pi i}}{1 - e^{2s\pi i}}$$

in I_1 , and

$$\frac{1}{e^{2s\pi i} - 1} = e^{-2s\pi i} + e^{-4s\pi i} + \dots + e^{-2(k-1)s\pi i} + \frac{e^{-2ks\pi i}}{1 - e^{-2s\pi i}}$$

in I_2 . If we observe that

$$\int_{C_1} e^{s^2 \pi i x + 2 \nu z \pi i} \cos 2 z \pi \theta dz + \int_{C_2} e^{s^2 \pi i x - 2 \nu z \pi i} \cos 2 z \pi \theta dz \\ = 2 \int_0^n e^{s^2 \pi i x} \cos 2 \nu z \pi \cos 2 z \pi \theta dz,$$

we see that (2. 1213) may be transformed into

$$(2. 1214) \quad \sum_0^n e^{\nu^2 \pi i x} \cos 2 \nu \pi \theta - 2 \sum_0^{k-1} \int_0^n e^{s^2 \pi i x} \cos 2 \nu \pi z \cos 2 z \pi \theta dz = K_1 + K_2,$$

where

$$K_1 = P \int_{C_1} e^{s^2 \pi i x} \cos 2 z \pi \theta \frac{e^{2 k z \pi i}}{1 - e^{2 z \pi i}} dz, \\ K_2 = P \int_{C_2} e^{s^2 \pi i x} \cos 2 z \pi \theta \frac{e^{-2 k z \pi i}}{1 - e^{-2 z \pi i}} dz.$$

2. 122. We shall now suppose that $H \rightarrow \infty$, so that the parts of C_1 and C_2 which are parallel to the axis of x go off to infinity. If $z = \xi + i\eta$, and η is large and positive, the modulus of the subject of integration in K_1 is very nearly equal to

$$\frac{1}{2} \exp \left\{ -2 \pi \eta (k + \xi x - \theta) \right\};$$

while if $z = \xi - i\eta$, and η is again large and positive, the modulus of the subject of integration in K_2 is very nearly equal to

$$\frac{1}{2} \exp \left\{ -2 \pi \eta (k - \xi x - \theta) \right\}.$$

From this it follows immediately that, if

$$(2. 1221) \quad k > n x + \theta,$$

the contributions to K_1 and K_2 of the parts of C_1 and C_2 which we are causing to tend to infinity will tend to zero.

We are now left with two integrals each of which is composed of two parts taken along rectilinear contours, and we may write

$$K_1 = \left(P \int_0^{i\infty} - P \int_n^{n+i\infty} \right) e^{s^2 \pi i x} \cos 2z \pi \theta \frac{e^{2kz \pi i}}{1 - e^{2s \pi i}} dz$$

$$K_2 = \left(P \int_0^{-i\infty} - P \int_n^{n-i\infty} \right) e^{s^2 \pi i x} \cos 2z \pi \theta \frac{e^{-2kz \pi i}}{1 - e^{-2s \pi i}} dz.$$

Of the four rectilinear integrals thus obtained two, viz. the two taken along the imaginary axis, cancel one another. In the other two we write

$$z = n + it, \quad z = n - it$$

respectively, and then unite the two into a single integral with respect to t ; and when we substitute the result in (2. 1214) we obtain

$$\sum_0^n e^{\nu^2 \pi i x} \cos 2\nu \pi \theta - 2 \sum_0^{k-1} \int_0^n e^{s^2 \pi i x} \cos 2\nu \pi z \cos 2z \pi \theta dz = K,$$

where

$$K = i \int_0^\infty e^{\pi i x (n^2 - t^2)} \frac{e^{-2knt}}{1 - e^{-2\pi t}} \{ e^{2n\pi t} \cos 2(n - it) \pi \theta - e^{-2n\pi t} \cos 2(n + it) \pi \theta \} dt.$$

2. 123. We now write

$$K = i \int_0^\infty = i \int_0^1 + i \int_1^\infty = K' + K'';$$

and we proceed to show that

$$(2. 1231) \quad K'' = O \sqrt{\frac{1}{x}}$$

uniformly in respect to θ , by which we imply that there is an absolute constant A such that

$$|K''| < \frac{A}{\sqrt{x}}$$

for $0 < x < 1$, $0 \leq \theta \leq 1$, all values of n , and all values of k subject to the inequality (2. 1221).

We may plainly ignore the factor $i e^{n^2 \pi i x}$ in K . The factor in curly brackets is equal to

$$2(\cos 2n\pi\theta \cosh 2t\pi\theta \sinh 2nx\pi t + i \sin 2n\pi\theta \sinh 2t\pi\theta \cosh 2nx\pi t).$$

The factor $e^{-t^2 \pi i x}$ we separate into its real and imaginary parts. When we multiply these two factors together our integral splits up into four, of which the integral

$$(2. 1232) \quad \int_1^{\infty} \cos t^2 \pi x \cosh 2t\pi\theta \sinh 2nx\pi t \frac{e^{-2k\pi t}}{1 - e^{-2\pi t}} dt$$

is typical; and it will be sufficient to consider this integral, the same arguments applying to all four.

The function $1/(1 - e^{-2\pi t})$ decreases steadily as t increases from 1 to ∞ . Hence, by the second mean value theorem, the integral (2. 1232) may be written in the form

$$(2. 1233) \quad A \int_1^T \cos t^2 \pi x \cosh 2t\pi\theta \sinh 2nx\pi t e^{-2k\pi t} dt,$$

where A (as always in this part of the paper) denotes an absolute numerical constant, and $T > 1$. In (2. 1233) we replace the hyperbolic functions by their expressions in terms of exponentials; and the integral then splits up into four, of which we need only consider

$$(2. 1234) \quad A \int_1^T \cos t^2 \pi x e^{-2\pi t(k - nx - \theta)} dt,$$

the arguments which we apply to this integral applying *a fortiori* to the rest. The integral (2. 1234) may, by another application of the second mean value theorem, be transformed into

$$(2. 1235) \quad A \int_1^T \cos t^2 \pi x dt,^1 \quad (1 < T' < T).$$

Now, if T' and T'' are any positive numbers whatever, we have

$$\int_{T'}^{T''} \cos t^2 \pi x dt = \frac{1}{\sqrt{x}} \int_{T'\sqrt{x}}^{T''\sqrt{x}} \cos \pi u^2 du;$$

and the integral last written is less in absolute value than an absolute constant. We have therefore proved the equation (2. 1231), and it follows that

$$(2. 1236) \quad \sum_0^n e^{\nu^2 \pi i x} \cos 2 \nu \pi \theta - 2 \sum_0^{k-1} \int_0^n e^{z^2 \pi i x} \cos 2 \nu \pi z \cos 2 z \pi \theta dz = K' + O \sqrt{\frac{1}{x}}.$$

2. 124. The next step in the proof consists in showing that, in the equation (2. 1236), k may be regarded as capable of variation to an extent $O(1)$ on either side, that is to say that we may replace k by any other integer k' lying between $k - A$ and $k + A$, without affecting the truth of the equation. That this is so if k is increased is obvious from what precedes, as the inequality (2. 1221) is still satisfied; but when k is decreased an independent proof is required.

We consider separately the effects of such a variation on the two sides of the equation (2. 1236). As regards the left hand side, it is plain that our assertion will be true if

$$\int_0^n e^{z^2 \pi i x} \cos 2 z \pi \alpha dz = O \sqrt{\frac{1}{x}}$$

uniformly for all values of n and α , and therefore certainly true if

$$\int_0^n e^{z^2 \pi i x + 2 z \pi i \alpha} dz = O \sqrt{\frac{1}{x}}.$$

¹ The A in this formula is of course not the *same* numerical constant as before.

But

$$\begin{aligned} \int_0^n e^{2\pi i x + 2s\pi i a} dz &= e^{-\pi i a^2/x} \int_0^n e^{\pi i x(x+a/x)^2} dz \\ &= e^{-\pi i a^2/x} \int_{a/x}^{n+a/x} e^{2\pi i x} dz \\ &= \frac{1}{\sqrt{x}} e^{-\pi i a^2/x} \int_{a/\sqrt{x}}^{n\sqrt{x}+a/\sqrt{x}} e^{\pi i u^2} du; \end{aligned}$$

and this expression is evidently of the form desired.

We have now to consider the effect of a variation of k on the right hand side of (2. 1236). The difference produced by such a variation is plainly of the form

$$\begin{aligned} O \int_0^1 \left| \frac{e^{-2k\pi t} - e^{-2k'\pi t}}{1 - e^{-2\pi t}} \right| e^{2\pi t(n\pi + \theta)} dt \\ = O \int_0^1 e^{-2\pi t(k - n\pi - \theta)} dt \\ = O(1) = O \sqrt{\frac{1}{x}}. \end{aligned}$$

Thus finally we may regard the k which occurs on either side of (2. 1236) as capable of variation to an extent $O(1)$.

2. 125. We proceed now to replace the integrals which occur on the left hand side of (2. 1236) by integrals over the range $(0, \infty)$. We write

$$I_\nu = \int_0^n e^{2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta dz = \int_0^\infty - \int_n^\infty = I'_\nu - I''_\nu.$$

Now consider the integral

$$\int e^{2\pi i x} \cos 2\nu\pi z \cos 2z\pi\theta dz,$$

taken round the rectangular contour whose angular points are $n, n + N,$

$n + N + iH, n + iH$. The modulus of the subject of integration is less than a constant multiple of

$$e^{-2\pi\eta(\xi x - \nu - \theta)},$$

and from this it is easily deduced that, if

$$\nu + \theta < nx,$$

the contributions of the sides $(n + N, n + N + iH)$ and $(n + N + iH, n + iH)$ tend to zero as N and H tend to infinity, and so that the second integral which occurs in our expression for I_ν may be replaced by one taken along the line $(n, n + i\infty)$. In order that this transformation may be legitimate for $\nu = 0, 1, \dots, k' - 1$ we must have

$$(2. 1251) \quad k' < nx + 1 - \theta.$$

It is important to observe that this condition and the condition (2. 1221) cannot always be satisfied with $k = k'$; but that the difference between the least k such that $k > nx + \theta$ and the greatest k' such that $k' < nx + 1 - \theta$ cannot be greater than 1.¹

On the assumption that (2. 1251) is satisfied, we have

$$\begin{aligned} 2 \sum_0^{k'-1} I_\nu'' &= \int_n^{n+i\infty} e^{x^2 \pi i z} \cos 2z \pi \theta \frac{\sin (2k' - 1) \pi z}{\sin \pi z} dz \\ &= i \int_0^\infty e^{(n^2 - t^2) \pi i x - 2nxt} \cos 2(n + it) \pi \theta \frac{\sinh (2k' - 1) \pi t}{\sinh \pi t} dt \\ &= L, \end{aligned}$$

say; and so, bearing in mind the results of the analysis of 2. 124,

$$(2. 1252) \quad \sum_0^n e^{\nu^2 \pi i x} \cos 2\nu \pi \theta - 2 \sum_0^{k'-1} \int_0^\infty e^{x^2 \pi i z} \cos 2\nu \pi z \cos 2z \pi \theta dz \\ = K' - L + O \sqrt{\frac{1}{x}}.$$

¹ It is these facts which render necessary the analysis of 2. 124.

2. 126. We next write

$$L = \int_0^{\infty} = \int_0^1 + \int_1^{\infty} = L' + L'',$$

and we proceed to show that

$$L'' = O \sqrt{\frac{1}{x}},$$

so that L may be replaced by L' in (2. 1252). The argument is practically the same as that of 2. 123. We have to consider a number of integrals of which

$$(2. 1261) \quad \int_1^{\infty} \cos t^2 \pi x \cosh 2t\pi\theta e^{-2n\pi t} \frac{\sinh (2k' - 1)\pi t}{\sinh \pi t} dt$$

is typical. Writing $2e^{-\pi t} / (1 - e^{-2\pi t})$ for cosech πt , observing that the factor $1 / (1 - e^{-2\pi t})$ is monotonic, and using the second mean value theorem as in 2. 123, we arrive at the result desired.

We may accordingly replace L by L' in (2. 1252). And our next step is to show that the k' which occurs in this modified form of (2. 1252) may be regarded as capable of variation to an extent $O(1)$. Here again our analysis is practically the same as some of our previous work (in 2. 124), and there is therefore no need to insist on its details. We may now write (2. 1252) in the form

$$(2. 1262) \quad \sum_0^n e^{v^2 \pi i x} \cos 2v\pi\theta - 2 \sum_0^{k-1} \int_0^{\infty} e^{z^2 \pi i x} \cos 2v\pi z \cos 2z\pi\theta dz$$

$$= \mathfrak{R} - \mathfrak{Q} + O \sqrt{\frac{1}{x}},$$

where

$$(2. 1263) \quad \left\{ \begin{aligned} \mathfrak{R} &= i \int_0^1 e^{\pi i x (n^2 - t^2)} \frac{e^{-2k\pi t}}{1 - e^{-2\pi t}} \{ e^{2n\pi t} \cos 2(n - it)\pi\theta - e^{-2n\pi t} \cos 2(n + it)\pi\theta \} dt, \\ \mathfrak{Q} &= i \int_0^1 e^{\pi i x (n^2 - t^2) - 2n\pi t} \cos 2(n + it)\pi\theta \frac{\sinh (2k - 1)\pi t}{\sinh \pi t} dt; \end{aligned} \right.$$

and, as the k 's which occur in these equations may all be regarded as capable of variation to an extent $O(1)$, there is no longer any reason to distinguish between k and k' .

2. 127. Again

$$(2. 1271) \quad \mathfrak{R} - \mathfrak{I} = \frac{1}{2} i \int_0^1 \frac{e^{\pi i x (n^2 - t^2)}}{\sinh \pi t} Q dt,$$

where

$$\begin{aligned} Q &= e^{-(2k-1)\pi t} \{ e^{2n\pi t} \cos 2(n-it)\pi\theta - e^{-2n\pi t} \cos 2(n+it)\pi\theta \} \\ &\quad - 2 e^{-2n\pi t} \sinh (2k-1)\pi t \cos 2(n+it)\pi\theta \\ &= 2 \cos 2n\pi\theta \cosh 2t\pi\theta \sinh (2nx - 2k + 1)\pi t \\ &\quad + 2i \sin 2n\pi\theta \sinh 2t\pi\theta \cosh (2nx - 2k + 1)\pi t. \end{aligned}$$

We select the value of k for which

$$-1 < 2nx - 2k + 1 < 1;$$

and the integral (2. 1271) splits up into two, of which it will be sufficient to consider the first, viz.

$$(2. 1272) \quad i \cos 2n\pi\theta \int_0^1 e^{\pi i x (n^2 - t^2)} \cosh 2t\pi\theta \frac{\sinh (2nx - 2k + 1)\pi t}{\sinh \pi t} dt.$$

This is of the form

$$(2. 1273) \quad O(1) \int_0^1 e^{-\beta n i x} \cosh 2t\pi\theta \frac{\sinh \alpha \pi t}{\sinh \pi t} dt,$$

where $\alpha = |2nx - 2k + 1|$. It will be enough to consider the real part of this integral, the imaginary part being amenable to similar treatment.

The function

$$\frac{\sinh \alpha \pi t}{\sinh \pi t} \quad (0 < \alpha < 1)$$

decreases steadily from α as t increases from zero. Hence

$$\int_0^1 \cos \pi x t^2 \cosh 2t\pi\theta \frac{\sinh \alpha\pi t}{\sinh \pi t} dt = \alpha \int_0^{\tau} \cos \pi x t^2 \cosh 2t\pi\theta dt$$

$$= \alpha \cosh 2\tau\pi\theta \int_{\tau'}^{\tau} \cos \pi x t^2 dt,$$

τ and τ' denoting positive numbers less than 1. Since $0 < \alpha < 1$, $0 \leq \theta \leq 1$, the first factor here is of the form $O(1)$; and the second is (cf. 2. 123) of the form $O\sqrt{\frac{1}{x}}$. Hence finally

$$\mathfrak{R} - \mathfrak{I} = O\sqrt{\frac{1}{x}},$$

and so the left hand side of (2. 1262) is itself of the form $O\sqrt{\frac{1}{x}}$.

2. 128. But

$$\int_0^{\infty} e^{v^2\pi iz} \cos 2\nu\pi z \cos 2z\pi\theta dz = \frac{1}{2} \sqrt{\frac{i}{x}} e^{-\pi i(\theta^2 + \nu^2)x} \cos \frac{2\nu\pi\theta}{x}.$$

Substituting this expression in (2. 1262), and observing that k may now be supposed to be the integral part of nx , we obtain

Theorem 2. 128 *If $0 < x < 1$, $0 \leq \theta \leq 1$, then*

$$\sum^n e^{v^2\pi ix} \cos 2\nu\pi\theta - \sqrt{\frac{i}{x}} e^{-\pi i\theta^2 x} \sum^{nx} e^{-v^2\pi i|x} \cos \frac{2\nu\pi\theta}{x} = O\sqrt{\frac{1}{x}},$$

where $O\sqrt{\frac{1}{x}}$ denotes a function of n , x , and θ which is in absolute value less than a constant multiple of $\sqrt{\frac{1}{x}}$.

We have omitted the lower limits of summation, and the dashes, which are now plainly irrelevant.

We can also prove, by arguments of the same character as those of §§ 2. 121 *et seq.*,

¹ LINDELÖF, *l. c.*, p. 44.

Theorem 2. 1281. *Under similar conditions*

$$\sum^n e^{(v-\frac{1}{2})^2 \pi i x} \cos (2v-1) \pi \theta - \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} \sum^n (-1)^v e^{-v^2 \pi i/x} \cos \frac{2v\pi\theta}{x} = O \sqrt{\frac{1}{x}},$$

$$\sum^n (-1)^v e^{v^2 \pi i x} \cos 2v\pi\theta - \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} \sum^n e^{(v-\frac{1}{2})^2 \pi i x} \cos \frac{(2v-1)\pi\theta}{x} = O \sqrt{\frac{1}{x}}.$$

It will hardly be necessary for us to exhibit any details of the proofs, and we will only remark that the integral

$$\int e^{z^2 \pi i x} \cos 2z\pi\theta \cot \pi z \, dz$$

of 2. 121 is replaced by one or other of the integrals

$$\int e^{z^2 \pi i x} \cos 2z\pi\theta \tan \pi z \, dz, \quad \int e^{z^2 \pi i x} \cos 2z\pi\theta \operatorname{cosec} \pi z \, dz.$$

It is on the transformation formulae contained in Theorems 2. 128 and 2. 1281 that all the results of this part of the paper will depend.

2. 13. We have the following system of formulae:

$$s_n^2(x+1, \theta) = \sqrt{i} s_n^2(x, \theta),$$

$$s_n^3(x+1, \theta) = s_n^4(x, \theta),$$

$$s_n^4(x+1, \theta) = s_n^3(x, \theta),$$

$$s_n^2(-x, \theta) = \bar{s}_n^2(x, \theta),$$

$$s_n^3(-x, \theta) = \bar{s}_n^3(x, \theta),$$

(2. 131)

$$s_n^4(-x, \theta) = \bar{s}_n^4(x, \theta),$$

$$s_n^2(x, \theta) = \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} s_{n^2}^4\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O \sqrt{\frac{1}{x}},$$

$$s_n^3(x, \theta) = \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} s_{n^3}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O \sqrt{\frac{1}{x}},$$

$$s_n^4(x, \theta) = \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} s_{n^4}^2\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O \sqrt{\frac{1}{x}}.$$

Here \bar{s}_n denotes the conjugate of s_n . It will be convenient in what follows to write $O\sqrt{\frac{1}{x}}$ in the equivalent form

$$\frac{O(1)}{\sqrt{x}}.$$

Now suppose that x is expressed in the form of a simple continued fraction

$$(2. 132) \quad \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \dots,$$

and write

$$(2. 133) \quad x = \frac{1}{a_1 + x_1}, \quad x_1 = \frac{1}{a_2 + x_2}, \dots,$$

$$\theta_1 = \frac{\theta}{x} - \left[\frac{\theta}{x} \right], \quad \theta_2 = \frac{\theta_1}{x_1} - \left[\frac{\theta_1}{x_1} \right], \dots,$$

so that

$$0 < x_r < 1, \quad 0 \leq \theta_r < 1$$

for all values of r . Further, let λ_r denote an unspecified index chosen from the numbers 2, 3, 4; and let ω denote a number whose modulus is unity but whose exact value will vary from equation to equation.

This being so, we have

$$\begin{aligned} s_n^\lambda(x, \theta) &= \frac{\omega}{\sqrt{x}} s_{nx}^{\lambda_1} \left(-\frac{1}{x}, \frac{\theta}{x} \right) + \frac{O(1)}{\sqrt{x}} \\ &= \frac{\omega}{\sqrt{x}} s_{nx}^{\lambda_1} (-a_1 - x_1, \theta_1) + \frac{O(1)}{\sqrt{x}} \\ &= \frac{\omega}{\sqrt{x}} s_{nx}^{\lambda_1} (-x_1, \theta_1) + \frac{O(1)}{\sqrt{x}} \\ &= \frac{\omega}{\sqrt{x}} \bar{s}_{nx}^{\lambda_1}(x_1, \theta_1) + \frac{O(1)}{\sqrt{x}}. \end{aligned}$$

Transforming $s_{nx}^{\lambda_1}(x_1, \theta_1)$ in the same way, we obtain

$$s_n^\lambda(x, \theta) = \frac{\omega}{\sqrt{x x_1}} s_{n x x_1}^{\lambda_2}(x_2, \theta_2) + O(1) \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x x_1}} \right\}.$$

Repeating the argument, we find

$$\begin{aligned}
 s_n^\lambda(x, \theta) &= \frac{\omega}{\sqrt{x x_1 \dots x_{\nu-1}}} s_{n x x_1 \dots x_{\nu-1}}^{\lambda_\nu}(x_\nu, \theta_\nu) \\
 &+ O(1) \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x x_1}} + \dots + \frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}} \right\} \\
 (2. 134) \quad &= \frac{\omega}{\sqrt{x x_1 \dots x_{\nu-1} x_\nu}} s_{n x x_1 \dots x_{\nu-1} x_\nu}^{\lambda_{\nu+1}}(x_{\nu+1}, \theta_{\nu+1}) \\
 &+ O(1) \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x x_1}} + \dots + \frac{1}{\sqrt{x x_1 \dots x_{\nu-1} x_\nu}} \right\}.
 \end{aligned}$$

Now

$$\begin{aligned}
 x_r &\leq \frac{1}{1 + x_{r+1}}, \\
 (2. 135) \quad x_r x_{r+1} &\leq \frac{x_{r+1}}{1 + x_{r+1}} < \frac{1}{2},
 \end{aligned}$$

and so $x x_1 \dots x_r \rightarrow 0$ as $r \rightarrow \infty$. We may therefore define ν by the inequalities

$$(2. 136) \quad n x x_1 \dots x_{\nu-1} x_\nu < 1 \leq n x x_1 \dots x_{\nu-1}.$$

This being so, the first of the equations (2. 134) gives

$$\begin{aligned}
 (2. 137) \quad s_n^\lambda(x, \theta) &= O(n \sqrt{x x_1 \dots x_{\nu-1}}) \\
 &+ O(1) \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x x_1}} + \dots + \frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}} \right\},
 \end{aligned}$$

and the second gives

$$(2. 137I) \quad s_n^\lambda(x, \theta) = O(1) \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x x_1}} + \dots + \frac{1}{\sqrt{x x_1 \dots x_{\nu-1} x_\nu}} \right\}.$$

We have thus two inequalities for $s_n^\lambda(x, \theta)$, the further study of which depends merely on an analysis of the continued fraction (2. 132). These inequalities,

however, may be simplified. For, by (2. 135), $x_r x_{r+1} < \frac{1}{2}$, and so

$$\begin{aligned}
 &\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x x_1}} + \dots + \frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}} \\
 &= \frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}} (1 + \sqrt{x_{\nu-1}} + \sqrt{x_{\nu-2} x_{\nu-1}} + \dots + \sqrt{x_1 x_2 \dots x_{\nu-1}})
 \end{aligned}$$

$$\begin{aligned} &< \frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}} \left(1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} + \dots \right) \\ &< \frac{K}{\sqrt{x x_1 \dots x_{\nu-1}}}. \end{aligned}$$

Hence (2. 137) may be replaced by

$$(2. 138) \quad s_n^2(x, \theta) = O(n\sqrt{x x_1 \dots x_{\nu-1}}) + O\frac{1}{\sqrt{x x_1 \dots x_{\nu-1}}};$$

and similarly (2. 137I) may be replaced by

$$(2. 138I) \quad s_n^2(x, \theta) = O\frac{1}{\sqrt{x x_1 \dots x_{\nu-1} x_\nu}}.$$

2. 14. From (2. 138) and (2. 138I) we can very easily deduce the principal results of this part of the paper.

Theorem 2. 14. *We have*

$$s_n(x, \theta) = o(n)$$

for any irrational x , and uniformly for all values of θ . In particular, if $\theta = 0$, we have

$$s_n = o(n)$$

Since $n x x_1 \dots x_{\nu-1} \geq 1$, the second term on the right hand side of (2. 138) is of the form $O(\sqrt{n})$. And since $x x_1 \dots x_{\nu-1} \rightarrow 0$ as $\nu \rightarrow \infty$, the first is of the form $o(n)$. Thus the theorem is proved.

Theorem 2. 141. *If the partial quotients a_n in the expression of x as a continued fraction are limited, then*

$$s_n(x, \theta) = O(\sqrt{n}),$$

uniformly in respect to θ ; and in particular

$$s_n = O(\sqrt{n}).$$

These results hold, for example, when x is any quadratic surd, pure or mixed.

For, if $a_n < K$, x_ν lies between

$$\frac{1}{K}, \quad \frac{K}{K+1}$$

and so

$$x x_1 \dots x_{\nu-1} x_\nu > x x_1 \dots x_{\nu-1} / K > 1 / (n K).$$

Using (2. 1381), the result of the theorem follows.

Theorem 2. 142. *If $a_n = O(n^e)$, then*

$$s_n(x, \theta) = O \left\{ n^{\frac{1}{2}} (\log n)^{\frac{1}{2}e} \right\}.$$

Theorem 2. 143. *If $a_n = O(e^{\rho n})$, where $\rho < \frac{1}{2} \log 2$, then*

$$s_n(x, \theta) = O \left(n^{\frac{1}{2} + \frac{\rho}{\log 2} + \epsilon} \right),$$

for any positive value of ϵ .

For

$$x x_1 \dots x_\nu < \frac{1}{n} \leq x x_1 \dots x_{\nu-1} < 2^{-\frac{1}{2}\mu},$$

where $\mu = \nu$ or $\mu = \nu - 1$, according as ν is even or odd. Hence

$$n > 2^{\frac{1}{2}\mu},$$

$$\nu < \frac{(2 + \epsilon) \log n}{\log 2}.$$

But

$$x x_1 \dots x_\nu > H \nu^{-e} x x_1 \dots x_{\nu-1},$$

where H is a constant, and so

$$\frac{1}{\sqrt{x x_1 \dots x_\nu}} = O \left(\nu^{\frac{1}{2}e} \sqrt{n} \right) = O \left\{ n^{\frac{1}{2}} (\log n)^{\frac{1}{2}e} \right\}.$$

This proves Theorem 2. 142. Similarly, under the conditions of Theorem 2. 143, we have

$$\frac{1}{\sqrt{x x_1 \dots x_\nu}} = O \left(e^{\frac{1}{2}e\nu} \sqrt{n} \right) = O \left(n^{\frac{1}{2} + \frac{\rho}{\log 2} + \epsilon} \right).$$

2. 15. Suppose now that $\varphi(n)$ is a logarithmico-exponential function¹ (L -function) of n such that the series

$$(2. 151) \quad \sum \frac{1}{\varphi(n)}$$

is, to put it roughly, near the boundary between convergence and divergence, so that the increase of $\varphi(n)$ is near to that of n . Then, arguing as in 2. 14, we see that, if $a_n = O\{\varphi(n)\}$,

$$x x_1 \dots x_\nu > \frac{H}{\varphi(\nu)} x x_1 \dots x_{\nu-1},$$

$$\frac{1}{V x x_1 \dots x_\nu} = O V n \varphi(\nu) = O V n \varphi(\log n).$$

Now it has been proved by BOREL and BERNSTEIN² that the set of values of x for which

$$a_n = O\{\varphi(n)\}$$

is of measure zero when the series (2. 151) is divergent, and of measure unity when the series is convergent. Hence we obtain

Theorem 2. 15. *If $\varphi(n)$ is a logarithmico-exponential function of n such that*

$$\sum \frac{1}{\varphi(n)}$$

is convergent, then

$$s_n = O V n \varphi(\log n)$$

for almost all values of x . In particular, if δ is positive, then

$$s_n = O \left\{ n^{\frac{1}{2}} (\log n)^{\frac{1}{2} + \delta} \right\}$$

for almost all values of x .

It was this last result to which reference was made in 2. 11.

¹ HARDY, *Orders of Infinity*, p. 17.

² See BOREL, *Rendiconti di Palermo*, Vol. 27, p. 247, and *Math. Annalen*, Vol. 72, p. 578; BERNSTEIN, *Math. Annalen*, Vol. 71, p. 417 and Vol. 72, p. 585.

2. 16. Suppose that a series $\sum u_n$ possesses the property that

$$s_n = u_1 + u_2 + \dots + u_n = O\{\psi(n)\},$$

ψ being a function which tends steadily to infinity with n ; and let φ be a function which tends steadily to zero as $n \rightarrow \infty$, and satisfies the condition that

$$\sum \psi(n) \Delta \frac{\varphi(n)}{\psi(n)}$$

is convergent. Then it follows immediately, by an elementary application of ABEL'S transformation, that the series

$$\sum \frac{\varphi(n)}{\psi(n)} u_n$$

is convergent. This obvious remark may be utilised to deduce a number of corollaries from some of our theorems. To give one instance only, it follows from Theorem 2. 15 that the series

$$\sum n^{-\alpha} e^{n^2 \pi i x} \cos 2n\pi\theta \quad \left(\alpha > \frac{1}{2}\right)$$

is convergent for almost all values of x , and, for any particular x , uniformly with respect to θ .

A rather more subtle deduction can be made from Theorem 2. 14. It does not follow that, because $s_n = o(n)$, the series $\sum \frac{u_n}{n}$ is convergent; and indeed we shall see later that it is not true that (*e. g.*) the series

$$(2. 161) \quad \sum \frac{e^{n^2 \pi i x}}{n}$$

is convergent for all irrational values of x . But it is true that, if $s_n = o(n)$, the series $\sum \frac{u_n}{n}$ is either convergent or not summable by any of CESÀRO'S means¹; and this conclusion accordingly holds of the series (2. 161). Similarly, if x is such that $a_n = O(1)$, the series

$$\sum \frac{e^{n^2 \pi i x}}{\sqrt{n}}$$

¹ HARDY and LITTLEWOOD, *Proc. Lond. Math. Soc.*, Vol. 11, p. 433.

possesses the same property. We shall see later that it is the second alternative which is true.

2. 17. So far we have dealt with series in which the parameter θ occurs in a cosine $\cos 2n\pi\theta$ or $\cos(2n-1)\pi\theta$. It is naturally suggested that similar results should hold for the corresponding series involving $\sin 2n\pi\theta$ and $\sin(2n-1)\pi\theta$; and this is in fact the case. These series are, from the point of view of the theory of functions, of a less elementary character: they are not limiting forms of series which occur in the theory of elliptic functions. But it is not difficult to make the necessary modifications in our analysis.

We write

$$(2. 171) \quad \begin{cases} \sigma_n^2(x, \theta) = \sum_{\nu \leq n} e^{(\nu - \frac{1}{2})^2 \pi i x} \sin(2\nu - 1)\pi\theta \\ \sigma_n^3(x, \theta) = \sum_{\nu \leq n} e^{\nu^2 \pi i x} \sin 2\nu\pi\theta \\ \sigma_n^4(x, \theta) = \sum_{\nu \leq n} (-1)^\nu e^{\nu^2 \pi i x} \sin 2\nu\pi\theta \end{cases}$$

Theorem 2. 17. *If $0 < x < 1$, $0 \leq \theta \leq 1$, then*

$$\sigma_n^2(x, \theta) = \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} \sigma_{nx}^4\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

$$\sigma_n^3(x, \theta) = \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} \sigma_{nx}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

$$\sigma_n^4(x, \theta) = \sqrt{\frac{i}{x}} e^{-\pi i \theta^2/x} \sigma_{nx}^2\left(-\frac{1}{x}, \frac{\theta}{x}\right) + O\sqrt{\frac{1}{x}},$$

uniformly in respect to θ .

Let us consider, for example, the second of these equations. We start from the integral

$$\int e^{z^2 \pi i x} \sin 2z\pi\theta \pi \cot \pi z \, dz,$$

and we arrive, by arguments practically the same as those of 2. 121--2. 127, at the equation

$$(2. 172) \quad \sum_{\nu \leq n} e^{\nu^2 \pi i x} \sin 2\nu\pi\theta - 2 \sum_0^{nx} \int_0^\infty e^{z^2 \pi i x} \cos 2\nu\pi z \sin 2z\pi\theta \, dz = O\sqrt{\frac{1}{x}}.$$

The only substantial differences between the reasoning required for the proof of this equation and those which we used before lie in the facts, first that some of the signs of the principal value which we then used are now unnecessary, and secondly that the two integrals along the axis of imaginaries no longer cancel one another. These integrals, however, are of the form

$$\int_0^{\infty} e^{-\rho^2 \pi i x} \sinh 2t\pi\theta \frac{e^{-2k\pi t}}{1 - e^{-2\pi t}} dt,$$

and are easily seen to be small when k is large. They are accordingly without importance in our argument.

The integrals which occur in (2. 172), unlike the corresponding cosine integrals, cannot be evaluated in finite form. We have, however,

$$(2. 173) \quad 2 \int_0^{\infty} e^{z^2 \pi i x} \cos 2\nu\pi z \sin 2z\pi\theta dz = I(\nu + \theta) - I(\nu - \theta),$$

where

$$(2. 174) \quad I(A) = \int_0^{\infty} e^{z^2 \pi i x} \sin 2z\pi A dz.$$

Now let us consider the integral

$$\int e^{z^2 \pi i x + 2z\pi A} dz \quad (A > 0)$$

taken round the contour defined by the positive halves of the axes and a circle of radius R . It is easy to show, by a type of argument familiar in the theory of contour integration, that the contribution of the curved part of the contour tends to zero as $R \rightarrow \infty$. Hence we deduce

$$\int_0^{\infty} e^{z^2 \pi i x + 2z\pi A} dz = i \int_0^{\infty} e^{-\rho^2 \pi i x - 2\pi A} dt;$$

and so

$$\begin{aligned}
 I(A) &= \frac{1}{i} \int_0^{\infty} e^{2\nu iz} (e^{2z\pi i A} - \cos 2z\pi A) dz \\
 &= i \int_0^{\infty} e^{2\nu iz} \cos 2z\pi A dz + \int_0^{\infty} e^{-2\nu iz - 2t\pi A} dt.
 \end{aligned}$$

Again, it is easy to show that

$$\int_0^{\infty} e^{-2\nu iz - 2t\pi A} dt = \frac{\beta}{A} + O\left(\frac{1}{A^3}\right),$$

where $\beta = 1/2\pi$. Hence

$$\begin{aligned}
 (2. 175) \quad I(\nu + \theta) - I(\nu - \theta) &= i \int_0^{\infty} e^{2\nu iz} \{ \cos 2(\nu + \theta)\pi z - \cos 2(\nu - \theta)\pi z \} dz \\
 &+ \frac{\beta}{\nu + \theta} - \frac{\beta}{\nu - \theta} + O\left(\frac{1}{\nu^3}\right) \\
 &= \sqrt{\frac{i}{x}} e^{-(\theta^2 + \nu^2)\pi ix} \sin \frac{2\nu\pi\theta}{x} + O\left(\frac{1}{\nu^3}\right).
 \end{aligned}$$

From (2. 171), (2. 173), and (2. 175) we at once deduce the second equation of Theorem 2. 17; and the others may be established similarly.

2. 18. From Theorem 2. 17 follow the analogues for the sums σ of those already established for the sums s . Thus we have

Theorems 2. 18, 2. 181—4. *The results established in Theorems 2. 14, 2. 141—3, 2. 15, for series involving cosines, are true also for the corresponding series involving sines.*

2. 19. The preceding results have a very interesting application to the theory of TAYLOR'S series.

Let

$$f(z) = \sum a_n z^n$$

be a power series whose radius of convergence is unity, and let, as usual, $M(r)$ denote the maximum of $|f|$ along a circle of radius r less than 1. Further, suppose that

$$M(r) = O(1-r)^{-a},$$

and let

$$g(r) = \sum |a_n| r^n.$$

Then it is known that¹

$$g(r) = O(1-r)^{-a-\frac{1}{2}}.$$

Further, it is known that the number $\frac{1}{2}$ occurring in the last formula cannot be replaced by any smaller number, that is to say that, if δ is any positive number, a function $f(z)$ can be found such that the difference between the orders of $g(r)$ and $M(r)$ is $\frac{1}{2} - \delta$.² But so far as we are aware, no example has been given of a function $f(z)$ such that the orders of $g(r)$ and $M(r)$ differ by as much as $\frac{1}{2}$. We are now in a position to supply such an example.

Let

$$f(z) = \sum e^{n^2 \pi i \xi} z^n,$$

where ξ is an irrational of the type considered in Theorem 2. 141, so that the partial quotients in its expression as a continued fraction are limited. Then, if $z = r e^{2\pi i \theta}$, we have, by Theorems 2. 141 and 2. 181,

$$S_n = \sum_{v=0}^n e^{v^2 \pi i \xi + 2v\pi i \theta} = O(\sqrt{n}),$$

uniformly in θ ; and from this it follows that

$$f(z) = f(r e^{2\pi i \theta}) = \sum r^n e^{n^2 \pi i \xi + 2n\pi i \theta} = O\sqrt{\frac{1}{1-r}},$$

uniformly in θ . Hence

$$M(r) = O\sqrt{\frac{1}{1-r}},$$

while

$$g(r) = \sum r^n = \frac{1}{1-r}.$$

¹ HARDY, *Quarterly Journal*, Vol. 44, p. 147.

² HARDY, *l. c.*, p. 156.

Thus the orders of $g(r)$ and $M(r)$ differ by exactly $\frac{1}{2}$. If we consider, instead of $f(z)$, the function

$$\sum n^{\alpha-\frac{1}{2}} e^{n^2 i \xi} z^n \quad \left(0 < \alpha < \frac{1}{2}\right),$$

we obtain in the same way an example of a function such that

$$M(r) = O(1-r)^{-\alpha},$$

$$g(r) \infty \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{(1-r)^{\alpha+\frac{1}{2}}}.$$

These examples show that the equation

$$M(r) = O(1-r)^{-\alpha} \quad (\alpha > 0)$$

does *not* involve

$$g(r) = o(1-r)^{-\alpha-\frac{1}{2}};$$

a possibility which had before remained open.¹

2. 19. Theorems 2. 14 etc. also enable us to make a number of interesting inferences as to the behaviour of the modular functions

$$\sum q^{\left(n-\frac{1}{2}\right)^2}, \quad \sum q^{n^2}, \quad \sum (-1)^n q^{n^2}$$

as q tends along a radius vector² to an irrational place $e^{n i \xi}$ on the circle of convergence. Thus from Theorem 2. 14 we can easily deduce that, if $f(q)$ denotes any one of these functions, then

$$f(q) = o(1-|q|)^{-\frac{1}{2}};$$

and from Theorem 2. 141 that, if ξ is an irrational of the class there considered, then

$$f(q) = O(1-|q|)^{-\frac{1}{4}}.$$

¹ HARDY, *l. c.*, p. 150.

² Or along any 'regular path' which does not touch the circle of convergence.

These results are, however, more easily proved by a more direct method, which enables us at the same time to assign certain *lower* limits for the magnitude of $|f(q)|$, and to show that Theorems 2. 14 *et seq* are in a certain sense the best possible of their kind. It is to the development of this method, which depends on a direct use of the ordinary formulae for the linear transformation of the \mathcal{J} -functions, that the greater part of the rest of the paper will be devoted.

2. 2. — \mathcal{Q} Theorems.

2. 20. We have occupied ourselves, so far, with the determination of certain upper limits for the magnitude of sums of the type s_n . Thus we proved that $s_n = o(n)$ for any irrational x , and that $s_n = O(\sqrt{n})$ for an important class of such irrationals, including for example the class of quadratic surds. But we have done nothing to show that these results are the best of their kind that are true. The theorems which follow will show that this is the case.

We shall begin, however, by proving a theorem of a more elementary character which involves no appeal to the formulae of the transformation theory.

Theorem 2. 20. *Suppose that $\varphi(n)$ is a positive decreasing function of n , such that the series $\sum \varphi(n)$ is divergent. Then it is possible to find irrationals x such that the series*

$$\sum \varphi(n) e^{n^2 \pi i x}$$

is not convergent. The same is true of the series

$$\sum \varphi(n) e^{\left(n - \frac{1}{2}\right)^2 \pi i x}, \quad \sum (-1)^n \varphi(n) e^{n^2 \pi i x},$$

and of the real and imaginary parts of all these series.

Consider, for example, the real part of the first series. We shall suppose that, among the convergents p_ν/q_ν to x , there are infinitely many of the form $2\lambda/(4\mu + 1)$. Let (q_ν) be a subsequence selected from the denominators of these convergents. We are clearly at liberty to suppose that the increase of $a_{\nu+1}$, when compared with that of any number which depends only on q_ν and the function φ , is as rapid as we please.

We shall consider the sum

$$S_\nu = \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n) \cos(n^2 \pi x),$$

where A_ν is an integer large compared with q_ν but small compared with $q_{\nu+1}/q_\nu$. We shall suppose A_ν so chosen that

$$(2. 201) \quad q_\nu^{-\frac{3}{2}} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) \rightarrow \infty$$

$$(2. 202) \quad a_{\nu+1} / A_\nu^2 q_\nu \rightarrow \infty;$$

and we shall show that, in these circumstances, $|S_\nu|$ tends to infinity with ν , and hence that the series

$$\sum \varphi(n) \cos(n^2 \pi x)$$

cannot converge.

We may consider, instead of S_ν , the sum

$$(2. 203) \quad S'_\nu = \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n) \cos(n^2 \pi p_\nu / q_\nu).$$

For

$$S_\nu - S'_\nu = \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n) \{ \cos(n^2 \pi x) - \cos(n^2 \pi p_\nu / q_\nu) \}.$$

Now

$$\left| n^2 \left(x - \frac{p_\nu}{q_\nu} \right) \right| = \frac{n^2}{q_\nu q'_{\nu+1}} < \frac{A_\nu^2 q_\nu}{q'_{\nu+1}},$$

where $a'_{\nu+1}$ is the complete quotient corresponding to the partial quotient $a_{\nu+1}$, and $q'_{\nu+1} = a'_{\nu+1} q_\nu + q_{\nu-1}$; and from this it follows that $|S_\nu - S'_\nu|$ is less than a constant multiple of

$$\frac{A_\nu^2 q_\nu}{q'_{\nu+1}} \sum_{q_\nu}^{A_\nu q_\nu - 1} \varphi(n),$$

and so of

$$A_\nu^2 q_\nu^2 / q'_{\nu+1} < A_\nu^2 q_\nu / a_{\nu+1}.$$

Thus $S_\nu - S'_\nu \rightarrow 0$ as $\nu \rightarrow \infty$, in virtue of (2. 202).

We may write S'_ν in the form

$$S'_\nu = \sum_{r=1}^{A_\nu-1} \sum_{s=0}^{q_\nu-1} \varphi(rq_\nu + s) \cos(s^2 \pi p_\nu / q_\nu).$$

If in this sum we replace $\varphi(rq_\nu + s)$ by $\varphi(rq_\nu)$, the error introduced is not greater than

$$\begin{aligned} \sum_{r=1}^{A_\nu-1} \sum_{s=0}^{q_\nu-1} \{\varphi(rq_\nu) - \varphi(rq_\nu + s)\} &\leq q_\nu \sum_{r=1}^{A_\nu-1} \{\varphi(rq_\nu) - \varphi[(r+1)q_\nu]\} \\ &\leq q_\nu \varphi(q_\nu). \end{aligned}$$

Thus, with an error not greater than $q_\nu \varphi(q_\nu)$, and *a fortiori* not greater than $q_\nu \varphi(1)$, we can replace S'_ν by

$$(2. 204) \quad S''_\nu = \sum_{r=1}^{A_\nu-1} \varphi(rq_\nu) \sum_{s=0}^{q_\nu-1} \cos(s^2 \pi p_\nu / q_\nu) = \pm V \sqrt{q_\nu} \sum_{r=1}^{A_\nu-1} \varphi(rq_\nu).$$

Now

$$\begin{aligned} \varphi(q_\nu) + \varphi(2q_\nu) + \dots + \varphi\{(A_\nu-1)q_\nu\} &\geq \frac{1}{q_\nu} \sum_{2q_\nu}^{A_\nu q_\nu} \varphi(n) \\ &\geq \frac{1}{q_\nu} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) - \varphi(1), \end{aligned}$$

and so

$$|S''_\nu| \geq \frac{1}{V \sqrt{q_\nu}} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) - V \sqrt{q_\nu} \varphi(1).$$

Hence

$$\frac{|S''_\nu|}{q_\nu \varphi(1)} \geq \frac{1}{\varphi(1)} q_\nu^{-\frac{3}{2}} \sum_{q_\nu}^{A_\nu q_\nu} \varphi(n) - \frac{1}{V \sqrt{q_\nu}},$$

which tends to infinity with ν , in virtue of (2. 201). Hence S'_ν , and so S_ν , tends to infinity with ν ; which proves the theorem.

In particular it is possible to find irrational values of x for which the series

$$\sum \frac{\cos(n^2 \pi x)}{n}, \quad \sum \frac{\cos(n^2 \pi x)}{n \log n}, \dots$$

are not convergent.

2. 21. We shall find it convenient at this stage to introduce a new notation. We define the equation

$$f = \Omega(\varphi),$$

where φ is a positive function of a variable, which may be integral or continuous but which tends to a limit, as meaning that there exists a constant H and a sequence of values of the variable, themselves tending to the limit in question, such that

$$|f| > H\varphi$$

for each of these values. In other words, $f = \Omega(\varphi)$ is the negation of $f = o(\varphi)$. In the notation of Messrs WHITEHEAD and RUSSELL we should write

$$f = \Omega(\varphi) . = . \infty (f = o(\varphi)). \quad Df.$$

2. 22. We shall now prove the following theorems.

Theorem 2. 22. *If x is irrational, then*

$$s_n = \Omega(\sqrt{n}).$$

Theorem 2. 221. *If φ is any positive function of n , which tends to zero as $n \rightarrow \infty$, then it is possible to find irrationals x such that*

$$s_n = \Omega(n\varphi).$$

These theorems show that the equation

$$s_n = O(\sqrt{n}),$$

established by Theorem 2. 141 for a particular class of values of x , cannot possibly be replaced by any better equation; and that the equation

$$s_n = o(n)$$

of Theorem 2. 14 is the best that is true of *all* irrationals. We shall deduce these theorems from certain results concerning the elliptic modular functions.

2. 23. We write

$$\begin{aligned} q &= e^{\pi i x} = e^{\pi i(x+iy)} = e^{-\pi y + \pi i x} \\ &= r e^{\pi i x} \quad (x > 0, y > 0, 0 < r < 1). \end{aligned}$$

$$\mathfrak{D}_2(0, \tau) = 2 \sum_1^{\infty} q^{\left(n-\frac{1}{2}\right)^2},$$

$$\mathfrak{D}_3(0, \tau) = 1 + 2 \sum_1^{\infty} q^{n^2},$$

$$\mathfrak{D}_4(0, \tau) = 1 + 2 \sum_1^{\infty} (-1)^n q^{n^2}.$$

We suppose that p_n/q_n is a convergent to

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \dots,$$

and write

$$p_{n-1}q_n - p_nq_{n-1} = \eta_n = \pm 1.$$

We shall consider a linear transformation

$$T = \frac{c + d\tau}{a + b\tau},$$

where

$$\left. \begin{aligned} a &= p_n, & b &= -q_n, \\ c &= \eta_n p_{n-1}, & d &= -\eta_n q_{n-1}, \end{aligned} \right\} (p_n \text{ odd}),$$

$$\left. \begin{aligned} a &= -p_n, & b &= q_n, \\ c &= -\eta_n p_{n-1}, & d &= \eta_n q_{n-1}, \end{aligned} \right\} (p_n \text{ even}).$$

In either case $ad - bc = \eta_n^2 = 1$.

Finally, if a'_{n+1} is the complete quotient corresponding to a_{n+1} , we write

$$q'_{n+1} = a'_{n+1}q_n + q_{n-1},$$

and we take

$$y = 1/(q_n q'_{n+1}).$$

When

$$p_{n-1} \text{ is even, } p_n \text{ is odd,}$$

$$q_{n-1} \text{ is odd, } q_n \text{ is even,}$$

we shall say that the convergents p_{n-1}/q_{n-1} , p_n/q_n form a system of type

$$\begin{pmatrix} E & O \\ O & E \end{pmatrix}.$$

There are six possible types of system, viz.

$$\begin{pmatrix} E & O \\ O & E \end{pmatrix}, \begin{pmatrix} O & O \\ O & E \end{pmatrix}, \begin{pmatrix} E & O \\ O & O \end{pmatrix}, \begin{pmatrix} O & O \\ E & O \end{pmatrix}, \begin{pmatrix} O & E \\ E & O \end{pmatrix}, \begin{pmatrix} O & E \\ O & O \end{pmatrix},$$

which we number

$$1^{\circ}, 2^{\circ}, 3^{\circ}, 4^{\circ}, 5^{\circ}, 6^{\circ}.$$

The following remark is of fundamental importance for our present purpose. *In any continued fraction whatever, one or other of the systems 1^o, 2^o, 5^o, 6^o must occur infinitely often.* This appears from the fact that the second column in cases 3^o and 4^o is O, O, and that all cases in which the first column is O, O fall under 1^o, 2^o, 5^o, or 6^o.

2. 24. In cases 1^o, 2^o, 5^o, or 6^o we have

$$\mathfrak{J}_3(o, \tau) = \frac{1}{\omega \sqrt{a + b\tau}} \mathfrak{J}(o, T),$$

where ω is an 8-th root of unity, and \mathfrak{J} stands for one or other of \mathfrak{J}_3 and \mathfrak{J}_4 .¹ Now

$$|a + b\tau| = |p_n - q_n x - q_n i y| = \frac{|\pm 1 - i|}{q'_{n+1}} = \frac{\sqrt{2}}{q'_{n+1}}.$$

Also, if $Q = e^{niT}$, we have

$$|Q| = e^{-\pi\lambda},$$

where

$$\begin{aligned} \lambda = \mathbf{I}(T) &= \mathbf{I} \left(\frac{c + d\tau}{a + b\tau} \right) = \mathbf{I} \left\{ \frac{d}{b} - \frac{1}{b(a + b\tau)} \right\} \\ &= \frac{y}{(\mathbf{I}/q'_{n+1})^2 + q_n^2 y^2} = \frac{q'_{n+1}}{2q_n} > \frac{1}{2}. \end{aligned}$$

Hence

$$\begin{aligned} |Q| &< e^{-\frac{1}{2}\pi} < 1/(4 \cdot 8) < .21, \\ 2|Q| + 2|Q|^4 + \dots &< 2(.21) + 2(.21)^4 + \dots \\ &< \frac{1}{2}, \end{aligned}$$

¹ *T. and M.*, Vol. 2, p. 262 (Table XLII).

$$|\mathcal{G}(0, T)| = |1 \pm 2Q + 2Q^4 \pm \dots| > \frac{1}{2}.$$

Consequently

$$|\mathcal{G}_s(0, \tau)| > K\sqrt{q'_{n+1}} > K\sqrt[4]{q_n q'_{n+1}} = K\sqrt[4]{1/y}.$$

From this follows at once

Theorem 2. 24. *If $q = re^{nix}$, where x is irrational, then*

$$1 + 2 \sum_1^{\infty} q^{n^2} = \Omega \left\{ \sqrt[4]{\frac{1}{1-r}} \right\}$$

as $r \rightarrow 1$.

From this we can deduce Theorem 2. 22 as a corollary. For if we had

$$s_n = o(\sqrt{n}),$$

the series

$$1 + 2 \sum_1^{\infty} e^{n^2nix} r^{n^2} = \sum u_n r^n$$

would satisfy the condition

$$u_0 + u_1 + \dots + u_n = o(\sqrt{n}),$$

and so we should have

$$\begin{aligned} \sum u_n r^n &= (1-r) \sum (u_0 + u_1 + \dots + u_n) r^n \\ &= (1-r) \sum o(\sqrt{n}) r^n \\ &= o \left\{ \sqrt[4]{\frac{1}{1-r}} \right\}; \end{aligned}$$

an equation which Theorem 2. 24 shows to be untrue.

Again, let $\varphi(1/y)$ be any function which tends to zero with y . We have

$$|\mathcal{G}_s(0, \tau)| > K\sqrt{q'_{n+1}} = K\sqrt{1/q_n y}.$$

We choose a value of x such that, for an infinity of values of n corresponding to one of the favourable cases $1^\circ, 2^\circ, 5^\circ, 6^\circ$, we have

$$\sqrt{1/q_n} > \varphi(q_n q'_{n+1});$$

this may certainly be secured by supposing that a_{n+1} is sufficiently large. We have then

$$|\mathcal{G}_s(0, \tau)| > KV \sqrt{\frac{1}{y}} \varphi(1/y).$$

From this we deduce

Theorem 2. 241. *Given any function φ which tends to zero, it is possible to find irrational values of x such that*

$$1 + 2 \sum_1^{\infty} q^{n^2} = \Omega \left\{ \sqrt{\frac{1}{1-r}} \varphi \left(\frac{1}{1-r} \right) \right\}$$

when $q = re^{i\alpha}$ and $r \rightarrow 1$.

From this theorem Theorem 2. 221 follows as a corollary just as Theorem 2. 22 followed from Theorem 2. 24.

2. 25. It is interesting to consider a little more closely the case in which x is an irrational for which $a_n = O(1)$.

Let us, instead of considering only the special value $1/(q_n q'_{n+1})$ of y , consider the range R_n defined by

$$\frac{1}{q'^2_{n+1}} \leq y \leq \frac{1}{q_n^2}$$

or

$$\frac{\eta}{q_n q'_{n+1}} \leq y \leq \frac{1}{\eta q_n q'_{n+1}},$$

where $\eta = q_n/q'_{n+1}$. It is clear that, for different values of n , these ranges cover up the *whole* range of variation of y . If now $y = \zeta/(q_n q'_{n+1})$, so that $\eta \leq \zeta \leq 1/\eta$, we have

$$\lambda = \frac{y}{(1/q'_{n+1})^2 + q_n^2 y^2} = \frac{\zeta}{1 + \zeta^2} \frac{q'_{n+1}}{q_n}.$$

The least values of λ correspond to $\zeta = \eta, 1/\eta$; and then

$$\lambda = \frac{q'^2_{n+1}}{q_n^2 + q'^2_{n+1}} > \frac{1}{2}.$$

Suppose first that n corresponds to a system of one of the types 1°, 2°, 5°, 6°. Then the argument of 2. 24 shows that the absolute value of $\mathcal{G}(0, T)$ lies between $\frac{1}{2}$ and $\frac{3}{2}$. If on the other hand n corresponds to a system of type 3° or 4°, we have

$$\mathcal{G}_3(o, \tau) = \frac{1}{\omega\sqrt{a+b\tau}} \mathcal{G}_2(o, T).$$

Now

$$\mathcal{G}_2(o, T) = 2Q^{\frac{1}{2}} (1 + Q^2 + Q^6 + \dots),$$

and the absolute value of the second factor lies between $\frac{3}{4}$ and $\frac{5}{4}$. On the other hand λ lies between $q'_{n+1}/(q_n^2 + q'_{n+1})$ and $q'_{n+1}/2q_n$, and *a fortiori* between $\frac{1}{2}$ and $\frac{1}{2}(K+1)$, where K is the greatest value of a partial quotient. Hence in this case also $|\mathcal{G}(o, T)|$ lies between fixed positive limits.

Thus, as the ranges R_n fill up the whole range of variation of y , we can determine two constants H_1, H_2 , so that

$$\frac{H_1}{\sqrt{|a+b\tau|}} < |\mathcal{G}_3(o, \tau)| < \frac{H_2}{\sqrt{|a+b\tau|}}.$$

But

$$\sqrt{|a+b\tau|} = \sqrt[4]{y \left(\frac{1}{q'_{n+1}y} + q_n^2 y \right)}$$

and it is easy to see that the second factor under the radical lies between fixed positive limits. Hence we obtain

Theorem 2. 25. *If $q = re^{i\pi x}$, the partial quotients to x being limited, and $r \rightarrow 1$, then*

$$\left| 1 + 2 \sum_1^{\infty} q^{n^2} \right| \asymp \sqrt[4]{\frac{1}{1-r}}.$$

2. 26. In the preceding discussion, the argument which showed that $\lambda > \frac{1}{2}$ was independent of any hypothesis as to the continued fraction. Hence we have in any case

$$\begin{aligned} |\mathcal{G}_3(o, \tau)| &< \frac{H_2}{\sqrt{|a+b\tau|}} = \frac{H_2}{\sqrt[4]{(1/q'_{n+1})^2 + q_n^2 y^2}} \\ &= O \left\{ \frac{1}{\sqrt{q_n y}} \right\} = o \left(\frac{1}{\sqrt{y}} \right), \end{aligned}$$

as $q_n \rightarrow \infty$. Hence we obtain

Theorem 2. 26. *For any irrational value of x , we have*

¹ The formula $f \asymp \varphi$ implies that $|f|/\varphi$ lies between fixed positive limits: see HARDY, *Orders of Infinity*, pp. 2, 5.

$$1 + 2 \sum_1^{\infty} r^{n^2} = o \left\{ \sqrt[4]{\frac{1}{1-r}} \right\}.$$

This result may of course also be proved as a corollary of Theorem 2. 14, by reasoning analogous to that used in 2. 24. But the direct proof is none the less interesting.

2. 27. The argument used in 2. 24, in deducing Theorem 2. 22 from Theorem 2. 24, may be adapted so as to prove an interesting generalisation of the former theorem. Let us write, as before

$$1 + 2 \sum_1^{\infty} e^{n^2 \pi i x} r^{n^2} = \sum u_n r^n,$$

and suppose that $k! S_n^k / n^k$ is one of CESÀRO'S means associated with the series $\sum u_n$. Then

$$(2. 27I) \quad S_n^k = \Omega \left(n^{k + \frac{1}{4}} \right).$$

For if this were not so, we should have

$$\begin{aligned} \sum u_n r^n &= (1-r)^{k+1} \sum S_n^k r^n = (1-r)^{k+1} \sum o \left(n^{k + \frac{1}{4}} \right) r^n \\ &= o \left\{ \sqrt[4]{\frac{1}{1-r}} \right\}. \end{aligned}$$

From (2. 27I) it follows that the series $\sum u_n$ cannot become summable (Ck) on the introduction of a convergence factor $n^{-\frac{1}{4}}$.¹ And from this we deduce

Theorem 2. 27. *The series*

$$\sum n^{-\alpha} e^{n^2 \pi i x} \quad \left(\alpha \leq \frac{1}{2} \right)$$

cannot be convergent, or summable by any of CESÀRO'S means, for any irrational x .

We need hardly remark that the same is true of

$$\sum n^{-\alpha} e^{\left(n - \frac{1}{2} \right)^2 \pi i x}, \quad \sum (-1)^n n^{-\alpha} e^{n^2 \pi i x}.$$

On the other hand, if $\alpha > \frac{1}{2}$, all these series converge *presque partout* (2. 11, 2. 16).

¹ HARDY and LITTLEWOOD, *Proc. Lond. Math. Soc.*, Vol. 11, p. 435.

2. 3. — An application to the theory of trigonometrical series.¹

2. 30. The problem of finding a trigonometrical series whose coefficients tend to zero, and which converges, if ever, only for a set of values of the argument of measure 0, was first formulated by FATOU² and first solved by LUSIN.³ The results of the earlier part of this paper have led us to a solution of FATOU's problem which seems to us to have considerable advantages over LUSIN's.

We can, in fact, prove the following theorem, which is an extension of Theorem 2. 27.

Theorem 2. 30. *The series*

$$\sum n^{-\alpha} \cos (n^2 \pi x), \quad \sum n^{-\alpha} \sin (n^2 \pi x),$$

where $0 < \alpha \leq \frac{1}{2}$, are never convergent, or summable by any of CESÀRO's means, for any irrational value of x .⁴

Considered simply as solutions of FATOU's problem, these series have, as against LUSIN's, two advantages. In the first place, they are series of a simple, natural, and elegant analytical form. In the second place, the problem of convergence is solved completely; there is no exceptional set of values of x for which doubt remains.⁵

2. 31. We proceed to the proof of Theorem 2. 30. This theorem is a corollary of

¹ An abstract of the contents of this part of the paper appeared, under the title 'Trigonometrical Series which Converge Nowhere or Almost Nowhere', in the *Records of Proceedings of the London Math. Soc.* for 13 Febr. 1913.

² *Acta Mathematica*, Vol. 30, p. 398.

³ *Rendiconti di Palermo*, Vol. 32, p. 386.

⁴ The cosine series converges when x is a rational of the form $(2\lambda + 1)/(2\mu + 1)$ or $2\lambda/(4\mu + 3)$, the sine series when x is a rational of the form $(2\lambda + 1)/(2\mu + 1)$ or $2\lambda/(4\mu + 1)$ (see 2. 01). In the abstract referred to above this part of the result (which is of course trivial) was stated incorrectly.

⁵ It is only since this paper was written that we have become aware of a different solution given by H. STEINHAUS (*Comptes Rendus de la Société Scientifique de Varsovie*, 1912, p. 223). STEINHAUS also solves the problem of convergence for his series completely; they converge, in fact, for no values of x . Thus in this respect our examples have no advantage over his; the advantage, if anywhere, is on his side. In respect of simplicity etc. our examples have the advantage over his as much as over LUSIN's.

Theorem 2. 31. *If $q = re^{ix}$, where x is irrational, then, as $r \rightarrow 1$, both the real and the imaginary parts of*

$$f(q) = 1 + 2 \sum_1^{\infty} q^{n^2}$$

are of the form $\Omega \left\{ \sqrt[4]{\frac{1}{1-r}} \right\}$.

In fact, when once this theorem has been established, Theorem 2. 30 follows from it in the same way as Theorem 2. 22 followed from Theorem 2. 24. And the proof of Theorem 2. 31 is in principle the same as that of Theorem 2. 24, though naturally more complicated.

Our notation will be the same as in 2. 23. We shall prove first that, in cases 1°, 2°, 5°, and 6°, we have

$$(i) \quad |\mathcal{G}_3(0, \tau)| > Ky^{-\frac{1}{4}},$$

$$(ii) \quad \left| \text{am } \mathcal{G}_3(0, \tau) - \frac{1}{2} m\pi \right| > \delta$$

for all integral values of m , K and δ being positive constants, provided either

$$(a) \quad a_{n+1} > 1$$

or

$$(\beta) \quad a_{n+1} = 1, a_{n+2} = 1.$$

We shall express this shortly by saying that 1°, 2°, 5°, 6° are *favourable cases*, except possibly when

$$a_{n+1} = 1, a_{n+2} > 1;$$

a 'favourable case' being one in which we can prove the inequalities

$$(2. 311) \quad |\mathbf{R}\{\mathcal{G}_3(0, \tau)\}| > Ky^{-\frac{1}{4}}, \quad |\mathbf{I}\{\mathcal{G}_3(0, \tau)\}| > Ky^{-\frac{1}{4}}.$$

We have

$$(2. 312) \quad \mathcal{G}_3(0, \tau) = \frac{1}{\omega \sqrt{a + b\tau}} \mathcal{G}(0, T).$$

If $a_{n+1} > 1$,

$$|Q| = e^{-\pi q'_{n+1}/2q_n} < e^{-\pi} < \frac{1}{23};$$

and if $a_{n+1} = 1, a_{n+2} = 1$,

$$\frac{q'_{n+1}}{q_n} = 1 + \frac{1}{a'_{n+2}} + \frac{q_{n-1}}{q_n} > \frac{3}{2},$$

$$|Q| < e^{-\frac{3}{4}\pi} < \frac{1}{10}.$$

In either case

$$2|Q| + 2|Q|^4 + \dots < \frac{1}{4},$$

and so

$$(2. 313) \quad |\mathcal{G}(0, T)| > \frac{3}{4}, \quad |\text{am } \mathcal{G}(0, T)| < \text{arc tan } \frac{1}{4} < \frac{1}{12} \pi.$$

Again

$$a + b\tau = \pm (\eta_n + i) / q'_{n+1},$$

$$(2. 314) \quad |a + b\tau|^{-\frac{1}{2}} = 2^{-\frac{1}{4}} \sqrt{q'_{n+1}} > Ky^{-\frac{1}{4}},$$

$$(2. 315) \quad \text{am} \left\{ (a + b\tau)^{-\frac{1}{2}} \right\} \equiv -\frac{1}{8} \eta_n \pi \quad \left(\text{mod. } \frac{1}{2} \pi \right).$$

From (2. 312), (2. 313), (2. 314), and (2. 315) it follows, first that the modulus of $\mathcal{G}_3(0, \tau)$ is greater than a constant multiple of $y^{-\frac{1}{4}}$ (as has been shown already under 2. 24), and secondly that

$$(2. 316) \quad \text{am } \mathcal{G}_3(0, \tau) \equiv -\frac{1}{8} \eta_n \pi + \left\{ \frac{1}{12} \pi \right\} \quad \left(\text{mod. } \frac{1}{4} \pi \right),$$

where $\left\{ \frac{1}{12} \pi \right\}$ denotes a number whose absolute value is less than $\frac{1}{12} \pi$. Hence $\text{am } \mathcal{G}_3(0, \tau)$ must differ by at least

$$\frac{\pi}{8} - \frac{\pi}{12} = \frac{\pi}{24}$$

from any multiple of $\frac{1}{2} \pi$; and so the cases which we are considering are all favourable.

2. 32. We shall now prove that, as $n \rightarrow \infty$, favourable cases must recur infinitely often. This will complete the proof of Theorem 2. 31.

We represent the state of affairs, as regards the oddness or evenness of p_n and q_n , in a way which will be made most clear by an example. If every

p_n is odd, and q_n is alternately odd and even, we represent the continued fraction diagrammatically in the form

$$\begin{array}{ccccccc} O & O & O & O & O & \dots & \\ O & E & O & E & O & \dots & \end{array}$$

— and so in other cases.

Suppose first that $O O$ occurs infinitely often above. Then one or other of the systems

$$\begin{pmatrix} O & O \\ O & E \end{pmatrix}, \quad \begin{pmatrix} O & O \\ E & O \end{pmatrix}$$

must occur infinitely often. If the first, which is system 2°, either favourable cases recur infinitely often, or the ensuing partial quotient is always 1. We represent this state of affairs by the symbol

$$\begin{array}{cc|c} O & O & \\ O & E & \end{array}$$

In this case our diagram continues

$$\begin{array}{cc|c} O & O & E \\ O & E & O \end{array};$$

and as $\begin{pmatrix} O & E \\ E & O \end{pmatrix}$ is case 5°, either favourable cases recur continually, or the next quotient is also 1, so that we have

$$\begin{array}{cc|c} O & O & E \\ O & E & O \end{array}.$$

But then the first four letters represent a system of type 2° followed by two quotients $a_{n+1} = 1$, $a_{n+2} = 1$; and this is a favourable case. Thus if $\begin{pmatrix} O & O \\ O & E \end{pmatrix}$ recurs continually, favourable cases recur continually.

We consider next the result of supposing that $\begin{pmatrix} O & O \\ E & O \end{pmatrix}$ recurs continually. This is case 4°. If the diagram continues with an O above, it must continue in the form

$$\begin{array}{ccc} O & O & O \\ E & O & E \end{array}$$

and then we can repeat our previous argument. The only alternative is that it should continue

$$\begin{array}{ccc} O & O & E \\ E & O & O \end{array}$$

— and as the last four letters form a system of type 6°, the next quotient must (in the unfavourable case) be 1. Hence we obtain

$$\begin{array}{ccc|c} O & O & E & O \\ E & O & O & E \end{array}$$

The next quotient must also be 1; and so the system of type 6° gave in reality a favourable case.

We have thus proved that, whenever the succession OO recurs continually above, we obtain an infinity of favourable cases. It only remains to consider the hypothesis that p_n is alternately odd and even.

If we have OE above, we have one or other of the systems $\begin{pmatrix} O & E \\ O & O \end{pmatrix}$, $\begin{pmatrix} O & E \\ E & O \end{pmatrix}$; systems 5° and 6°. Thus we have a favourable case unless $a_{n+1} = 1$. If the system is of type 5°, we are led to

$$\begin{array}{cc|c} O & E & O \\ O & O & E \end{array}$$

— so that the system is favourable. On the other hand, if it is of type 6°, we are led to

$$\begin{array}{cc|c} O & E & O \\ E & O & O \end{array}$$

As the next numerator is even, the next denominator is odd. Hence the next system is $\begin{pmatrix} O & E \\ O & O \end{pmatrix}$, and we have seen that this case must be favourable.

We have now examined all possible hypotheses, and found that they all involve the continual recurrence of favourable cases. Thus Theorem 2. 31 is established.

2. 33. From this theorem we can, as was explained in 2. 31, deduce Theorem 2. 30 as a corollary. The latter theorem has an interesting consequence which we have not seen stated explicitly.

The series

$$\sum n^{-\alpha} \cos (n^2 \pi x), \quad \sum n^{-\alpha} \sin (n^2 \pi x),$$

where $\alpha \leq \frac{1}{2}$, are not FOURIER's series.

For if they were they would be summable (C_1) almost everywhere, by a theorem of LEBESGUE.¹ It follows that trigonometrical series exist, such that

$$\sum (|a_n|^{2+\delta} + |b_n|^{2+\delta})$$

is convergent for every positive δ ,² which are not FOURIER's series. This is of interest for the following reason. If $\sum (a_n^2 + b_n^2)$ is convergent, the series is the FOURIER's series of a function whose square is summable.³ Further if p is any odd integer, and

$$\sum (|a_n|^{1+\frac{1}{p}} + |b_n|^{1+\frac{1}{p}})$$

is convergent, then the function has its $(1+p)$ -th power summable.⁴ It would be natural to suppose that the RIESZ-FISCHER Theorem might be capable of extension in the opposite direction. One might expect, for example, to find that a series for which

$$\sum (|a_n|^{1+p} + |b_n|^{1+p})$$

is convergent must be the FOURIER's series of a function whose $(1 + \frac{1}{p})$ -th power is summable. That this is not true has been shown by YOUNG, by means of the series

¹ *Math. Annalen*, Vol. 61, p. 251. See also *Leçons sur les séries trigonométriques*, p. 94 where however the proof is inaccurate. A FOURIER's series is in fact summable (C_δ), for any positive δ , almost everywhere (HARDY, *Proc. Lond. Math. Soc.*, Vol. 12 p. 365). That our series are not FOURIER's series when $\alpha < \frac{1}{2}$ can in fact be inferred merely from their non-convergence, since to replace $n^{-\alpha}$ by $n^{-\beta}$, where β is any number greater than α , would, if they were FOURIER's series, render them convergent almost everywhere (YOUNG, *Comptes Rendus*, 23 Dec. 1912).

² Or even for which

$$\sum \frac{|a_n|^2 + |b_n|^2}{(\log n)^{1+\delta}}$$

is convergent.

³ This is the 'RIESZ-FISCHER Theorem'.

⁴ W. H. YOUNG, *Proc. Lond. Math. Soc.*, Vol. 12, p. 71.

$$\sum \frac{\cos nx + \sin nx}{n^{\frac{1}{2}} (\log n)^{\frac{1}{2}}}$$

— here $p=3$. Our examples however show a good deal more, viz. that as soon as the 2 which occurs in the RIESZ-FISCHER Theorem is replaced by any higher index, the series ceases to be necessarily a FOURIER's series at all.

2. 34. There are other classes of series the theory of which resembles in many respects that of the series studied in this paper. One such class comprises such series as

$$\sum \operatorname{cosec} n\pi x, \quad \sum (-1)^n \operatorname{cosec} n\pi x$$

and the corresponding series in which the cosecant is replaced by a cotangent: these series are limiting forms of q -series such as

$$\sum \frac{q^n}{1 - q^{2n}}.$$

Another class comprises the series

$$\sum \left\{ (nx) - \frac{1}{2} \right\}, \quad \sum (-1)^n \left\{ (nx) - \frac{1}{2} \right\}$$

and the corresponding series in which $(nx) - \frac{1}{2}$ is replaced by \overline{nx} . We have proved a considerable number of theorems, relating to these various series, of which we hope to give a systematic account on some future occasion.

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 - 2. 2. Ω Theorems.
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