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Irregular distribution of $\{n\beta\}$, n=1,2,3,..., quadrature of singular integrands, and curious basic hypergeometric series

by K.A. Driver[1], D.S. Lubinsky[1], G. Petruska[1] and P. Sarnak[2]

[1] *Department of Mathematics, University of the Witwatersrand, P.O. Wits 2050, Rep. of South Africa*
[2] *Department of Mathematics, Princeton University, Princeton NJ 08544, U.S.A.*

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ABSTRACT

Let $\beta\in(0,1)$ be irrational and let $\{n\beta\}$ denote the fractional part of $n\beta$, $n\geq 1$. The uniform distribution of $\{n\beta\}$, $n\geq 1$, implies that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} g(\{j\beta\}) = \int_0^1 g(t)\,dt,$$

for each bounded and Riemann integrable g. Hardy and Littlewood proved that this relation persists when g has integrable singularities at 0 and 1, under suitable conditions on g and β.

We show that by choosing suitable β, and g with an arbitrarily weak singularity at a suitable interior point $\alpha\in(0,1)$, one can ensure that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} g(\{j\beta\}) = \infty.$$

On the other hand, if the singularity lies at 0, then at least

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} g(\{j\beta\}) = \int_0^1 g(t)\,dt.$$

The motivation for these results lies in determination of the radius of convergence of the q or basic hypergeometric series

$$f(z) := 1+\sum_{j=1}^{\infty}\left\{\prod_{k=1}^{j}(A-q^k)\right\}z^j,$$

the solution of the functional equation

$$f(z)(1-zA) = 1-zqf(qz).$$

Especially for $|A|=|q|=1$, these power series are of interest in Padé approximation. Although the

radius of convergence is 1 for "most" A and q, we show that f may be a transcendental entire function for suitable $|A|=|q|=1$.

§1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\beta\in(0,1)$ be irrational and $\{n\beta\}$ denote the fractional part of $n\beta$, $n\geq 1$. The sequence $\{n\beta\}$, $n=1,2,3,\ldots$, is *uniformly distributed* in $[0,1]$: That is,

$$(1.1)\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} g(\{j\beta\})=\int_0^1 g(t)\,dt,$$

for each g bounded and Riemann integrable on $[0,1]$. In investigating certain power series, Hardy and Littlewood [5] extended (1.1) to g with integrable singularities at 0 and 1.

Suppose that $g(x)$ is bounded and Riemann-integrable in $[\delta,1-\delta]$ for all $0<\delta<1$, and increases steadily to ∞ as $x\to 0+$ or $x\to 1-$. Then they showed [5, p. 89, Thm. 5] that (1.1) persists if the integral there is finite, and if the sequence of partial quotients $\{a_j\}_{j=1}^{\infty}$ in the continued fraction expansion

$$(1.2)\qquad \beta=\frac{1\rfloor}{\lceil a_1}+\frac{1\rfloor}{\lceil a_2}+\frac{1\rfloor}{\lceil a_3}+\cdots$$

is bounded.

The set of $\beta\in(0,1)$ with this latter property has linear measure 0. As a supplementary result, Hardy and Littlewood [5, p. 89, Thm. 4] showed that (1.1) persists for almost all $\beta\in(0,1)$ if also

$$\int_0^1 g(t)\,dt\left[\log^2\frac{1}{t}+\log^2\frac{1}{1-t}\right]dt$$

converges.

Recently, consideration of certain power series has led the authors to investigation of the convergence of (1.1) for *all* irrational $\beta\in(0,1)$, and even for $n\to\infty$ through a subsequence. In the positive direction, we prove:

THEOREM 1.1. Let $\psi:(0,1]\to[0,\infty)$ be monotone decreasing with

$$(1.3)\qquad \lim_{t\to 0+}\psi(t)=\infty,$$

but

$$(1.4)\qquad \int_0^1\psi(t)\,dt<\infty.$$

Let $\beta\in(0,1)$ be irrational. Then

$$(1.5)\qquad \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\psi(\{j\beta\})=\int_0^1\psi(t)\,dt.$$

It is easy to show, and well known, that, no matter how weak is the singularity of ψ at 0, there exists β for which the lim inf cannot be replaced by lim. One of the essential features here is that the singularity of ψ lies at an endpoint of [0,1]. When the singularity lies inside (0,1), *not even a subsequence* of the quadrature sums need be bounded, let alone converge:

In formulating our result, we need the convergents p_j/q_j, $j=1,2,3,\ldots$ of the continued fraction (1.2). These may be defined by the relations $p_0:=0$; $p_1:=1$; $q_0:=1$; $q_1:=a_1$ and

$$p_n = a_n p_{n-1} + p_{n-2};\ q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 2.$$

THEOREM 1.2. Let ψ be as in Theorem 1.1. There exists irrational $\beta \in (0,1)$ and $\alpha \in (0,\beta)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi(|\{j\beta\} - \alpha|) = \infty. \tag{1.6}$$

The pair (α, β) may be any pair satisfying the following: Let p_j/q_j, $j=1,2,3,\ldots$, denote the convergents of the continued fraction (1.2) of β, and assume that

$$\lim_{j\to\infty} \psi\left(\frac{1}{q_{j+1}}\right) \Big/ \sum_{k=1}^{j} q_k = \infty. \tag{1.7}$$

Define α by

$$\alpha := \beta + \sum_{k=1}^{\infty} (q_k \beta - p_k). \tag{1.8}$$

The rather surprising feature here is that the full sequence in (1.6) diverges. This is in contrast to the situation for other procedures of numerical integration, such as Gauss quadratures, where at least a subsequence converges in the presence of an interior singularity [1,12,15]. There, as here, an interior singularity causes more difficulties than an endpoint singularity, as evidenced (in this case) by the more negative assertion of Theorem 1.2.

Note that it is possible to ensure (1.7) by assigning the partial quotients a_j in the continued fraction (1.2) to grow sufficiently rapidly. Since the condition (1.7) forces the denominators q_j of the convergents to grow fairly rapidly, the set $\mathcal{S}$ of β satisfying (1.7) is generally "quite thin". For example, if

$$\psi(t) := \log\frac{1}{t}, \quad t \in (0,1],$$

(1.7) implies that

$$\lim_{j\to\infty} (\log q_{j+1})/q_j = \infty,$$

and hence that (by standard estimates)

$$|\beta - p_j/q_j| \leq 1/(q_j q_{j+1}) \leq \exp(-\xi_j q_j), \quad j \geq 1,$$

where $\xi_j \to \infty$ as $j \to \infty$.

We can use the Jarnik-Besicovitch theorem [9, p. 510, Thm. 3] to deduce that not only does $\mathcal{S}$ have linear Lebesgue measure zero, but Hausdorff h-measure zero, for the function

$$h(t) = \left(\log \frac{1}{t}\right)^{-\beta}, \quad \beta > 2.$$

Hence $\mathscr{S}$ has Hausdorff dimension zero and logarithmic dimension at most 2 (see [16] for definitions). In the other direction, $\mathscr{S}$ will be dense in (0, 1) and will have positive Hausdorff h-measure for a suitable function h. Unfortunately, this is not an immediate consequence of classical theorems on diophantine approximation, but we shall not give further attention to this point.

As alluded to above, Hardy and Littlewood considered (1.1) for singular g, in their investigation of the radius of convergence of certain power series. They investigated, for example, the basic hypergeometric series

$$f(z) := \sum_{j=0}^{\infty} z^j / \prod_{k=1}^{j} (1-q^k)$$
$$= \sum_{j=0}^{\infty} z^j/(q;q)_j,$$

where

$$(1.9) \qquad (a;q)_k := \begin{cases} 1 & , \quad k=0. \\ (1-a)(1-aq)\ldots(1-aq^{k-1}), & k>0. \end{cases}$$

With the aid of the remarkable identity

$$f(z) = \exp\left(\sum_{n=1}^{\infty} z^n/(n(1-q^n))\right),$$

they established [5, p. 86] that the radius of convergence of f is

$$(1.10) \qquad \liminf_{n\to\infty} \left|\prod_{j=1}^{n} (1-q^j)\right|^{1/n} = \liminf_{n\to\infty} |1-q^n|^{1/n},$$

for any $|q| \leq 1$. This limit relation is itself of some interest.

Recently, there have been several studies of convergence of Padé approximants for certain q-hypergeometric series when $|q|=1$ [2, 3, 4, 13]. This was made possible by explicit formulae derived for the Padé approximants by P. Wynn, back in the 1960's [19]. The convergence problem is of particular interest for $|q|=1$, since in this case the power series usually have natural boundaries on their circle of convergence. While there are general theorems on convergence of Padé approximants for functions with essential singularities or branchpoints [14, 17], not so much is known for functions with natural boundaries.

One of the series considered [2, 3] was

$$(1.11) \qquad \begin{cases} F(A;q;z) := 1 + \sum_{j=1}^{\infty} \left(\prod_{k=1}^{j} (A-q^k)\right) z^j \\ \qquad = \sum_{j=0}^{\infty} (A^{-1}q;q)_j (Az)^j, \end{cases}$$

which satisfies the functional equation

$$(1.12)\qquad F(A;q;z)(1-zA) = 1 - zqF(A;q;qz).$$

The radius of convergence of this series turned out to be far more enigmatic than expected, and was the inspiration for this paper. To avoid trivialities, suppose that

$$(1.13)\qquad A \neq q^k, \quad k = 0, 1, 2, \ldots.$$

The interesting case is when $|A| = |q| = 1$ and $q = \exp(2\pi i\beta)$, $\beta \in (0,1)$ irrational. However, for completeness, we first enumerate the other cases: Let us denote the radius of convergence of $F(A;q;z)$ by $R(A;q)$. So

$$(1.14)\qquad 1/R(A;q) = \limsup_{n\to\infty} \Big|\prod_{j=1}^{n} (A - q^j)\Big|^{1/n}.$$

CASE I: $|q| > 1$. Then $R(A;q) = 0$.
CASE II: $|q| < 1$. Then $R(A;q) = 1/|A|$ and f is meromorphic in $\mathbb{C}$.
CASE III: $q = \exp(2\pi i\mu/\nu)$, where μ, ν are coprime integers. Then $F(A;q;z)$ is a rational function with simple poles contained in the set of νth roots of $A^\nu - 1$.
CASE IV: $q = \exp(2\pi i\beta)$ where $\beta \in (0,1)$ is irrational and $|A| \neq 1$. Then $R(A;q) = 1/\max\{1, |A|\}$, and the circle of convergence is the natural boundary of f.

For a proof of these relatively simple facts, we refer the reader to [2, 3]. We now turn to the remaining case. In formulating our result, we need logarithmic capacity, which may be defined as follows: Let $\mathscr{P}_n$ denote the set of polynomials of degree at most n. For compact $\mathscr{E} \subset \mathbb{C}$, we set

$$\operatorname{cap}(\mathscr{E}) := \lim_{n\to\infty} \Big(\min_{\substack{\text{degree}(P)=n \\ P \text{ monic}}} \|P\|_{L_\infty(\mathscr{E})} \Big)^{1/n}.$$

For arbitrary $\mathscr{F} \subset \mathbb{C}$, its (inner) logarithmic capacity is

$$\operatorname{cap}(\mathscr{F}) := \sup\{\operatorname{cap}(\mathscr{E}) \colon \mathscr{E} \subset \mathscr{F} \text{ and } \mathscr{E} \text{ compact}\}.$$

A set of cap 0 is thin indeed. It has planar Lebesgue measure 0. Moreover, its intersection with every line has linear Lebesgue measure 0, and it has Hausdorff dimension 0. See [7, 8, 18].

THEOREM 1.3. Consider the power series (1.11) subject to (1.13). Let

$$(1.15)\qquad q = \exp(2\pi i\beta), \quad \beta \in (0,1) \text{ irrational,}$$

and

$$(1.16)\qquad |A| = 1.$$

Then
(I) $R(A;q) \geq 1$.
(II) When $A = 1$, $R(A;q) = 1$.
(III) There exists a set $\mathscr{S}_q$, depending on q, with $\operatorname{cap}(\mathscr{S}_q) = 0$ such that $R(A;q) = 1$ whenever $A \notin \mathscr{S}_q$.

(IV) There exist A and q satisfying (1.13), (1.15) and (1.16) such that $R(A;q)=\infty$, that is, $F(A;q;z)$ is a transcendental entire function. For the pair (A,q), we require only that the convergents p_j/q_j, $j\geq 1$, of the continued fraction (1.2) of β satisfy

(1.17) $\quad \lim_{j\to\infty} (\log q_{j+1})/q_j = \infty,$

and that

(1.18) $\quad A = \exp(2\pi i\alpha),$

where α is defined by (1.8).

Note that in the cases (II) and (III), f has a natural boundary on its circle of convergence [2,3]. It seems certain that given $r\in(1,\infty)$, there exists a pair (A,q) as above for which $R(A;q)=r$. The set of such (A,q) should be "thin" in $\{z\colon |z|=1\}^2$, as indicated by the above theorem.

We prove Theorems 1.1 and 1.2 in Section 2, and prove Theorem 1.3 in Section 3.

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§2. PROOFS OF THEOREMS 1.1 AND 1.2

We begin by recalling some elementary properties of continued fractions:

LEMMA 2.1. Let $\beta\in(0,1)$ be irrational, with continued fraction (1.2). Let p_j/q_j, $j=1,2,3,\ldots$, denote the sequence of convergents to the continued fraction of β. Then

(a)

(2.1) $\quad |\beta-p_j/q_j|\leq 1/(q_jq_{j+1}), \quad j\geq 1.$

(b) $(-1)^j(q_j\beta-p_j)=|q_j\beta-p_j|>0$, $j\geq 1$, and $|q_j\beta-p_j|$ decreases strictly as j increases. Moreover, p_j, q_j are coprime for $j\geq 1$.

(c) q_j increases strictly as j increases, and

(2.2) $\quad \beta>1/(q_1+1).$

PROOF. (a), (b) See [6, pp. 137-8].

(c) See [6, p. 132] for the proof that q_j is increasing. Also, from (1.2),

$$\beta>\frac{1}{a_1+1}=\frac{1}{q_1+1}. \qquad \square$$

We let $\langle x\rangle$ denote the integer part of $x\in\mathbb{R}$, that is, the greatest integer $\leq x$.

PROOF OF THEOREM 1.1. First let $\varepsilon \in (0,1)$ and

$$g(t) := \begin{cases} \psi(t), & t \in [\varepsilon, 1]. \\ 0, & \text{otherwise.} \end{cases}$$

Then g is bounded and Riemann integrable in $[0,1]$, so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) = \int_0^1 g(t)\,dt = \int_\varepsilon^1 \psi(t)\,dt.$$

Since $\psi \geq g$ and $\varepsilon > 0$ is arbitrary, we deduce from Lebesgue's Monotone Convergence Theorem that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi(\{j\beta\}) \geq \int_0^1 \psi(t)\,dt. \tag{2.3}$$

The other direction is more difficult. Choose j even, and let $p/q = p_j/q_j$ be an even order convergent to the continued fraction expansion of β. Then by (2.1),

$$0 \leq \beta - p/q \leq 1/q^2, \tag{2.4}$$

and

$$(p,q) = 1. \tag{2.5}$$

One immediate consequence of (2.5) is that

$$\{\{jp/q\}\colon 1 \leq j \leq q-1\} = \{j/q\colon 1 \leq j \leq q-1\}. \tag{2.6}$$

Hence for $1 \leq j \leq q-1$,

$$1/q \leq \{jp/q\} \leq 1 - 1/q. \tag{2.7}$$

Then in view of (2.4), and (2.7),

$$\begin{aligned} 0 &\leq \{j(\beta - p/q)\} + \{jp/q\} \\ &\leq j(\beta - p/q) + 1 - 1/q \\ &\leq j/q^2 + 1 - 1/q < 1/q + 1 - 1/q = 1. \end{aligned}$$

Hence for $1 \leq j \leq q-1$,

$$\begin{aligned} \{j\beta\} &= \{j(\beta - p/q) + jp/q\} \\ &= \{\{j(\beta - p/q)\} + \{jp/q\}\} \\ &= \{j(\beta - p/q)\} + \{jp/q\}. \end{aligned}$$

Then (2.4) and the monotonicity of ψ ensure that

$$\psi(\{j\beta\}) \leq \psi(\{jp/q\}), \quad 1 \leq j \leq q-1. \tag{2.8}$$

Now, (2.6) ensures that

$$\frac{1}{q-1} \sum_{j=1}^{q-1} \psi(\{j\beta\}) \leq \frac{1}{q-1} \sum_{j=1}^{q-1} \psi(\{jp/q\})$$

$$= \frac{q}{q-1} \frac{1}{q} \sum_{j=1}^{q-1} \psi(j/q)$$

$$\leq \frac{q}{q-1} \int_0^{1-1/q} \psi(t)\,dt \quad \text{(by monotonicity of } \psi\text{)}$$

$$\to \int_0^1 \psi(t)\,dt,$$

as $q \to \infty$. We deduce that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi(\{j\beta\}) \leq \int_0^1 \psi(t)\,dt.$$

Together with (2.3), this proves the theorem. □

In proving Theorem 1.2, we shall need:

LEMMA 2.2. Assume the notation of Lemma 2.1. Let $q_0 := 1$, and

$$(2.9) \qquad s_j := \sum_{k=0}^{j} q_k, \quad j \geq 1.$$

Then (a)

$$(2.10) \qquad \{s_j \beta\} = \beta + \sum_{k=1}^{j} (q_k \beta - p_k) \in (0, \beta), \quad j \geq 1.$$

(b) There exists

$$\lim_{j\to\infty} \{s_j \beta\} = \beta + \sum_{k=1}^{\infty} (q_k \beta - p_k) =: \alpha \in (0,1).$$

PROOF. (a) By Lemma 2.1, $\sum_{k=1}^{j} (q_k \beta - p_k)$ is an alternating finite series of terms that decrease in magnitude. Since

$$q_1 \beta - p_1 < 0,$$

we have

$$q_1 \beta - p_1 < \sum_{k=1}^{j} (q_k \beta - p_k) < 0.$$

Now

$$\beta > \beta + q_1 \beta - p_1 \geq \beta - 1/q_2 \quad \text{(by (2.1))}$$

$$\geq \beta - 1/(q_1 + 1) > 0,$$

by (2.2). Hence

$$\beta > \beta + \sum_{k=1}^{j} (q_k \beta - p_k) > \beta + q_1 \beta - p_1 > 0.$$

Then

$$\{s_j \beta\} = \{\beta + \sum_{k=1}^{j} q_k \beta\}$$

$$= \{\beta + \sum_{k=1}^{j} (q_k \beta - p_k)\} = \beta + \sum_{k=1}^{j} (q_k \beta - p_k).$$

(b) This follows directly from the alternating series test. □

PROOF OF THEOREM 1.2. Let $(s_j)_{j=1}^{\infty}$ be as in Lemma 2.2. Let $j \geq 2$, and let

$$s_{j-1} \leq n < s_j.$$

Then

$$\Sigma_n := \frac{1}{n} \sum_{k=1}^{n} \psi(|\{k\beta\} - \alpha|)$$
$$> \frac{1}{s_j} \psi(|\{s_{j-1}\beta\} - \alpha|).$$

But by Lemma 2.2,

$$(2.11) \quad \begin{cases} |\{s_{j-1}\beta\} - \alpha| = |\sum_{k=j}^{\infty} (q_k\beta - p_k)| \\ \qquad \leq |q_j\beta - p_j| \leq 1/q_{j+1}, \end{cases}$$

by Lemma 2.1. Then

$$\Sigma_n > \frac{1}{s_j} \psi(1/q_{j+1})$$
$$= \psi(1/q_{j+1})/(1 + \sum_{k=1}^{j} q_k) \to \infty,$$

as $j \to \infty$, by (1.7). □

§3. PROOF OF THEOREM 1.3

PROOF OF THEOREM 1.3(I). Let μ_n be the unit measure assigning mass $1/n$ to q^j, $1 \leq j \leq n$. One way to reformulate the uniform distribution of $(q^j)_{j=1}^{\infty}$ on the unit circle, is that μ_n converges weakly as $n \to \infty$ to normalized Lebesgue measure on the circle:

$$d\mu_n(\theta) \xrightarrow{*} d\theta/(2\pi), \quad n \to \infty.$$

Let us consider the associated potential functions

$$U(z; \mu_n) = \int \log |z - t|^{-1} d\mu_n(t),$$

and note that

$$|\prod_{j=1}^{n} (z - q^j)|^{1/n} = \exp(-U(z; \mu_n)).$$

Let $\varepsilon > 0$. By the maximum modulus principle,

$$1/R(A; q) = \limsup_{n \to \infty} |\prod_{j=1}^{n} (A - q^j)|^{1/n}$$
$$\leq \limsup_{n \to \infty} (\max_{|z| = 1+\varepsilon} |\prod_{j=1}^{n} (z - q^j)|)^{1/n}$$
$$= \exp(-\liminf_{n \to \infty} (\min_{|z| = 1+\varepsilon} U(z; \mu_n))).$$

Since $K := \{z: |z| = 1+\varepsilon\}$ is a compact set lying at a positive distance to the support of $(\mu_n)_{n=1}^{\infty}$ and $d\theta/(2\pi)$, equicontinuity of $(U(z;\mu_n))_{n=1}^{\infty}$ on K, and the Arzela-Ascoli Theorem shows that

$$\begin{aligned}
&\lim_{n\to\infty} \min_{|z|=1+\varepsilon} U(z;\mu_n)) \\
&\quad = \min_{|z|=1+\varepsilon} U(z; d\theta/(2\pi)) \\
&\quad = \min_{|z|=1+\varepsilon} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|z-e^{i\theta}|^{-1}\, d\theta \\
&\quad = \log(1+\varepsilon)^{-1}.
\end{aligned}$$

Hence

$$1/R(A;q) \le 1+\varepsilon.$$

Now let $\varepsilon \to 0+$. □

PROOF OF THEOREM 1.3(II). Now in view of (1.14),

$$(3.1)\quad \begin{cases} 1/R(1;q) = \limsup_{n\to\infty} \left|\prod_{j=1}^{n}(1-q^j)\right|^{1/n} \\ \qquad\qquad = \exp\left[-\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(\{j\beta\})\right], \end{cases}$$

where

$$(3.2)\quad g(t) := -\log|1-e^{2\pi i t}| = -\log(2\sin \pi t), \quad t \in (0,1).$$

Let r/s be an even order convergent in the continued fraction expansion of β. Then, as at (2.7),

$$(3.3)\quad 1/s \le \{jr/s\} \le 1-1/s, \quad 1 \le j \le s-1,$$

and

$$(3.4)\quad 0 \le \beta - r/s \le 1/s^2; \quad (r,s)=1.$$

If first,

$$(3.5)\quad \frac{\log s}{s} \le \{jr/s\} \le 1-\frac{\log s}{s},$$

then by elementary inequalities,

$$\begin{aligned}
&|g(\{j\beta\}) - g(\{jr/s\})| \\
&\quad = \left|\log\left|1+\frac{\sin(\pi\{j\beta\})-\sin(\pi\{jr/s\})}{\sin(\pi\{jr/s\})}\right|\right| \\
&\quad \le \log\left|1-\left|\frac{j\pi(\beta-r/s)}{\sin(\pi(\log s)/s)}\right|\right|^{-1} \\
&\quad \le \log\left|1-\frac{j\pi/s}{2(\log s)}\right|^{-1}
\end{aligned}$$

$$\le \pi/(s \log s),$$

for large enough s. If $\mathscr{I}_s$ denotes the set of indices j satisfying (3.5), we obtain for large enough s,

$$\text{(3.6)} \qquad \frac{1}{s-1} \sum_{j \in \mathscr{I}_s} |g(\{j\beta\}) - g(\{jr/s\})| \le \frac{\pi}{\log s}.$$

Let $\mathscr{K}_s = \{1, 2, \dots, s-1\} \setminus \mathscr{I}_s$. Now exactly as in the proof of (2.6) and (2.7), we can show that

$$|\{j_1 r/s\} - \{j_2 r/s\}| \ge 1/s, \quad 1 \le j_1 \ne j_2 \le s-1.$$

Hence the number of elements of $\mathscr{K}_s$ is at most $2(\log s + 1)$. Moreover, if $j \in \mathscr{K}_s$, (3.3) shows, as above, that for large enough s,

$$|g(\{j\beta\}) - g(\{jr/s\})| \le \log \left| 1 - \left| \frac{j\pi(\beta - r/s)}{\sin(\pi/s)} \right| \right|^{-1}$$

$$\le \log \left| 1 - \frac{j\pi/s^2}{(\pi/s)(1 - 2/s^2)} \right|^{-1}$$

$$\le \log \left| 1 - \frac{1 - 1/s}{1 - 2/s^2} \right|^{-1}$$

$$\le \log(2s).$$

Hence, for large enough s,

$$\text{(3.7)} \qquad \begin{cases} \dfrac{1}{s-1} \displaystyle\sum_{j \in \mathscr{K}_s} |g(\{j\beta\}) - g(\{jr/s\}| \\ \quad \le \dfrac{2}{s-1} (\log s + 1)(\log(2s)). \end{cases}$$

Thus, by (3.6) and (3.7),

$$\text{(3.8)} \qquad \left| \frac{1}{s-1} \sum_{j=1}^{s-1} g(\{j\beta\}) - \frac{1}{s-1} \sum_{j=1}^{s-1} g(\{jr/s\}) \right| = o(1), \quad s \to \infty.$$

Now since

$$\{\{jr/s\} \colon 1 \le j \le s-1\} = \{j/s \colon 1 \le j \le s-1\},$$

we have

$$\exp\left(\frac{1}{s-1} \sum_{j=1}^{s-1} g(\{jr/s\}) \right) = \exp\left(\frac{1}{s-1} \sum_{j=1}^{s-1} g(j/s) \right)$$

$$= |\prod_{j=1}^{s-1} (1 - e^{2\pi i j/s})|^{-1/(s-1)} = s^{-1/(s-1)},$$

where we have used (3.2), and the polynomial

$$P(z) := \prod_{j=1}^{s-1} (z - e^{2\pi i j/s}) = (z^s - 1)/(z - 1), \quad z \ne 1,$$

which satisfies $P(1)=s$. Combined with (3.8), this yields

$$\lim_{\substack{s\to\infty \\ s\in\mathcal{N}}} \frac{1}{s-1} \sum_{j=1}^{s-1} g(\{j\beta\}) = 0, \tag{3.9}$$

where $\mathcal{N}$ is the set of all integers s that are the denominator in an even order convergent to β. Finally, g is bounded below by $-\log 2$, so introducing the truncations

$$g_N(t) := \min\{g(t), N\}, \quad t\in(0,1),\ n\geq 1,$$

we obtain that g_N is bounded and Riemann-integrable, so

$$\begin{aligned}\liminf_{n\to\infty} &\frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) \\ &\geq \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g_N(\{j\beta\}) \\ &= \int_0^1 g_N(t)\,dt.\end{aligned}$$

By monotone convergence, as $N\to\infty$, this last integral increases to

$$\int_0^1 g(t)\,dt = -\int_0^1 \log|1-e^{2\pi i t}|\,dt = 0.$$

Together with (3.9), this shows that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} g(\{j\beta\}) = 0.$$

By (3.1), $R(1;q)=1$. □

PROOF OF THEOREM 1.3(III). Now with the notation used in the proof of Theorem 1.3(I), we have

$$R(A;q) = \exp(-\liminf_{n\to\infty} U(A;\mu_n)).$$

But the weak convergence

$$d\mu_n(\theta) \xrightarrow{*} \frac{d\theta}{2\pi}, \quad n\to\infty,$$

and the lower envelope theorem of potential theory [11, Thm. 3.8] shows that there exists a set $\mathcal{S}_q$ of cap 0 such that for $A\notin\mathcal{S}_q$,

$$\liminf_{n\to\infty} U(A;\mu_n) = U(A;d\theta/(2\pi)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|A-e^{i\theta}|^{-1}\,d\theta = 0.$$

Hence for $|A|=1$, $A\notin\mathcal{S}_q$, $R(A;q)=1$. □

PROOF OF THEOREM 1.3(IV). Let us assume the notation of Lemma 2.2, and in particular the definition (2.9) of $(s_j)_{j=1}^{\infty}$. If $s_{j-1}\leq n<s_j$, then

$$
\begin{aligned}
&(\prod_{k=1}^{n} |A-q^k|)^{1/n} \\
&\quad \le 2^{(n-1)/n}|A-q^{s_{j-1}}|^{1/n} \\
&\quad = 2\,|\sin(\pi(\alpha-s_{j-1}\beta))|^{1/n} && \text{(by (1.15) and (1.18))} \\
&\quad = 2\,|\sin(\pi(\alpha-\{s_{j-1}\beta\}))|^{1/n} \\
&\quad \le 2(\pi/q_{j+1})^{1/n} && \text{(by (2.11))} \\
&\quad \le 2(\pi/q_{j+1})^{1/s_j} \\
&\quad \le 2\pi\exp(-\log q_{j+1}/(1+\sum_{k=1}^{j} q_k)) \to 0,
\end{aligned}
$$

as $j \to \infty$, by (1.17). □

REFERENCES

1. Davis, P.J. and P. Rabinowitz - Methods of Numerical Integration. Academic Press, New York, 1975.
2. Driver, K.A. - Convergence of Padé Approximants for Some q-Hypergeometric Series (Wynn's Power Series I, II and III). Ph.D. Thesis, Witwatersrand University, Johannesburg, 1991.
3. Driver, K.A. and D.S. Lubinsky - Convergence of Padé Approximants for a q-Hypergeometric Series (Wynn's Power Series I). Aequationes Mathematicae **42**, 85-106 (1991).
4. Driver, K.A. and D.S. Lubinsky - Convergence of Padé Approximants for a q-Hypergeometric Series (Wynn's Power Series II). To appear in Proc. International Conference on Approx. Theory, Kecskemet, Hungary, 1990.
5. Hardy, G.H. and J.E. Littlewood - Notes on the Theory of Series (XXIV): A Curious Power Series. Proc. Camb. Phil. Soc. **42**, 85-90 (1946).
6. Hardy, G.H. and E.M. Wright - An Introduction to the Theory of Numbers. 4th edn., Oxford University Press, Oxford, 1975.
7. Hayman, W.K. and P.B. Kennedy - Subharmonic Functions. Vol. 1, Academic Press, London, 1976.
8. Hille, E. - Analytic Function Theory. Vol. II, Ginn and Blaisdell, Waltham, 1962.
9. Jarnik, V. - Über die Simultanen Diophantischen Approximationen. Math. Zeit. **33**, 505-543 (1931).
10. Kuipers, L. and H. Niederreiter - Uniform Distribution of Sequences. Wiley, New York, 1974.
11. Landkof, N.S. - Foundations of Modern Potential Theory. Grundlehren der Mathematischen Wissenschaften, Vol. 190, Springer-Verlag, New York, 1972.
12. Lubinsky, D.S. and P. Rabinowitz - Rates of Convergence of Gaussian Quadrature for Singular Integrands. Math. Comp. **43**, 219-242 (1987).
13. Lubinsky, D.S. and E.B. Saff - Convergence of Padé Approximants of Partial Theta Functions and the Rogers-Szegö Polynomials. Constr. Approx. **3**, 331-361 (1987).
14. Nuttall, J. - The Convergence of Padé Approximants of Meromorphic Functions. J. Math. Anal. Appl. **31**, 147-153 (1970).
15. Rabinowitz, P. - Gaussian Integration in the Presence of a Singularity. SIAM J. Numer. Anal. **4**, 197-201 (1967).
16. Rogers, C.A. - Hausdorff Measures. Cambridge University Press, Cambridge, 1970.
17. Stahl, H. - General Convergence Results for Padé Approximants. Approximation Theory VI (eds. C.K. Chui et al.), Academic Press, San Diego, 1989, p. 605-634.
18. Tsuji, M. - Potential Theory in Modern Function Theory. Chelsea, New York, 1975.
19. Wynn, P. - A General System of Orthogonal Polynomials. Quart. J. Math. Oxford **18**, 81-96 (1967).