Indag. Mathem., N.S., 3 (3), 353-364 September 21, 1992

On the radius of convergence of q-series

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Communicated by Prof. R. Tijdeman at the meeting of February 24, 1992

ABSTRACT

Completing the description of the possible radii of convergence of q-series

$$F(z) = 1 + \sum_{j=1}^{\infty} \Big(\prod_{k=1}^{j} (A - q^k) \Big) z^j,$$

it is shown in this paper that, for any given $1 \le R \le \infty$ there is a q-series with $A = \exp(2\pi i\alpha)$, $q = \exp(2\pi i\beta)$ such that its radius convergence is R. If α is a rational number, then for any irrational β, the radius is always 1.

1. The study and applications of the q- or basic hypergeometric series

$$(1) \qquad F(z) = 1 + \sum_{j=1}^{\infty} \Big(\prod_{k=1}^{j} (A - q^k) \Big) z^j = \sum_{j=0}^{\infty} (a;q)_j w^j \quad (a = A^{-1}q,\ w = Az)$$

is a standard subject in approximation theory (see [GR]). The radii of convergence of these and related series have been investigated in several papers (see, p.e. [DL], [DLPS], [HL]). In the most interesting case $|A| = |q| = 1$ it has been proved in [DLPS], that the radius is 1, if $A = 1$, and also that choosing the irrationals α, β in a suitable way and putting $A = \exp(2\pi i\alpha)$, $q = \exp(2\pi i\beta)$, one can obtain a transcendental entire function in (1). Our aim in this paper is to fill the gap between 1 and ∞ by showing, that any $1 < R < \infty$ can be prescribed as the radius of convergence, which makes the list of radii in [DLPS] complete. We also prove (Theorem 5), that the case of rational α is analogous to $A = 1$, that is the radius of convergence is 1 for any irrational β. Our main result is the following theorem.

THEOREM 1. For every $1<R<\infty$ there exists a pair of irrationals α, β such that with $A=\exp(2\pi i\alpha)$, $q=\exp(2\pi i\beta)$, the radius of convergence of the power series in (1) is R.

2. For any given real number x we denote by $a_n=a_n(x)$ $(n=0,1,\dots)$ the partial quotients of the simple continued fraction expansion of $x=[a_0;a_1,a_2,\dots]$. The corresponding approximants in their lowest terms are denoted by $p_n(x)/q_n(x)=p_n/q_n$. This notation will be applied throughout in this paper. The recursions

$$(2)\qquad \begin{cases} p_k=a_kp_{k-1}+p_{k-2},\ q_k=a_kq_{k-1}+q_{k-2},\ q_kp_{k-1}-p_kq_{k-1}=(-1)^k, \\ \qquad (k\geq 2) \end{cases}$$

and relations

$$(3)\qquad q_kx-p_k=\frac{(-1)^k}{q_{k+1}+\alpha_{k+1}q_k}\qquad (0<\alpha_{k+1}<1),$$

as well as the best approximation property: if a,b are integers and $a<q_{k+1}$, then

$$(4)\qquad |ax-b|\geq|q_kx-p_k|$$

are well known facts in the theory of continued fractions (see [HW] or [NI]).

LEMMA 1. Let a_n,q_n,p_n denote as above and let $s_n=\sum_{j=0}^{n}q_j$. Then
(i) $s_n<3q_n$,
(ii) and if $a_n\to\infty$, then $s_n/q_n\to 1$.

PROOF. Adding the inequalities $q_{j+1}=a_{j+1}q_j+q_{j-1}\geq q_j+q_{j-1}$, $(j=1,\dots,n-1)$, we obtain $q_n\geq q_1+s_{n-2}>s_{n-2}$ and hence $s_n<3q_n$. Thus $s_n=s_{n-1}+q_n<3q_{n-1}+q_n$. This implies

$$1<\frac{s_n}{q_n}<3\frac{q_{n-1}}{q_n}+1<\frac{3}{a_n}+1,$$

and the statement follows.

LEMMA 2. For any given $c>0$ there exists an irrational number $x=[0;a_1,a_2,\dots]$ which is defined inductively by

$$(5)\qquad a_{k+1}=\left[\frac{e^{cq_k}}{q_k}\right],\quad k=1,2,\dots,$$

such that x admits the following properties:

$$(6)\qquad a_n<a_{n+1},\quad n=1,2,\dots;$$

$$(7)\qquad \lim_{n\to\infty}\frac{\log q_{n+1}}{q_n}=c.$$

Any irrational satisfying (7) also satisfies

$$\prod_{j=1}^{n-1} q_j \le q_n^{\varepsilon}, \quad n \ge n_0(\varepsilon). \tag{8}$$

PROOF. Let $c>0$ be given. The partial quotients a_n will be defined by induction. If c is large (say, $c \ge 1$), then the inductive definition is automatic by (5). If however c is small, we have to be more careful to make sure, that the fractions in (5) are large enough. Let $a_1 \ge 2$ so large, that

$$\frac{e^{ca_1}}{a_1^2+a_1} > 1, \quad \text{then } \frac{e^{ca_1}}{a_1} > \frac{a_1^2+a_1}{a_1} = a_1+1,$$

thus defining a_2 by (5) for $k=1$ and taking into account $q_1 = a_1$, we get $a_2 \ge a_1+1 \ge 3$. Suppose that for $1 \le k \le n$ we have defined a_k, q_k such that (5) and (6) hold for $1 \le k \le n$. To show that the inductive definition (5) makes sense for n and that (6) holds true for $n+1$ as well, we need to estimate the fraction in (5) from below. (5) implies $\exp(cq_k) \ge a_{k+1} q_k$, $1 \le k \le n-1$, (6) implies $a_k \ge 3$, $2 \le k \le n$, thus we obtain for any $2 \le k \le n$

$$\begin{aligned}\exp(cq_k) &= \exp(c(a_k q_{k-1} + q_{k-2})) = (\exp(cq_{k-1}))^{a_k} \cdot \exp(cq_{k-2}) \\ &> (a_k q_{k-1})^{a_k} = (a_k q_{k-1})^2 (a_k q_{k-1})^{a_k-2} > q_k(a_k+1).\end{aligned}$$

That is,

$$\frac{e^{cq_k}}{q_k} > a_k + 1,$$

and we can complete the induction by putting $k=n$. Notice that by (6) we obviously have $a_n > n$, in particular $a_n \to \infty$.

As an upper estimate for q_{n+1} we have

$$q_{n+1} = a_{n+1} q_n + q_{n-1} < e^{cq_n} + q_{n-1} < 2e^{cq_n}. \tag{9}$$

For the other side, notice first that

$$\frac{e^{cq_n}}{q_n} \ge a_{n+1} > n, \quad \text{which implies } e^{cq_n} > nq_n, \quad e^{cq_n} - q_n > \left(1 - \frac{1}{n}\right) e^{cq_n},$$

and

$$\frac{e^{cq_n}}{q_n} - 1 < a_{n+1}, \quad \text{that is} \quad e^{cq_n} - q_n < a_{n+1} q_n < q_{n+1},$$

and hence

$$\left(1 - \frac{1}{n}\right) e^{cq_n} < q_{n+1}.$$

By this and (9) we get

$$\log\left(1 - \frac{1}{n}\right) + cq_n < \log q_{n+1} < cq_n + \log 2,$$

and dividing by q_n and letting $n \to \infty$ we obtain (7). We have to show that (7) implies (8). Taking the logarithm in $a_{n+1}q_n < q_{n+1} < 2a_{n+1}q_n$ and dividing by q_n we obtain

$$\lim_{n \to \infty} \frac{\log a_{n+1}}{q_n} = c$$

as an obvious consequence of (7). This implies in particular $a_n \to \infty$, and

$$(10) \qquad e^{cq_n} < (a_{n+1})^{1+\varepsilon} < (q_{n+1})^{1+\varepsilon},$$

if n is large enough. Also

$$\log q_{n+1} < (1+\varepsilon)cq_n$$

holds for $n \geq N$, N fixed. Adding these inequalities we get

$$\log \prod_{j=N}^{n+1} q_j < (1+\varepsilon)cs_n,$$

and hence by Lemma 1 (ii)

$$\log \prod_{j=1}^{n+1} q_j < (1+2\varepsilon)cq_n,$$

if n is large enough. Finally by (10) we obtain

$$\prod_{j=1}^{n+1} q_j < (q_{n+1})^{1+3\varepsilon},$$

and the lemma is proved. We remark, that the same argument provides an analogous lower estimate for this product as well, thus (7) in fact implies

$$\lim_{n \to \infty} \frac{\sum_{j=1}^{n} \log q_j}{\log q_n} = 1.$$

LEMMA 3. For a real number x let $\|x\|$ denote its distance to the nearest integer, and let $0 \leq t < \|x\|$. Then

$$(11) \qquad |\sin \pi(x \pm t)| = |1+\mu| \cdot |\sin \pi x|, \text{ where } |\mu| \leq \frac{t^2}{2}\pi^2 + \frac{\pi t}{2\|x\|}.$$

PROOF. Since $|\sin \pi(x \pm t)| = |\sin \pi(b \pm x \pm t)|$, where b is an arbitrary integer, we can assume without loss of generality that $0 < x \leq 1/2$. Then $x = \|x\|$, $\sin \pi x \geq 2x$ and

$$\begin{aligned} |\sin \pi(x \pm t)| &= |\sin \pi x \cos \pi t \pm \cos \pi x \sin \pi t| \\ &= \left| 1 + \left(\cos \pi t - 1 + \frac{\cos \pi x}{\sin \pi x} \sin \pi t \right) \right| |\sin \pi x|. \end{aligned}$$

But

$$\left| \cos \pi t - 1 + \frac{\cos \pi x}{\sin \pi x} \sin \pi t \right| \leq \frac{t^2}{2}\pi^2 + \frac{\pi t}{2x},$$

and the lemma is proved.

LEMMA 4. Let x be an irrational number, then

$$\lim_{n \to \infty} \frac{1}{q_n - 1} \sum_{k=1}^{q_n - 1} \log 2|\sin \pi k x| = 0. \tag{12}$$

REMARK. This statement has been proved in [DLPS] for the special case, when n runs through the even indices. But this is just a minor technical advantage and a similar idea combined with Lemma 3 here works for the general case.

PROOF. Let $1 \le k \le q_n - 1$ and write

$$|\sin \pi k x| = \left| \sin \pi \left(k \frac{p_n}{q_n} + k \left(x - \frac{p_n}{q_n} \right) \right) \right|.$$

Since

$$\left\| k \frac{p_n}{q_n} \right\| \ge \frac{1}{q_n} \tag{12a}$$

(p_n and q_n are coprime), and by (3)

$$k \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_{n+1}},$$

we can apply Lemma 3 with

$$t := k \left| x - \frac{p_n}{q_n} \right| \quad \text{and} \quad x := k \frac{p_n}{q_n},$$

and obtain

$$|\sin \pi k x| = \left| \sin \pi k \frac{p_n}{q_n} \right| \cdot |1 + \mu_k|.$$

Since p_n and q_n are coprime, the numbers kp_n for $k = 1, \dots, q_n - 1$ run through the non-zero residue classes $\bmod q_n$, thus the set of fractions $\{k(p_n/q_n) : 1 \le k \le q_n - 1\}$ is identical mod 1 to the set $\{(k/q_n) : 1 \le k \le q_n - 1\}$. Therefore the number of indices k such that

$$\left\| k \frac{p_n}{q_n} \right\| \le \frac{1}{\sqrt{q_n}}$$

is $2\sqrt{q_n}$. Extending the summation for these indices and applying Lemma 3 and (12a) we obtain by trivial estimate

$$\sigma_1 = \sum_{\|k(p_n/q_n)\| \le 1/\sqrt{q_n}} \log(1 + \mu_k) < 2\sqrt{q_n} \log 3.$$

If, on the other hand

$$\left\| k \frac{p_n}{q_n} \right\| \ge \frac{1}{\sqrt{q_n}},$$

then by Lemma 3

$$|\mu_k| \le \frac{\pi^2}{2}\frac{1}{q_{n+1}^2} + \frac{\pi}{2}\frac{\sqrt{q_n}}{q_{n+1}} < \frac{3}{\sqrt{q_n}},$$

and extending the summation to these indices of second kind we get

$$\sigma_2 = \sum_{\|k(p_n/q_n)\| > 1/\sqrt{q_n}} \log(1+\mu_k) < q_n \log\left(1 + \frac{3}{\sqrt{q_n}}\right) < 3\sqrt{q_n}.$$

Thus

$$\frac{1}{q_n - 1}\sum_{k=1}^{q_n - 1} \log 2|\sin \pi k x| = \frac{1}{q_n - 1}\sum_{k=1}^{q_n - 1} \log 2\left|\sin \pi k \frac{p_n}{q_n}\right| + \frac{\sigma_1 + \sigma_2}{q_n - 1}$$

$$= \frac{1}{q_n - 1}\sum_{k=1}^{q_n - 1} \log 2\left|\sin \pi \frac{k}{q_n}\right| + o(1) = o(1),$$

since, more generally

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N-1} \log 2\left|\sin \pi \frac{k}{N}\right| = \int_0^1 \log 2|\sin \pi x|\, dx = 0$$

is well known.

3. Now we turn to the proof of Theorem 1. The statement is reformulated in a slightly stronger way in Theorem 2, which together with Lemma 2 obviously imply Theorem 1.

THEOREM 2. Let $R = e^c$, $c > 0$ be given. If an irrational number β satisfies condition (7), then there exists another (irrational) number α such that putting $A = \exp(2\pi i\alpha)$, $q = \exp(2\pi i\beta)$, the radius of convergence of (1) is R.

PROOF. The main tool in the proof (just as it was for $R = \infty$ in [DLPS]) is the function defined by

$$f(x) = \sum_{k=0}^{\infty} (q_k(x)x - p_k(x)). \tag{13}$$

This function has many interesting number theoretic and function theoretic properties, which will be studied independently in the forthcoming papers by J.N. Ridley and the author. Here we only deal with the irrationality of $f(x)$ in section 4.

Note that the series in (13) is an alternating series of terms decreasing in absolute value (properties (3) and (4)), therefore

$$\sum_{j=n+1}^{\infty} (q_j x - p_j) = \vartheta(q_{n+1}x - p_{n+1}) \quad (0 < \vartheta < 1). \tag{14}$$

If furthermore $a_n \to \infty$, or equivalently $(q_{n+1}/q_n) \to \infty$, then by (3)

$$\sum_{j=n+1}^{\infty} (q_j x - p_j) = \vartheta_{n+1}\frac{(-1)^{n+1}}{q_{n+2}}, \tag{15}$$

where $\vartheta_n \to 1$. For a given $1 < R < \infty$ let $c = \log R$, and let then $\beta = [0; a_1, a_2, \dots]$ denote an irrational with the properties as stated by Lemma 2. Put $\alpha = f(\beta)$ and let A, q be as in the statement of the theorem. We have to show that

$$\limsup_{n\to\infty} \Big(\prod_{k=1}^{n} |A - q^k| \Big)^{1/n} = \frac{1}{R},$$

or writing it in logarithmic form

$$\limsup_{n\to\infty} \frac{1}{n} \Big(\sum_{k=1}^{n} \log |A - q^k| \Big) = -c. \tag{16}$$

We write, as usual,

$$\log |A - q^k| = \log |e^{2\pi i\alpha} - e^{2\pi i k\beta}|$$
$$= \log |1 - e^{2\pi i(k\beta - \alpha)}| = \log 2 |\sin \pi(k\beta - \alpha)|,$$

and denote

$$\sigma_N = \frac{1}{N} \sum_{k=1}^{N} \log 2 |\sin \pi(k\beta - \alpha)|,$$

$$\sigma_N^{(1)} = \frac{1}{N} \sum_{\substack{1 \le k \le N \\ k \ne s_j}} \log 2 |\sin \pi(k\beta - \alpha)|,$$

$$\sigma_N^{(2)} = \frac{1}{N} \sum_{s_j \le N} \log 2 |\sin \pi(s_j \beta - \alpha)|,$$

then $\sigma_N = \sigma_N^{(1)} + \sigma_N^{(2)}$ (recall that $s_n = \sum_0^n q_k$). Let first $k \ne s_j$, say $s_n < k < s_{n+1}$. Then $k = s_n + m$, where $1 \le m \le q_{n+1} - 1$, and by (15)

$$|\sin \pi(k\beta - \alpha)| = |\sin \pi(m\beta - \sum_{j=n+1}^{\infty} (q_j \beta - p_j))| = \left| \sin \pi \left(m\beta \pm \frac{\vartheta_{n+1}}{q_{n+2}} \right) \right|.$$

Since $m < q_{n+1}$, (4) implies $\|m\beta\| \ge |q_n \beta - p_n| > (1/2q_{n+1})$, thus applying Lemma 3 we obtain

$$\log 2 |\sin \pi(k\beta - \alpha)| = \log 2 |\sin \pi m\beta| + \log(1 + \mu_{n+1,m}), \tag{17}$$

where

$$|\mu_{n+1,m}| \le \frac{4q_{n+1}}{q_{n+2}},$$

independent of m. Suppose $s_\nu \le N < s_{\nu+1}$. Writing (17) in $\sigma_N^{(1)}$ we obtain

$$N\sigma_N^{(1)} = \sum_{n=1}^{\nu} \sum_{m=1}^{q_n - 1} \log 2 |\sin \pi m\beta| + \sum_{m=1}^{N - s_\nu} \log 2 |\sin \pi m\beta| + L_N,$$

where

$$L_N = \sum_{n=1}^{\nu} \sum_{m=1}^{q_n - 1} \log(1 + \mu_{n,m}) + \sum_{m=1}^{N - s_\nu} \log(1 + \mu_{\nu+1,m}).$$

Since $a_n \to \infty$ implies $\log(1 + \mu_{n,m}) \to 0$, we obtain $L_N = o(N)$ for $N \to \infty$. Put briefly

$$S_n = \frac{1}{n} \sum_{j=1}^{n} \log 2 |\sin \pi j \beta|,$$

then

$$(18) \qquad \sigma_N^{(1)} = \frac{s_\nu - \nu}{N} \cdot \frac{1}{s_\nu - \nu} \sum_{n=1}^{\nu} (q_n - 1) S_{q_n - 1} + \frac{N - s_\nu}{N} \cdot S_{N - s_\nu} + o(1).$$

By Lemma 4 we have $S_{q_n - 1} \to 0$. Thus in the first part of (18), we form averages of the terms of a null sequence, therefore that part tends to 0 with $\nu \to \infty$. It is easy and well known (see formula (4.4) in [HL]), that

$$\limsup_{n \to \infty} S_n \leq 0$$

holds for any irrational β. This implies immediately, that

$$\limsup_{N \to \infty} \frac{N - s_\nu}{N} S_{N - s_\nu} \leq 0,$$

and hence $\limsup \sigma_N^{(1)} \leq 0$. Taking $N = s_{\nu+1} - 1$ we obtain by Lemma 4

$$(19) \qquad \limsup_{N \to \infty} \sigma_N^{(1)} = \lim_{\nu \to \infty} \sigma_{s_\nu - 1}^{(1)} = 0.$$

It remains to study the behaviour of $\sigma_N^{(2)}$. Applying (15) again

$$\begin{aligned} \log 2 |\sin \pi (s_n \beta - \alpha)| &= \log 2 \left| \sin \pi \frac{\vartheta_{n+1}}{q_{n+2}} \right| \\ &= \log \frac{2\pi}{q_{n+2}} + \log \left(\frac{q_{n+2}}{\pi} \sin \pi \frac{\vartheta_{n+1}}{q_{n+2}} \right) \\ &= \log 2\pi - \log q_{n+2} + \lambda_n, \end{aligned}$$

where $\lambda_n \to 0$. Hence assuming $s_\nu \leq N < s_{\nu+1}$ we obtain

$$\sigma_N^{(2)} = \frac{\nu}{N} \log 2\pi - \frac{1}{N} \log \prod_{j=1}^{\nu} q_{j+2} + \frac{1}{N} \sum_{j=1}^{\nu} \lambda_j,$$

or,

$$\sigma_N^{(2)} = -\frac{1}{N} \log \prod_{j-1}^{\nu} q_{j+2} + o(1),$$

where we have made use of the obvious relations

$$\frac{\nu}{N} \leq \frac{\nu}{s_\nu} \to 0 \text{ and } \lambda_n \to 0.$$

The estimate

$$\frac{1}{N} \log \prod_{j=1}^{\nu} q_{j+2} \geq \frac{1}{s_{\nu+1} - 1} \log \prod_{j=1}^{\nu} q_{j+2}$$

is trivial, and hence by (7), (8) and Lemma 1 we get

$$(20) \qquad \limsup_{N \to \infty} \frac{1}{N} \log \prod_{j=1}^{\nu} \frac{1}{q_{j+2}} = \lim_{\nu \to \infty} \frac{1}{s_{\nu+1} - 1} \log \prod_{j=1}^{\nu} \frac{1}{q_{j+2}} = \lim_{\nu \to \infty} \frac{-\log q_{\nu+2}}{q_{\nu+1}} = -c.$$

Since $\limsup_{N\to\infty}\sigma_N \le \limsup_{N\to\infty}\sigma_N^{(1)} + \limsup_{N\to\infty}\sigma_N^{(2)}$, and both limsups on the right hand side are attained on the same sequence $N = s_\nu - 1$, we get by (19) and (20)

$$\limsup_{N\to\infty}\sigma_N = \lim_{\nu\to\infty}\sigma_{s_\nu - 1} = -c,$$

and the proof is complete, apart from the irrationality of α, which is irrelevant to the fact that the radius can be prescribed.

The question of the rationality of α will be discussed in the next two sections.

4. It is not true that $f(x)$ maps any irrational number x onto an irrational value. Moreover, it can be shown that any number (in particular, any rational number) in $(0, 1)$ is taken by f in an irrational point. However, in the theorems below, we prove that $\alpha = f(\beta)$ considered in section 3 is indeed irrational.

THEOREM 3. Let x be a given irrational. Let $q_n \le a < q_{n+1}$ and b be integer numbers, and define

$$\mu = \min\left\{\frac{a}{q_n}, \max\left(1, \frac{q_{n+1}-a}{q_n}\right)\right\}.$$

Then

$$|ax - b| \ge \mu |q_n x - p_n|.$$

PROOF. The weaker estimate $|ax-b| \ge |q_n x - p_n|$ is just the best approximation property (4). A slightly refined version of its standard proof (see for instance [NI]) will suffice for our theorem. Given a and b, the linear system

$$a = uq_n + vq_{n+1} \quad b = up_n + vp_{n+1}$$

has a unique and integer solution for u and v, since the determinant $q_n p_{n+1} - q_{n+1} p_n = \pm 1$. Since $q_n \le a < q_{n+1}$, we have either $v = 0$, $u > 0$, or $uv < 0$. But also

$$(q_n x - p_n)(q_{n+1} x - p_{n+1}) < 0,$$

thus

$$\begin{aligned} |ax - b| &= |u(q_n x - p_n) + v(q_{n+1}x - p_{n+1})| \\ &= |u(q_n x - p_n)| + |v(q_{n+1}x - p_{n+1})| \ge |u| \cdot |q_n x - p_n|. \end{aligned}$$

If $v = 0$, then

$$u = |u| = \frac{a}{q_n} \ge \mu.$$

If $v > 0$, then $u < 0$ and

$$|u| = \frac{vq_{n+1} - a}{q_n} \ge \max\left\{1, \frac{q_{n+1}-a}{q_n}\right\} \ge \mu.$$

If finally $v<0$, then $u>0$ and

$$|u| = u = \frac{|v|q_{n+1}+a}{q_n} \geq \mu,$$

i.e. in all three cases we obtain $|u| \geq \mu$ and hence the statement follows.

THEOREM 4. Suppose $0<x<1$ is irrational. If

(i) $f(x)=P/Q$, or more generally

(ii) $P+Qf(x)+Rx=0 \quad (P,R \in \mathbb{Z},\ Q \in \mathbb{N})$, then

$$a_n(x)<4Q \quad (n = 1,2,\cdots)$$

in the first case, and

$$\limsup a_n(x) \leq 4Q$$

in the second. That is, if $\{a_n(x), n=1,2,\ldots\}$ is unbounded, then $f(x)$ is irrational, moreover $1, x, f(x)$ are rationally independent numbers.

PROOF. Suppose $f(x)=P/Q$. Then for any n we have

$$\left| \sum_{j=0}^{n} (q_j x - p_j) - \frac{P}{Q} \right| = \left| \sum_{j=n+1}^{\infty} (q_j x - p_j) \right| < |q_{n+1}x - p_{n+1}|.$$

This can be written as

$$|s_n Qx - E| < Q|q_{n+1}x - p_{n+1}|, \tag{21}$$

where E is an integer. Assuming (ii), the same reasoning gives

$$|(s_n Q + R)x - E| < Q|q_{n+1}x - p_{n+1}|. \tag{22}$$

If $R \neq 0$, then for a given $\varepsilon>0$ choose n so large, that $\varepsilon q_{n-1} > |R|$, and suppose $a_{n+1} \geq (4+\varepsilon)Q$. If $R=0$, then simply choose any n with $a_{n+1} \geq 4Q$ and put $\varepsilon=0$. In either case, by Lemma 1 (i) we obtain

$$\begin{aligned} q_n < q_n Q < s_n Q - |R| < s_n Q + |R| \\ < (3+\varepsilon) q_n Q < (4+\varepsilon) Q q_n \leq a_{n+1} q_n < q_{n+1}, \end{aligned}$$

thus Theorem 3 applies with $a = s_n Q + R$. By $(s_n Q + R)/q_n > Q$ and $q_{n+1} - s_n Q - R > q_n Q$ we get both from (21) and (22)

$$Q|q_n x - p_n| < |s_n Qx - E| < Q|q_{n+1}x - p_{n+1}|,$$

i.e.

$$|q_n x - p_n| < |q_{n+1}x - p_{n+1}|,$$

a contradiction.

COROLLARY. If β is the irrational number we considered in Theorem 2, then $\alpha = f(\beta)$ is irrational as well.

5. In this section we are going to show that, if $\alpha = P/Q$ is a rational number, and β is an arbitrary irrational, then with the usual notations for A and q the radius of convergence of (1) is 1. This extends Theorem 1.3 (II) of [DLPS], where the statement was validated for $\alpha = 0$. Besides, this section provides a second and indirect proof of the fact, that $f(\beta)$ constructed in the proof of Theorem 2 is irrational.

THEOREM 5. Let $0 < P/Q < 1$, $(P, Q) = 1$ be a given rational, and x an irrational number. Then the radius of convergence of the q-series with $A = \exp(2\pi i(P/Q))$, $q = \exp(2\pi i x)$ is 1.

PROOF. It is enough to show that

$$\limsup \frac{1}{q_n - 1} \sum_{j=1}^{q_n - 1} \log 2 \left| \sin \pi \left(\frac{P}{Q} - jx \right) \right| = 0. \tag{23}$$

This statement generalizes Lemma 4 by allowing P/Q to appear, but it is weaker than Lemma 4 because of the limsup in place of lim. Nevertheless, a similar proof applies. Since q_n and q_{n+1} are coprime numbers for any n (see (2)), there are infinitely many q_n such that Q is not a divisor of q_n. In what follows, this will always be supposed. Then $Pq_n - jQ \neq 0$ for any j, and hence

$$\left| \frac{P}{Q} - \frac{j}{q_n} \right| = \frac{|Pq_n - jQ|}{q_n Q} \geq \frac{1}{q_n Q}. \tag{24}$$

Since

$$\left\{ \frac{P}{Q} - k\frac{p_n}{q_n} : 1 \leq k \leq q_n - 1 \right\} = \left\{ \frac{P}{Q} - \frac{k}{q_n} : 1 \leq k \leq q_n - 1 \right\} \pmod 1, \tag{25}$$

we have by (24)

$$\min_k \left\| \frac{P}{Q} - k\frac{p_n}{q_n} \right\| = \min_k \left\| \frac{P}{Q} - \frac{k}{q_n} \right\| \geq \frac{1}{q_n Q}. \tag{26}$$

Let $1 \leq k \leq q_n - 1$ and write

$$\left| \sin \pi \left(\frac{P}{Q} - kx \right) \right| = \left| \sin \pi \left(\left(\frac{P}{Q} - k\frac{p_n}{q_n} \right) - k\left(x - \frac{p_n}{q_n} \right) \right) \right|,$$

then apply Lemma 3 to obtain

$$\left| \sin \pi \left(\frac{P}{Q} - kx \right) \right| = \left| \sin \pi \left(\frac{P}{Q} - k\frac{p_n}{q_n} \right) \right| \cdot |1 + \mu_k|.$$

Making use of (26) and the same reasoning we applied in the proof of Lemma 4, we get

$$\lim_{n \to \infty} \frac{1}{q_n - 1} \sum_{k=1}^{q_n - 1} \log |1 + \mu_k| = 0. \tag{27}$$

Thus by (27) and (25)

$$(28)\quad \begin{cases} \limsup \dfrac{1}{q_n-1}\sum\limits_{k=1}^{q_n-1} \log 2\left|\sin\pi\left(\dfrac{P}{Q}-kx\right)\right| \\ = \limsup \dfrac{1}{q_n-1}\sum\limits_{k=1}^{q_n-1} \log 2\left|\sin\pi\left(\dfrac{P}{Q}-k\dfrac{p_n}{q_n}\right)\right| \\ = \limsup \dfrac{1}{q_n-1}\sum\limits_{k=1}^{q_n-1} \log 2\left|\sin\pi\left(\dfrac{P}{Q}-\dfrac{k}{q_n}\right)\right|. \end{cases}$$

By (26) there is a j such that

$$\frac{j}{q_n}<\frac{P}{Q}<\frac{j+1}{q_n},$$

and for these indices

$$(29)\quad \left|\log 2\left|\sin\pi\left(\frac{P}{Q}-j\frac{1}{q_n}\right)\right|\right| \le \left|\log 2\sin\pi\frac{1}{q_nQ}\right| \le \log 2\pi q_nQ,$$

and the same estimate holds for the term of index $j+1$. Separating these two terms from the summation we write

$$(30)\quad \begin{cases} \dfrac{1}{q_n-1}\sum\limits_{k=1}^{q_n-1} \log 2\left|\sin\pi\left(\dfrac{P}{Q}-\dfrac{k}{q_n}\right)\right| \\ = \dfrac{1}{q_n-1}\sum\limits_{k\neq j,j+1} \log 2\left|\sin\pi\left(\dfrac{P}{Q}-\dfrac{k}{q_n}\right)\right| + O\left(\dfrac{\log q_n}{q_n}\right). \end{cases}$$

But

$$\lim_{n\to\infty}\frac{1}{q_n-1}\sum_{k\neq j,j+1} \log 2\left|\sin\pi\left(\frac{P}{Q}-\frac{k}{q_n}\right)\right| = \int_0^1 \log 2\left|\sin\pi\left(\frac{P}{Q}-t\right)\right| dt$$

$$= \int_0^1 \log 2\left|\sin\pi t\right| dt = 0,$$

is immediate by the piecewise monotone property of the integrand, thus by (30) the proof is complete.

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