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# Localization of eigenfunctions at zero for certain almost periodic operators

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*Abstract*. For every irrational number $\alpha$ satisfying the property $\lim_{n\to\infty} |\sin \pi\alpha n|^{-1/n} = 1$ and for every number $\beta > 1$, it is shown that the difference equation

$$\xi_{n+1} + \xi_{n-1} + 2\beta\cos(2\pi\alpha n + \theta)\xi_n = 0, \quad n \in \mathbb{Z}$$

has a non-trivial solution $\{\xi_n\}$ satisfying $\overline{\lim}_{|n|\to\infty} |\xi_n|^{1/|n|} \leq |\beta|^{-1}$ if and only if $\theta = 2\pi\alpha n + 2\pi k \pm \pi/2$ for some $n, k \in \mathbb{Z}$.

## 1. *Introduction*

The problem to prove the existence of (exponentially) localized eigenfunctions for almost periodic operators is a very difficult one, which has been solved to date in certain special cases only (see [**S, CS, FSW**] for some of the more recent contributions). Most of the known proofs establishing the existence of localized eigenfunctions are probabilistic in nature, exploiting the ergodicity of an underlying dynamical system. It follows, for instance, from the theorems proven in [**S**] and [**FSW**] for a particular class of one-dimensional almost periodic difference operators, known as almost Mathieu operators, that for almost all numbers in the spectrum of these operators (with respect to the integrated density of states) there is a localized eigenfunction. It would be desirable, beyond these generic results, to identify specific points in these spectra, and exhibit localized eigenfunctions associated with them by an explicit construction. This is the objective of the present paper. We will show that for rotation numbers satisfying a diophantine condition the almost Mathieu operators have an exponentially localized eigenfunction at zero, whenever the (averaged) Lyapunov index is strictly positive. Our approach utilizes power series expansions and is inspired by a paper of Hardy and Littlewood [**HL**]. We will also discuss the question for which values of the underlying phase shift may a localized eigenfunction exist. The arguments involved in answering this question are $C^*$-algebraic in nature and are valid for any point $x$ in the spectrum which is known to have at least one eigenfunction.

2.

For any irrational number $\alpha$ the rotation $C^*$-algebra $\mathcal{A}$ associated with $\alpha$ is (up to isomorphisms) determined by two unitary generators $u$ and $v$ satisfying the relation $uv = \lambda vu$ where $\lambda = e^{2\pi\alpha i}$. Throughout this paragraph we will assume that $\alpha$ satisfies the diophantine condition $\lim_{n\to\infty} |\sin \pi\alpha n|^{-1/n} = 1$.

THEOREM 2.1. *For every complex number $\beta$, $|\beta| > 1$, there exist analytic functions $f$ and $g$ on the open annulus $\{z \in \mathbb{C} \mid |\beta|^{-1} < |z| < |\beta|\}$ such that*

$$e^{f(uv^*)}(u + u^* + \beta(v + v^*))e^{-f(uv^*)} = e^{g(uv^*)}(v + v^*).$$

*Proof.* Letting $h = u + u^* + \beta(v + v^*)$, we have

$$h = \beta(\beta^{-1}uv^* + \mathbf{1})v + \beta v^*(\beta^{-1}vu^* + \mathbf{1}).$$

But

$$\beta^{-1}z + 1 = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{z}{\beta}\right)^n\right), \quad \text{for } |z| < |\beta|.$$

So, defining

$$f_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1 - \lambda^{-n})^{-1}\left(\frac{z}{\beta}\right)^n,$$

$$\widetilde{f}_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1 - \lambda^{n})^{-1}\left(\frac{z}{\beta}\right)^n,$$

and observing that $uv = \lambda vu$, we obtain

$$\beta^{-1}h = e^{f_0(uv^*)}ve^{-f_0(uv^*)} + e^{-\widetilde{f}_0(vu^*)}v^*e^{\widetilde{f}_0(vu^*)}.$$

Note that the diophantine condition imposed on $\alpha$ insures that $f_0$ and $\widetilde{f}_0$ are analytic in $\{z \in \mathbb{C} \mid |z| < |\beta|\}$. Now, letting

$$f_1(z) = f_0(z) + \widetilde{f}_0(z^{-1}) = \sum_{|n|=1}^{\infty} a_n z^n, \quad (|\beta|^{-1} < |z| < |\beta|),$$

we have

$$\begin{aligned}\beta^{-1}e^{-f_0(uv^*)}he^{f_0(uv^*)} &= v + e^{-f_1(uv^*)}v^*e^{f_1(uv^*)}\\ &= v + e^{f_2(uv^*)}v^*\\ &= (\mathbf{1} + e^{f_2(uv^*)}v^{-2})v,\end{aligned}$$

where

$$f_2(z) = \sum_{|n|=1}^{\infty}(\lambda^n - 1)a_n z^n = \sum_{|n|=1}^{\infty} b_n z^n.$$

Next, letting

$$f_3(z) = \sum_{|n|=1}^{\infty}(1 - \lambda^{2n})^{-1}b_n z^n,$$

we have

$$\begin{aligned}\beta^{-1}e^{-f_0(uv^*)}he^{f_0(uv^*)} &= (\mathbf{1}+e^{f_3(uv^*)}v^{-2}e^{-f_3(uv^*)})v\\ &= e^{f_3(uv^*)}(\mathbf{1}+v^{-2})e^{-f_3(uv^*)}v\\ &= e^{f_3(uv^*)}(v+v^*)e^{-f_3(\lambda uv^*)}.\end{aligned}$$

Note that $\lim_{n\to\infty}|\sin\pi\alpha n|^{-1/n}=1$ implies $\lim_{n\to\infty}|\sin 2\pi\alpha n|^{-1/n}=1$, so that $f_3$ is analytic in $\{z\in\mathbb{C}\mid |\beta|^{-1}<|z|<|\beta|\}$. Finally, letting $f(z)=-f_3(\lambda z)-f_0(z)$ and $g(z)=f_3(z)-f_3(\lambda z)+c$, where $e^c=\beta$,

$$e^{f(uv^*)}he^{-f(uv^*)}=e^{g(uv^*)}(v+v^*),$$

as claimed. □

*Remarks.*

(1) The similarity transformation in Theorem 2.1 is analytic in $\beta$.

(2) The operators $f_3(uv^*)$ and $g(uv^*)$ are self-adjoint whenever $\beta$ is real.

(3) Explicit Laurent expansions can be obtained for the functions involved by tracking their construction. This has been done by Hardy and Littlewood in [**HL**] for functions of the type $e^{f_0}$ and $e^{\widetilde{f}_0}$.

(4) It is clear from the proof that it is possible to trade off a narrower annulus for a moderately relaxed diophantine condition for $\alpha$. In other words, the similarity transformation in Theorem 2.1 with analytic functions $f$ and $g$ is still possible for sufficiently large $\beta$ as long as $\alpha$ satisfies the condition $\lim_{n\to\infty}|\sin\pi\alpha n|^{-1/n}<\infty$.

COROLLARY 2.2. *For every complex number $\beta$, $|\beta|>1$, the difference equation $\xi_{n+1}+\xi_{n-1}+2\beta\cos(2\pi\alpha n+\theta)\xi_n=0$ has a non-trivial solution satisfying $\overline{\lim}_{|n|\to\infty}|\xi_n|^{1/|n|}\le|\beta|^{-1}$ whenever $\theta=2\pi\alpha m\pm\pi/2$ for some $m\in\mathbb{Z}$.*

*Proof.* For a given $\theta=2\pi\alpha m\pm\pi/2$ we consider the representation $\rho_\theta$ of the $C^*$-algebra $\mathcal{A}$ on the Hilbert space $\ell^2(\mathbb{Z})$ which is determined by the assignments

$$\rho_\theta(u)\xi_n=\xi_{n+1},\quad (\rho_\theta(v)\xi)_n=e^{\theta i}\lambda^{-n}\xi_n.$$

Since $\rho_\theta(v+v^*)\delta_m=0$, where

$$\delta_m(n)=\begin{cases}1 & \text{for } n=m,\\ 0 & \text{for } n\neq m,\end{cases}$$

Theorem 2.1 yields that $\rho_\theta(e^{-f(uv^*)})\delta_m$ is the desired solution. Expanding the function $e^{-f}$ into a Laurent series shows that this solution decays exponentially for $|n|\to\infty$ as claimed. □

*Remarks.*

(1) Employing the formalism developed in [**R**], one can show that the inequality sign in Corollary 2.2 can be replaced by an equality sign.

(2) It is well known that without any diophantine condition for $\alpha$ there may be no localized eigenfunction at zero at all for the almost Mathieu operator $h$, even when the Lyapunov index is strictly positive. However, it can be shown (for real $\beta$) that, regardless of the diophantine nature of $\alpha$, the (averaged) Lyapunov index at zero always equals $\max\{0,\log|\beta|\}$.

3.

We now turn to the second question raised in the introduction. Henceforth, $\alpha$ is an arbitrary irrational number and $\beta$ is real.

LEMMA 3.1. *For any two real numbers $\theta_1$ and $\theta_2$ the corresponding representations $\rho_{\theta 1}$ and $\rho_{\theta 2}$ of the $C^*$-algebra $\mathcal{A}$ are equivalent if and only if $\theta_2 - \theta_1 \in 2\pi\alpha\mathbb{Z} + 2\pi\mathbb{Z}$.*

*Proof.* If $\theta_2 - \theta_1 = 2\pi\alpha n + 2\pi k$,

$$\rho_{\theta 2}(a) = \rho_{\theta 1}(u)^{-n}\rho_{\theta 1}(a)\rho_{\theta 1}(u)^n.$$

Thus $\rho_{\theta 1}$ and $\rho_{\theta 2}$ are equivalent. Now suppose that $\rho_{\theta 1}$ and $\rho_{\theta 2}$ are equivalent. Then there exists a unitary operator $w$ on $\ell^2(\mathbb{Z})$ such that

$$\rho_{\theta 2}(a) = w^*\rho_{\theta 1}(a)w \quad \text{for every } \ a \in \mathcal{A}.$$

Substituting $v$ for $a$ and letting

$$\delta_n(k) = \begin{cases} 1 & \text{for } k = n, \\ 0 & \text{for } k \neq n, \end{cases}$$

we get

$$e^{i\theta_2}\lambda^{-n}\delta_n = w^*\rho_{\theta 1}(v)w\delta_n,$$

or

$$\rho_{\theta 1}(v)(w\delta_n) = e^{i\theta_2}\lambda^{-n}(w\delta_n).$$

In other words, $w\delta_n$ is an eigenvector and $e^{i\theta_2}\lambda^{-n}$ is an eigenvalue for $\rho_{\theta 1}(v)$. In particular, $\{e^{i\theta_1}\lambda^n \mid n \in \mathbb{Z}\} = \{e^{i\theta_2}\lambda^n \mid n \in \mathbb{Z}\}$ which is equivalent with $\theta_2 - \theta_1 \in 2\pi\alpha\mathbb{Z} + 2\pi\mathbb{Z}$. □

THEOREM 3.2. *If $\xi^{(k)} \in \ell^2(\mathbb{Z})$ is an eigenfunction of $\rho_{\theta k}(h)$ for the eigenvalue $\chi$ ($k = 1, 2$), then $\theta_2 - \theta_1 \in 2\pi\alpha\mathbb{Z} + 2\pi\mathbb{Z}$ or $\theta_1 + \theta_2 \in 2\pi\alpha\mathbb{Z} + 2\pi\mathbb{Z}$.*

*Proof.* We recall from [**R**] that an eigenstate of $h$ for $\chi$ is a state $\psi$ on the $C^*$-algebra $\mathcal{A}$ satisfying

$$\psi(ah) = \psi(ha) = \chi\psi(a) \quad \text{for all } a \in \mathcal{A}.$$

So the functionals

$$\varphi_k(a) = \langle \rho_{\theta k}(a)\xi^{(k)}, \xi^{(k)}\rangle \quad a \in \mathcal{A}, \quad k \in \{1, 2\},$$

are pure eigenstates for $\chi$. It was shown in [**R**] that there are at most two pure eigenstates for $\chi$. If $\varphi_1 = \varphi_2$, then the representations $\rho_{\theta 1}$ and $\rho_{\theta 2}$ must be equivalent [**P**, §3.3.7], and Lemma 3.1 yields $\theta_2 - \theta_1 \in 2\pi\alpha\mathbb{Z} + 2\pi\mathbb{Z}$. If $\varphi_1 \neq \varphi_2$, then $\rho_{\theta 1}$ and $\rho_{\theta 2}$ are not equivalent and $\varphi_2 = \varphi_1 \circ \sigma$, where $\sigma$ is the automorphism of $\mathcal{A}$ determined by the assignments $\sigma(u) = u^*$ and $\sigma(v) = v^*$. This implies that $\theta_1 + \theta_2 \in 2\pi\alpha\mathbb{Z} + 2\pi\mathbb{Z}$. □

*Note added in proof.* This paper was submitted for publication for the first time in 1995.

## REFERENCES

**[CS]** V. A. Chulaevsky and Ya. G. Sinai. Anderson localization for the 1-D discrete Schrödinger operator with two-frequency potential. *Commun. Math. Phys.* **125** (1989), 91–112.

**[FSW]** J. Fröhlich, T. Spencer and P. Wittwer. Localization for a class of one-dimensional quasi-periodic Schrödinger operators. *Commun. Math. Phys.* **132** (1990), 5–25.

**[HL]** G. H. Hardy and J. E. Littlewood. Notes on the theory of series (XXIV): a curious power-series. *Proc. Cambridge Phil. Soc.* **42** (1946), 85–90.

**[P]** G. K. Pedersen. *$C^*$-algebras and their Automorphism Groups.* Academic, New York, 1979.

**[R]** N. Riedel. Almost Mathieu operators and rotation $C^*$-algebras. *Proc. London Math. Soc.* **56**(3) (1988), 281–302.

**[S]** Ya. G. Sinai. Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential. *J. Stat. Phys.* **46** (1987), 861–909.