Extrema of stable processes and number theory

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June 18, 2018

1. Introduction

2. Extrema of stable processes

Stable processes

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  where $\Psi(z)$ is the *characteristic exponent.*
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$$\Psi(z) = \frac{\sigma^2}{2}z^2 - az - \int_{\mathbb{R}} (e^{\mathrm{i}zx} - 1 - \mathrm{i}z\mathbf{1}_{\{|x|<1\}})\Pi(\mathrm{d}x)$$

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- For stable processes, the Lévy measure $\Pi(\mathrm{d}x)$ has density given by

$$\pi(x) = c_1 x^{-1-\alpha}\mathbf{1}_{\{x>0\}} + c_2|x|^{-1-\alpha}\mathbf{1}_{\{x<0\}}.$$

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- After choosing c in the right way, the characteristic exponent becomes

$$\Psi(z) = |z|^\alpha \left(e^{\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z>0\}} + e^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z<0\}}\right).$$

Distribution of the supremum

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D. A. Darling.
The maximum of sums of stable random variables.
Transactions of the American Mathematical Society, 83(1):pp. 164–169 (1956)

C. C. Heyde.
On the maximum of sums of random variables and the supremum functional for stable processes.
Journal of Applied Probability, 6(2):pp. 419–429 (1969)

N. H. Bingham.
Maxima of sums of random variables and suprema of stable processes.
Probability Theory and Related Fields, 26:273–296 (1973)

Distribution of the supremum

Let us define $\overline{X}_t = \sup\{X_s \ : \ 0 \le s \le t\}$.

- The density of $\overline{W}_1$ is

$$p(x) = \sqrt{2/\pi} e^{-x^2/2} = \sqrt{2/\pi} \sum_{n \ge 0} \frac{(-1)^n 2^{-n}}{\Gamma(n+1)} x^{2n}$$

- For stable processes we have the formula for $\overline{X}_1$ in form

$$p(x) = \sum_{m \ge 0} \sum_{n \ge 0} \frac{b_m \times c_n}{\Gamma(...)\Gamma(...)} x^{c_0 + c_1 m + c_2 n}$$

F. Hubalek and A. Kuznetsov (2011)
"A convergent series representation for the density of the supremum of a stable process."
Elect. Comm. in Probab., 16, 84-95.

Density of S_1: $\alpha = 1.1944446...$

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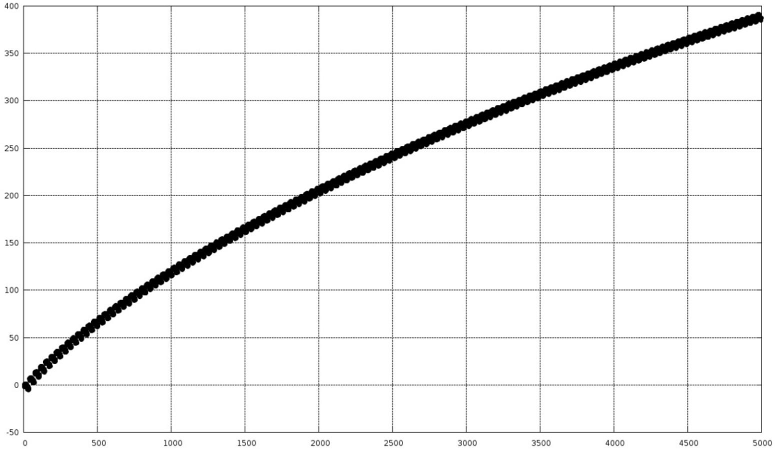
Logarithms of coefficients b_m



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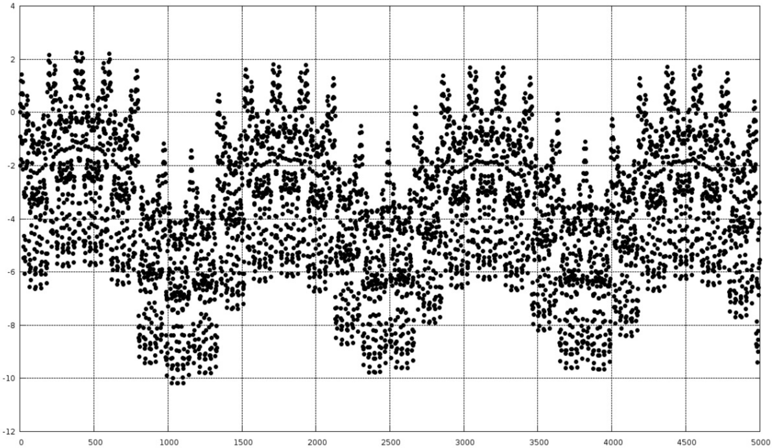

Wilson?

The Great Question of Stable Processes

Why do minor changes in α lead to drastic modifications in the qualitative behavior of the parameters which define the density of the supremum?

The Great Question of Life, the Universe and Everything

The hitchhiker's guide to the Galaxy

Both of the men had been trained for this moment, their lives had been a preparation for it, they had been selected at birth as those who would witness the answer, but even so they found themselves gasping and squirming like excited children.

"And you're ready to give it to us?" urged Loonquawl.

"I am."

"Now?"

"Now," said Deep Thought.

They both licked their dry lips.

"Though I don't think," added Deep Thought, "that you're going to like it."

"Doesn't matter!" said Phouchg. "We must know it! Now!"

"Now?" inquired Deep Thought.

"Yes! Now ..."

"Alright," said the computer and settled into silence again. The two men fidgeted. The tension was unbearable.

"You're really not going to like it," observed Deep Thought.

"Tell us!"

"Alright," said Deep Thought. "The Answer to the Great Question ..."

"Yes ...!"

"Of Life, the Universe and Everything ..." said Deep Thought.

"Yes ...!"

"Is ..." said Deep Thought, and paused.

"Yes ...!"

"Is ..."

"Yes ...!!!...?"

"Forty-two," said Deep Thought, with infinite majesty and calm.

The Answer to the Great Question of Stable Processes

The Answer to the Great Question of Stable Processes

The answer is 4242.

Outline

pssMp

Positive self-similar Markov processes: (pssMp)

$$\{kX_{k^{-\alpha}t}, \quad t>0,\ X_0=x\} \overset{d}{=} \{X_t, \quad t>0,\ X_0=kx\}$$

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Examples:

(i) Bessel processes ($\alpha = 2$).

(ii) Stable process killed on the first exit from $(0, \infty)$ ($\alpha \in (0, 2)$). Stable process conditioned to stay positive, or stable process conditioned to hit zero continuously ($\alpha \in (0, 2)$).

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Can we describe all PSSMPs?

Lamperti's transformation

Let ξ be a Lévy process. Define

$$I(t) = \int_0^t \exp(\alpha \xi_s) \mathrm{d}s,$$

and $\phi(t) = I^{-1}(t)$ for $0 \le t < I(\infty)$.

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$$X_t = x \exp(\xi_{\phi(tx^{-\alpha})})$$

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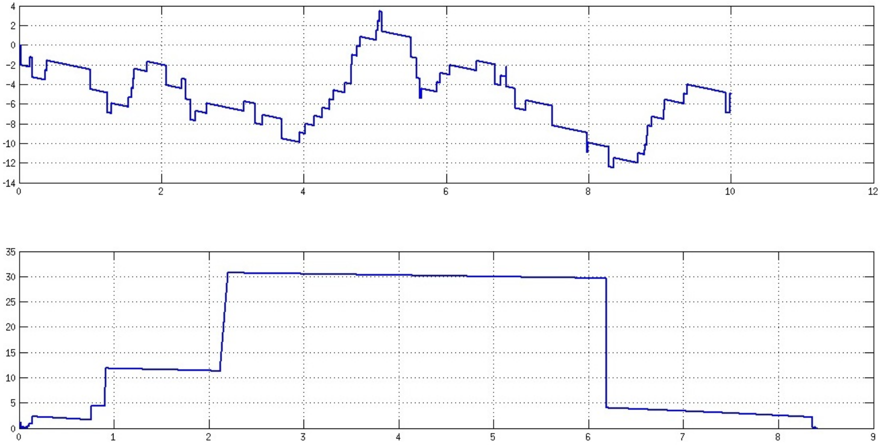
$$\text{Any pssMp } X \longleftrightarrow \text{Lévy process } \xi$$

Example



Let X be a stable process with stability parameter α and $\rho := \mathbb{P}(X_1 > 0)$.

Proposition

We have

$$\left(\overline{X}_1\right)^{-\alpha} \stackrel{d}{=} \int_0^{\zeta} e^{\xi_s} \mathrm{d}s,$$

where ξ is a killed Lévy process with the Laplace exponent

$$\psi(z) = \ln \mathbb{E}[e^{z\xi_1}] = -\frac{\Gamma(1+\alpha z)\Gamma(\alpha - \alpha z)}{\Gamma(1-\alpha\rho+\alpha z)\Gamma(\alpha\rho - \alpha z)}.$$

A. Kuznetsov and J.C. Pardo (2013)
"Fluctuations of stable processes and exponential functionals of hypergeometric Levy processes"
Acta Applicandae Mathematicae, 123(1):113-139.

Definitions and notations

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- *The exponential functional* is defined as

$$I_q := \int_0^{\mathrm{e}(q)} e^{X_s} \mathrm{d}s;$$

- We define the Mellin transform of I_q as

$$\mathcal{M}(s) := \mathbb{E}[I_q^{s-1}].$$

Main result

Theorem

Assume that Cramér's condition is satisfied: there exists $\theta > 0$ such that $\psi(\theta) = q$. Then

$$\mathcal{M}(s+1) = \frac{s}{q - \psi(s)} \mathcal{M}(s)$$

for all s in the strip $0 < \mathrm{Re}(s) < \theta$.

K. Maulik and B. Zwart (2006)
" Tail asymptotics for exponential functionals of Lévy processes."
Stoch. Proc. Appl., 116(2):156-177.

V. Rivero (2007)
"Recurrent extensions of self-similar Markov processes and Cramér's condition."
Bernoulli, 13(4):1053-1070.

To find the Mellin transform of $\overline{X}_1$ we need to find $f(s)$ such that

$$f(s+1) = f(s) \times s \times \frac{\Gamma(1-\alpha\rho+\alpha z)\Gamma(\alpha\rho-\alpha z)}{\Gamma(1+\alpha z)\Gamma(\alpha-\alpha z)}.$$

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How do we solve this type of equations?

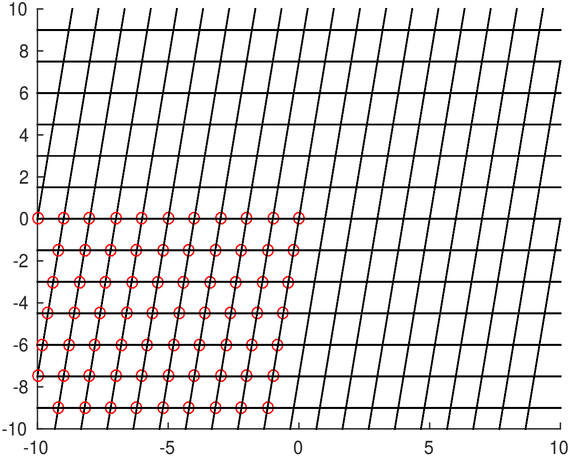
Barnes double gamma function

- Double gamma function satisfies $G(1;\tau)=1$ and

$$\begin{aligned} G(z+1;\tau) &= \Gamma\left(\frac{z}{\tau}\right)G(z;\tau), \\ G(z+\tau;\tau) &= (2\pi)^{\frac{\tau-1}{2}}\tau^{-z+\frac{1}{2}}\Gamma(z)G(z;\tau). \end{aligned}$$

- $G(z;\tau)$ is an entire function in z and has simple zeros on the lattice $m\tau+n$, $m\le 0$, $n\le 0$.
- Many other properties: infinite product representations, transformations of $G(z;1/\tau)$, etc.

Barnes double gamma function: zeros



Mellin transform of $\overline{X}_1$

Define $\mathcal{M}(s) := \mathbb{E}[(\overline{X}_1)^{s-1}]$.

Theorem

For $s \in \mathbb{C}$

$$\mathcal{M}(s) = \alpha^{s-1} \frac{G(\alpha\rho;\alpha)}{G(\alpha(1-\rho)+1;\alpha)} \times \frac{G(\alpha(1-\rho)+2-s;\alpha)}{G(\alpha\rho-1+s;\alpha)} \times \frac{G(\alpha-1+s;\alpha)}{G(\alpha+1-s;\alpha)}$$

Dealing with uniqueness

Proposition

Assume Cramér's condition and

(i) $f(1) = 1$ *and* $f(s+1) = sf(s)/(q - \psi(s))$ *for all* $s \in (0, \theta)$,

(ii) $f(s)$ *is analytic and zero-free in the strip* $\mathrm{Re}(s) \in (0, 1 + \theta)$,

(iii) $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ *as* $\mathrm{Im}(s) \to \infty$, $\mathrm{Re}(s) \in (0, 1 + \theta)$,

then $\mathbb{E}[I_q^{s-1}] \equiv f(s)$ *for* $\mathrm{Re}(s) \in (0, 1 + \theta)$.

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Density of S_1: asymptotics

Assume that $\alpha \notin \mathbb{Q}$. Define sequences $\{a_{m,n}\}_{m\geq 0, n\geq 0}$ and $\{b_{m,n}\}_{m\geq 0, n\geq 1}$ as

$$a_{m,n} = \frac{(-1)^{m+n}}{\Gamma\left(1-\rho-n-\frac{m}{\alpha}\right)\Gamma(\alpha\rho+m+\alpha n)} \times \prod_{j=1}^{m} \frac{\sin\left(\frac{\pi}{\alpha}(\alpha\rho+j-1)\right)}{\sin\left(\frac{\pi j}{\alpha}\right)} \times \prod_{j=1}^{n} \frac{\sin(\pi\alpha(\rho+j-1))}{\sin(\pi\alpha j)},$$

$$b_{m,n} = \frac{\Gamma\left(1-\rho-n-\frac{m}{\alpha}\right)\Gamma(\alpha\rho+m+\alpha n)}{\Gamma\left(1+n+\frac{m}{\alpha}\right)\Gamma(-m-\alpha n)} a_{m,n}.$$

Density of S_1: asymptotics

Theorem

Assume that $\alpha \notin \mathbb{Q}$. Then we have the following asymptotic expansions:

$$\begin{aligned} p(x) &\sim x^{\alpha\rho-1} \sum_{n\geq 0} \sum_{m\geq 0} a_{m,n} x^{m+\alpha n}, \quad x \to 0^+, \\ p(x) &\sim x^{-1-\alpha} \sum_{n\geq 0} \sum_{m\geq 0} b_{m,n+1} x^{-m-\alpha n}, \quad x \to +\infty. \end{aligned}$$

A. Kuznetsov
On extrema of stable processes.
The Annals of Probability, 39(3), 1027-1060, (2011)

Hardy and Littlewood (1946)

Theorem

For almost all θ we have

$$\lim_{n\to+\infty}\prod_{k=1}^{n}|\sin(k\pi\theta)|^{1/n}=\frac{1}{2}$$

G.H. Hardy and J.E. Littlewood
Notes on the theory of series (XXIV): a curious power-series.
Proc. Cambridge Phil. Soc., 42, pp. 85–90, (1946)

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Compare this with

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\ln|\sin(k\pi\theta)|=\int_0^1\ln(|\sin(\pi x)|)\mathrm{d}x=-\ln(2)$$

Hardy and Littlewood (1946)

Theorem

If the radius of convergence of

$$\sum_{n\geq 1} \frac{z^n}{\sin(n\theta\pi)}$$

is r (where $r \in (0,1]$), then the radius of convergence of

$$\sum_{n\geq 1} \frac{z^n}{\sin(\theta\pi)\sin(2\theta\pi)\dots\sin(n\theta\pi)}$$

is $r/2$.

The proof is based on the identity

$$\sum_{n\geq 0} \frac{z^n}{(1-q)(1-q^2)\dots(1-q^n)} = \exp\left(\sum_{n\geq 1} \frac{z^n}{n(1-q^n)}\right)$$

Density of S_1: convergent series

Theorem

For almost all α

$$p(x) = \begin{cases} x^{-1-\alpha} \sum_{n\geq 0} \sum_{m\geq 0} b_{m,n+1} x^{-m-\alpha n}, & \text{if } \alpha \in (0,1), \\ x^{\alpha\rho-1} \sum_{n\geq 0} \sum_{m\geq 0} a_{m,n} x^{m+\alpha n}, & \text{if } \alpha \in (1,2). \end{cases}$$

for all $x > 0$.

Diophantine approximations

The main question: how closely can we approximate a given irrational number by rational numbers?

Theorem (Liouville, 1840s)

If x is an irrational algebraic number of degree n over the rational numbers, then

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^n}$$

is satisfied only by finitely many integers p, q.

Corollary (Liouville, 1840s)

The number $\sum_{n\geq 1} 10^{-n!}$ is transcendental!

Diophantine approximations

Definition

Irrationality measure of an irrational number x is the smallest μ such that the inequality

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

is satisfied only by finitely many integers p, q.

Note that $\mu \geq 2$, since

Theorem (Borel (1903))

For any irrational x there exist infinitely many integers p, q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

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Almost all irrational numbers have irrationality measure equal to two.

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Theorem (Roth (1955))

Irrationality measure of any algebraic number is two.

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Theorem (Roth (1955))

Irrationality measure of any algebraic number is two.

- Lindemann (1882): π is transcendental (not algebraic).
- What is the irrationality measure of π?
- The current best result (due to Salikhov, 2008) is that $\mu < 7.6063$.

Continued fractions

- $355/113 = 3 + 16/113$

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$$355/113 = 3 + \cfrac{1}{7 + \cfrac{1}{16}} = [3; 7, 16]$$

Continued fractions: examples

- The golden ratio: $(1+\sqrt{5})/2 = [1; 1, 1, 1, 1, 1, 1, \ldots]$

Continued fractions: examples

- The golden ratio: $(1+\sqrt{5})/2 = [1; 1, 1, 1, 1, 1, 1, \dots]$
- $\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \dots]$

Continued fractions: examples

- The golden ratio: $(1+\sqrt{5})/2 = [1; 1, 1, 1, 1, 1, 1, \ldots]$
- $\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots]$
- $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$

Continued fractions

- $$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\cdots}}}}}$$

Continued fractions

- $$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \dots] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\dots}}}}}$$
- Truncating this infinite continued fraction gives very good approximations to π:

$$\pi \approx \frac{22}{7}, \quad \text{the error is } -0.0013\dots,$$
$$\pi \approx \frac{355}{113}, \quad \text{the error is } -2.66 \times 10^{-7}$$

Continued fractions

- The continued fraction representation of a real number x is defined as

$$x = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

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- p_n/q_n provides the best rational approximation to x in the following sense

$$\left| x - \frac{p_n}{q_n} \right| = \min \left\{ \left| x - \frac{p}{q} \right| \ : \ p \in \mathbb{Z}, 1 \leq q \leq q_n \right\}.$$

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- There is also a converse result: if integers p and q satisfy

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2},$$

then $p = p_n$ and $q = q_n$ for some n.

Defining the set $\mathcal{L}$

Definition

Let $\mathcal{L}$ be the set of all real irrational numbers x, for which there exists a constant $b > 1$ such that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{b^q}$$

is satisfied for infinitely many integers p and q.

Properties of the set $\mathcal{L}$

Proposition

(i) *If* $x \in \mathcal{L}$ *then* $zx \in \mathcal{L}$ *and* $z + x \in \mathcal{L}$ *for all* $z \in \mathbb{Q} \setminus \{0\}$.

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(iii) $x \notin \mathcal{L} \cup \mathbb{Q}$ *if and only if* $|\sin(n\pi x)|^{1/n} \to 1$ *as* $n \to +\infty$.

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(i) *If* $x \in \mathcal{L}$ *then* $zx \in \mathcal{L}$ *and* $z + x \in \mathcal{L}$ *for all* $z \in \mathbb{Q} \setminus \{0\}$.

(ii) $x \in \mathcal{L}$ *if and only if* $x^{-1} \in \mathcal{L}$.

(iii) $x \notin \mathcal{L} \cup \mathbb{Q}$ *if and only if* $|\sin(n\pi x)|^{1/n} \to 1$ *as* $n \to +\infty$.

(iv) $x \notin \mathcal{L} \cup \mathbb{Q}$ *if and only if*

$$\lim_{n\to+\infty} \prod_{k=1}^{n} |\sin(k\pi x)|^{1/n} = \frac{1}{2}$$

(v) $x = [a_0; a_1, a_2, \dots] \in \mathcal{L}$ *if and only if there exists* $b > 1$ *such that* $a_{n+1} > b^{q_n}$ *is satisfied for infinitely many* n.

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(vi) $\mathcal{L}$ *has Lebesgue measure zero and Haursdorff dimention zero.*

Density of S_1: convergent series

Theorem

Assume that $\alpha \notin \mathcal{L} \cup \mathbb{Q}$. *Then for all* $x > 0$

$$p(x) = \begin{cases} x^{-1-\alpha} \sum_{n \geq 0} \sum_{m \geq 0} b_{m,n+1} x^{-m-\alpha n}, & \text{if } \alpha \in (0,1), \\ x^{\alpha\rho-1} \sum_{n \geq 0} \sum_{m \geq 0} a_{m,n} x^{m+\alpha n}, & \text{if } \alpha \in (1,2). \end{cases}$$

F. Hubalek and A. Kuznetsov
A convergent series representation for the density of the supremum of a stable process.
Elect. Comm. in Probab., 16, 84-95, (2011)

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where $q_k = q_k(2/\alpha)$ *is the denominator of the* k*-th convergent for* $2/\alpha$.

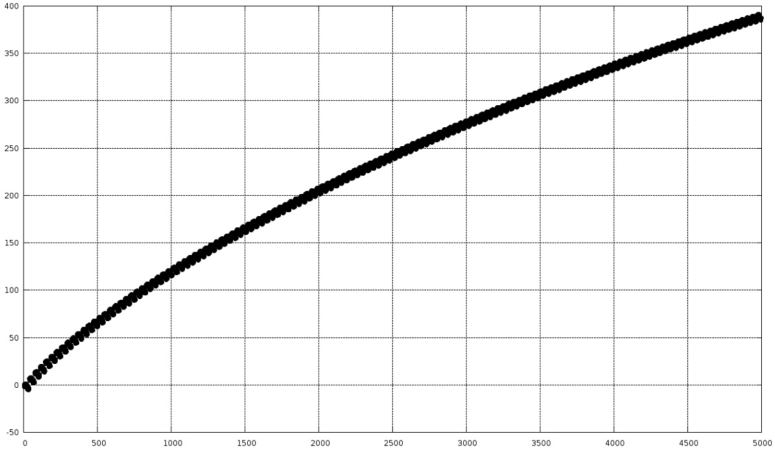
D. Hackmann and A. Kuznetsov (2013)
"A note on the series representation for the density of the supremum of a stable process"
Elect. Comm. in Probab., 18, article 42, 1-5.

The Great Question of Stable Processes

Why do minor changes in α lead to drastic modifications in the qualitative behavior of the parameters which define the density of the supremum?

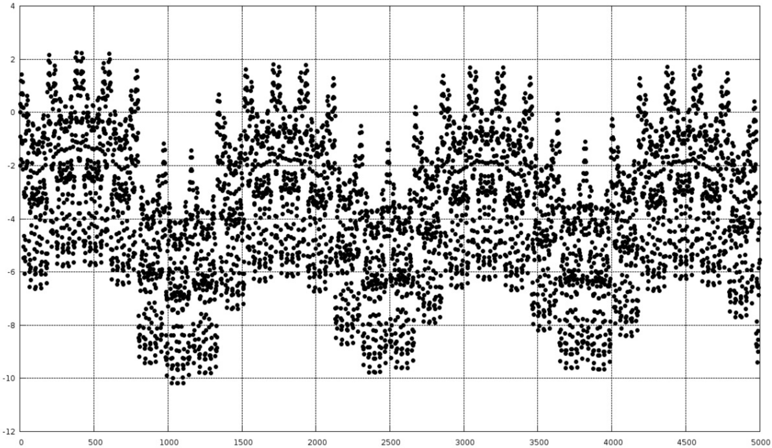
Density of S_1: $\alpha = 1.1944446...$

Logarithms of coefficients b_m



Density of S_1: $\alpha = 1.1945906...$

Logarithms of coefficients b_m



The answer to the Great Question: 4242

we have

$$1.194590640382233... = [1, 5, 7, 5, 7, 5, 7, 5, 7, 5, 7, ...]$$

The answer to the Great Question: 4242

we have

$$1.194590640382233... = [1, 5, 7, 5, 7, 5, 7, 5, 7, 5, 7, ...]$$

while

$$1.194444626329026... = [1, 5, 7, 4242, 7, 5, 7, 5, 7, ...]$$

Thank you!

www.math.yorku.ca/∼akuznets