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Estimates of the Averaged Sums of Fractional Parts

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Abstract—We establish asymptotically sharp estimates for the sums of the inverses (and more general sums) of the fractional parts $\{i\theta\}$ of irrational numbers θ, depending on the arithmetical characteristics of the numbers θ.

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1. INTRODUCTION

Let θ be a real number, $0 < \theta < 1$, and let $\{n\theta\}$ be the fractional parts of the numbers $n\theta$, $n = 1, 2, \dots$. One of the consequences of the Weyl criterion for a numerical sequence to have a uniform distribution (see, for example, [1]) is that, for each Riemann integrable function $f(x)$ on the closed interval $[0, 1]$, the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\{i\theta\}) = \int_0^1 f(x)\,dx \tag{1.1}$$

holds if and only if θ is an irrational number.

Hardy and Littlewood [2] studied the following question: For what values of θ, relation (1.1) holds for some class of unbounded functions f in neighborhoods of the points $x = 0$ and $x = 1$?

In a further study of this question, Oskolkov [3] considered the class H of functions

$$f(x), \quad x \in (0, 1), \qquad f(0+) = f(1-) = +\infty,$$

for which there exists a value $h = h(f) \in (0, 1/2)$ such that $f(x)$ is nonincreasing on $(0, h)$, nondecreasing on $(1 - h, 1)$, positive on these intervals, Riemann integrable on $[-h, h]$, and the integral $\int_0^1 f(x)\,dx$ converges.

The main result of [3] is the statement that, for $f \in H$, relation (1.1) holds if and only if

$$f(\{q_k\theta\}) = o(q_k), \qquad k \to \infty,$$

where q_k is the denominator of the convergent of order k of the continued fraction for the number θ.

Let us no longer require the convergence of the integral in the definition of the class H. The resulting new class of functions will be denoted by A.

It follows from the results obtained below that, for a function $f \in A$, the condition

$$f(\{q_k\theta\}) = o\biggl(q_k \int_{1/q_k}^{1-1/q_k} f(x)\,dx\biggr), \qquad k \to \infty,$$

is sufficient, while if

$$\delta(f(\delta) + f(1-\delta)) = o\biggl(\int_{\delta}^{1-\delta} f(x)\,dx\biggr), \qquad \delta \to +0,$$

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it is also necessary for the validity of the equality

$$\frac{1}{n}\sum_{i=1}^{n} f(\{i\theta\}) = (1+o(1))\int_{1/n}^{1-1/n} f(x)\,dx, \qquad n\to\infty.$$

In particular, if the integral of f on $[0,1]$ converges, then we obtain Oskolkov's result.

In the statements given below, we consider functions f decreasing on $(0,1]$. Obviously, the case of functions increasing on $[0,1)$ can also be reduced to this case. These statements can easily be extended to functions from A, because, in their proofs, only the monotonicity of functions in neighborhoods of the points $x=0$ and $x=1$ is essential.

2. AUXILIARY STATEMENTS

Lemmas 1 and 2 given below are only used to prove Lemma 3, which, in turn, serves to prove Lemma 4. Lemma 3 actually contains the arithmetical part of the problem. Similar statements whose form is not convenient for our purpose were given, for example, in [4, Chap. II, Sec. 3, Lemma 1] as well as in [3].

For the convergents p_k/q_k of a continued fraction $\theta = [0; a_1, a_2, \dots]$, we use the notation from [3]:

$$q_0 = 0, \qquad q_1 = 1, \qquad q_{k+1} = a_k q_k + q_{k-1}, \quad p_{k+1} = a_k p_k + p_{k-1}, \quad k = 1, 2, \dots,$$

which is in contrast, for example, to the notation from the book [5], in which

$$q_{k+1} = a_{k+1} q_k + q_{k-1}, \quad p_{k+1} = a_{k+1} p_k + p_{k-1}, \qquad k = 0, 1, \dots.$$

By $\|\xi\|$ we denote the distance from a real number ξ to the nearest integer.

Lemma 1. *Let s and i_0 be arbitrary integers, let $k \ge 2$, and let*

$$i_k(s) = q_k\left(1 - \left\{\frac{sq_{k-1}}{q_k}\right\}\right), \tag{2.1}$$

$$i_k(s, i_0) = i_0 + i_k(s), \tag{2.2}$$

$$\varepsilon_k(s, i_0) = \|q_k\theta\|\frac{i_k(s,i_0)}{q_k}. \tag{2.3}$$

Then

$$i_k(s,i_0)\theta = \frac{i_0 p_k}{q_k} + (-1)^{k-1}\left(\frac{s}{q_k} + \varepsilon_k(s,i_0)\right) + N_0, \tag{2.4}$$

where N_0 is an integer (N_0 depends on s, k, θ).

Proof. First, choose $i_0 = 0$ so that

$$i_k(s,i_0) = i_k(s), \qquad \varepsilon_k(s,i_0) = \eta_k(s) := \|q_k\theta\|\frac{i_k(s)}{q_k}.$$

Since

$$q_k\theta = p_k + (-1)^{k-1}\|q_k\theta\|, \qquad p_k q_{k-1} - q_k p_{k-1} = (-1)^k$$

(see [5]), it follows that

$$\begin{aligned}
i_k(s)\theta &= \frac{i_k(s)}{q_k}(p_k + (-1)^{k-1}\|q_k\theta\|) = (-1)^{k-1}\eta_k(s) + \left(1 - \left\{\frac{sq_{k-1}}{q_k}\right\}\right)p_k \\
&= (-1)^{k-1}\left(\frac{s}{q_k} + \eta_k(s)\right) + (p_k q_{k-1} - q_k p_{k-1})\frac{s}{q_k} + p_k - \left\{\frac{sq_{k-1}}{q_k}\right\}p_k \\
&= (-1)^{k-1}\left(\frac{s}{q_k} + \eta_k(s)\right) + \left(\frac{sq_{k-1}}{q_k} - \left\{\frac{sq_{k-1}}{q_k}\right\}\right)p_k - p_{k-1}s + p_k \\
&= (-1)^{k-1}\left(\frac{s}{q_k} + \eta_k(s)\right) + N_0.
\end{aligned}$$

Hence we have

$$i_k(s,i_0)\theta=(i_0+i_k(s))\theta=\frac{i_0}{q_k}(p_k+(-1)^{k-1}\|q_k\theta\|)+(-1)^{k-1}\biggl(\frac{s}{q_k}+\eta_k(s)\biggr)+N_0$$
$$=\frac{i_0p_k}{q_k}+(-1)^{k-1}\biggl(\frac{s}{q_k}+\|q_k\theta\|\frac{i_0+i_k(s)}{q_k}\biggr)+N_0,$$

which proves the assertion. □

Lemma 2. *In the notation of Lemma 1, if $i_0\ge 0$ and $i_0+q_k<q_{k+1}$, then*

$$\{i_k(s,i_0)\theta\}=\begin{cases}\left\{\dfrac{s+i_0p_k}{q_k}\right\}+\varepsilon_k(s,i_0) & \text{for } k \text{ odd},\\ 1-\left\{\dfrac{s-i_0p_k}{q_k}\right\}-\varepsilon_k(s,i_0) & \text{for } k \text{ even}.\end{cases}\tag{2.5}$$

Here

$$0<\varepsilon_k(s,i_0)<\frac{1}{q_k}\,.\tag{2.6}$$

Proof. Indeed, $\|q_k\theta\|<1/q_{k+1}$ (see [5]) and $i_k(s,i_0)=i_0+i_k(s)\le i_0+q_k$ (see (2.1)), so that, from the conditions $i_0\ge 0$, $i_0+q_k<q_{k+1}$ and from (2.3), we obtain (2.6). Relations (2.6), (2.4) and the definition of the fractional part of a number imply (2.5). □

Lemma 3. *Let $n\in\mathbb{N}$, let $q_m\le n<q_{m+1}$, $m\ge 2$, and let*

$$n=\sum_{s=1}^{m}b_sq_s,\tag{2.7}$$

be the Euclidean representation of the number n, where the b_s are integers, $0\le b_s\le a_s$, $b_m\ge 1$. Set

$$n_k=\sum_{s=1}^{k}b_sq_s,\qquad k=1,\dots,m,\quad n_m=n,\tag{2.8}$$

$$\delta=\delta_k(\nu)=\begin{cases}0 & \text{if } \left\{\dfrac{\nu q_{k-1}}{q_k}\right\}<1-\dfrac{n_{k-1}}{q_k},\\ 1 & \text{if } \left\{\dfrac{\nu q_{k-1}}{q_k}\right\}\ge 1-\dfrac{n_{k-1}}{q_k},\end{cases}\qquad k=2,\dots,m,\quad \nu=0,1,\dots,q_k-1.\tag{2.9}$$

For k such that $b_k\ne 0$, we also set

$$i_{k,j}(\nu)=q_k\biggl(j+\delta-\left\{\frac{\nu q_{k-1}}{q_k}\right\}\biggr),\qquad j=1,\dots,b_k,\tag{2.10}$$

$$\varepsilon_{k,j}(\nu)=\|q_k\theta\|\frac{i_{k,j}(\nu)}{q_k}=\|q_k\theta\|\biggl(j+\delta-\left\{\frac{\nu q_{k-1}}{q_k}\right\}\biggr).\tag{2.11}$$

Then $0<\varepsilon_{k,j}(\nu)<1/q_k$ and, for $i=i_{k,j}(\nu)$, $\nu=0,1,\dots,q_k-1$,

$$\{i\theta\}=\begin{cases}\dfrac{\nu}{q_k}+\varepsilon_{k,j}(\nu) & \text{for } k \text{ odd},\\ 1-\dfrac{\nu}{q_k}-\varepsilon_{k,j}(\nu) & \text{for } k \text{ even}.\end{cases}\tag{2.12}$$

Here the function $i=i_{k,j}(\nu)$ takes its values only from the sequence $i_0+1,i_0+2,\dots,i_0+q_k$, where

$$i_0:=n_{k-1}+(j-1)q_k,\tag{2.13}$$

whenever ν assumes the values $0,1,\dots,q_k-1$.

Proof. Let $s = 0, 1, \dots, q_k - 1$, and let $\nu = \nu(s)$ be the remainder of the division of $s + (-1)^{k-1} i_0 p_k$ by q_k, so that

$$s \equiv \nu + (-1)^k i_0 p_k \pmod{q_k}, \tag{2.14}$$

$$\left\{\frac{s + (-1)^{k-1} i_0 p_k}{q_k}\right\} = \frac{\nu}{q_k}, \qquad \nu \in [0, q_k - 1]. \tag{2.15}$$

Using (2.14), (2.15), and the equality $p_k q_{k-1} = (-1)^k + q_k p_{k-1}$ (see [5]), we obtain

$$sq_{k-1} \equiv (\nu + (-1)^k i_0 p_k) q_{k-1} \equiv \nu q_{k-1} + (-1)^k n_{k-1} p_k q_{k-1} \equiv \nu q_{k-1} + n_{k-1} \pmod{q_k}. \tag{2.16}$$

Therefore (see (2.1), (2.2), (2.13), (2.16), and (2.9)), for $b_k \neq 0$, we can write

$$\begin{aligned} \frac{i_k(s, i_0)}{q_k} &= \frac{i_0 + i_k(s)}{q_k} = \frac{n_{k-1}}{q_k} + (j-1) + \left(1 - \left\{\frac{sq_{k-1}}{q_k}\right\}\right) \\ &= \frac{n_{k-1}}{q_k} + j - \left\{\frac{\nu q_{k-1} + n_{k-1}}{q_k}\right\} = j + \delta - \left\{\frac{\nu q_{k-1}}{q_k}\right\}. \end{aligned}$$

Hence

$$i_k(s, i_0) = i_{k,j}(\nu), \tag{2.17}$$

$$\varepsilon_k(s, i_0) = \varepsilon_{k,j}(\nu) \tag{2.18}$$

(see, respectively, (2.10) and (2.3), (2.11)). Since

$$i_0 + q_k = n_{k-1} + jq_k \le n_{k-1} + b_k q_k = n_k < q_{k+1},$$

and $i_0 \ge 0$, we see that the assumptions of Lemma 2 hold. Therefore (see (2.16) and (2.18)), we have $0 < \varepsilon_{k,j}(\nu) < 1/q_k$, and, from (2.5), taking into account (2.17), (2.18), and (2.15), we obtain (2.12).

Obviously, $i_k(s, i_0)$ takes its values only from the series $i_0 + 1, \dots, i_0 + q_k$ whenever s assumes the values $0, 1, \dots, q_k - 1$ (see (2.1), (2.2)). At the same time, by (2.14), s is a function of ν ($s = s(\nu)$) taking its values $0, 1, \dots, q_k - 1$ whenever ν ranges over the sequence of values $0, 1, \dots, q_k - 1$. Therefore, the function $i = i_{k,j}(\nu) = i_k(s(\nu), i_0)$ takes its values $i_0 + 1, \dots, i_0 + q_k$ whenever ν assumes the values $0, 1, \dots, q_k - 1$. The lemma is proved. □

Lemma 4. *Suppose that $f(x)$ is a nonnegative nonincreasing function on $(0, 1]$, $\theta = [0; a_1, a_2, \dots]$ is an irrational number on $(0, 1)$, q_k are the denominators of its convergents,*

$$\delta_k = \min\left\{\{q_k\theta\}, \frac{1}{q_k}\right\};$$

n is an arbitrary natural number, and the number m is defined by the condition

$$q_m \le n < q_{m+1}, \qquad m \ge 2;$$

let

$$n = \sum_{k=1}^{m} b_k q_k \tag{2.19}$$

be the Euclidean representation of the number n, and let the b_k be integers, $0 \le b_k \le a_k$, $b_m \ge 1$. In that case, if

$$f(\delta_k) = o\left(q_k \int_{1/q_k}^{1} f(x)\,dx\right), \qquad k \to \infty, \tag{2.20}$$

then

$$\sum_{i=1}^{n} f(\{i\theta\}) = (1 + o(1)) \sum_{k=1}^{m} b_k q_k \int_{1/q_k}^{1} f(x)\,dx. \tag{2.21}$$

Proof. For $b_k \neq 0$, set

$$I_{k,j} = \{n_{k-1} + (j-1)q_k + 1, \dots, n_{k-1} + jq_k\}, \qquad j = 1, \dots, b_k, \quad k = 2, \dots, m.$$

By Lemma 3, for $i \in I_{k,j}$, the numbers $\{i\theta\}$ coincide, up to permutation, with the numbers

$$\frac{\nu}{q_k} + \varepsilon_{k,j}(\nu), \qquad \nu = 0, 1, \dots, q_k - 1,$$

for k odd and with the numbers

$$1 - \frac{\nu}{q_k} - \varepsilon_{k,j}(\nu), \qquad \nu = 0, 1, \dots, q_k - 1,$$

for k even; here $\varepsilon_{k,j}(\nu) \in (0, 1/q_k)$, $\varepsilon_{k,j}(0) = \|q_k\theta\| \cdot j$, $j = 1, \dots, b_k$.

Only one of the numbers $\{i\theta\}$, $i \in I_{k,j}$, satisfies the inequality $\{i\theta\} < 1/q_k$. This holds for $i = i_{k,j}(0) = q_k j$ and k odd; the number in question coincides with

$$\varepsilon_{k,j}(0) = \|q_k\theta\| \cdot j = \{q_k\theta\} \cdot j \geq \delta_k.$$

Combining this result with the fact that the function $f(x)$ is decreasing, for k odd, we obtain

$$\sum_{i \in I_{j,k}} f(\{i\theta\}) \leq f(\delta_k) + \sum_{\nu=1}^{q_k-1} f\left(\frac{\nu}{q_k}\right) \leq f(\delta_k) + f\left(\frac{1}{q_k}\right) + q_k \int_{1/q_k}^{1} f(x)\,dx$$
$$\leq 2f(\delta_k) + q_k \int_{1/q_k}^{1} f(x)\,dx. \tag{2.22}$$

Obviously, this inequality also remains valid for even k's (even when the summand $2f(\delta_k)$ is replaced by $f(\delta_k) = f(1/q_k)$).

Thus, (see (2.19) and (2.22)),

$$\sum_{i=1}^{n} f(\{i\theta\}) = \sum_{k=1}^{m} \sum_{j=1}^{b_k} \sum_{i \in I_{k,j}} f(\{i\theta\}) \leq 2 \sum_{k=1}^{m} b_k f(\delta_k) + \sum_{k=1}^{m} b_k q_k \int_{1/q_k}^{1} f(x)\,dx,$$

where, by condition (2.20),

$$f(\delta_k) = \frac{1}{2} \alpha_k q_k \int_{1/q_k}^{1} f(x)\,dx, \qquad \alpha_k \to 0 \quad \text{for} \quad k \to \infty.$$

Therefore, taking into account the fact that the function f is decreasing, we can write

$$\sum_{i=1}^{n} f(\{i\theta\}) \leq \left(\sum_{k=1}^{m} b_k q_k \cdot \alpha_k\right) \int_{1/q_m}^{1} f(x)\,dx + \sum_{k=1}^{m} b_k q_k \int_{1/q_k}^{1} f(x)\,dx. \tag{2.23}$$

Set

$$\widetilde{n} := \sum_{k=1}^{m-1} a_k q_k \equiv -1 + q_{m-1} + q_m$$

(recall that $a_k q_k = q_{k+1} - q_{k-1}$, $q_1 = 1$). Obviously,

$$\frac{1}{n} \sum_{k=1}^{m} b_k q_k \cdot \alpha_k \leq \frac{\widetilde{n}}{n} \left(\frac{1}{\widetilde{n}} \sum_{k=1}^{m-1} a_k q_k \cdot \alpha_k\right) + \alpha_m, \tag{2.24}$$

(the elements a_k are independent of n). Here the expression in parentheses is the arithmetic mean of the first $\widetilde{n}$ terms of the sequence

$$\underbrace{\alpha_1, \dots, \alpha_1}_{a_1 q_1 \text{times}}, \dots, \underbrace{\alpha_2, \dots, \alpha_2}_{a_2 q_2 \text{times}}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{a_k q_k \text{times}}, \dots,$$

tending to zero, and, therefore, it also tends to zero.

Since here $\widetilde{n} < 2n$, it follows that the left-hand side of inequality (2.24) is infinitely small as $n \to \infty$. Therefore, from (2.23), we obtain the inequality

$$\frac{1}{n}\sum_{i=1}^{n} f(\{i\theta\}) \le \frac{1}{n}\sum_{k=1}^{m} b_k q_k \int_{1/q_k}^{1} f(x)\,dx + o\biggl(\int_{1/q_m}^{1} f(x)\,dx\biggr), \qquad n \to \infty. \tag{2.25}$$

On the other hand, for any k (even and, especially, odd), independently of condition (2.20), using the fact that the function f is decreasing, we can write

$$\sum_{i\in I_{j,k}} f(\{i\theta\}) \ge \sum_{\nu=1}^{q_k} f\biggl(\frac{\nu}{q_k}\biggr) \ge q_k \int_{1/q_k}^{1} f(x)\,dx + f(1), \qquad f(1) \ge 0,$$

$$\sum_{i=1}^{n} f(\{i\theta\}) = \sum_{k=1}^{m}\sum_{j=1}^{b_k}\sum_{i\in I_{k,j}} f(\{i\theta\}) \ge \sum_{k=1}^{m} b_k q_k \int_{1/q_k}^{1} f(x)\,dx + f(1)\sum_{k=1}^{m} b_k.$$

Combining this with inequality (2.25), we obtain (2.21). The lemma is proved. □

3. MAIN STATEMENT. COROLLARIES

Theorem. *Let $f(x)$ be a nonnegative nonincreasing function on $(0,1]$, let θ be an irrational number on $(0,1)$, and let q_k be the denominators of its convergents. Then the condition*

$$f(\|q_k\theta\|) = o\biggl(q_k \int_{1/q_k}^{1} f(x)\,dx\biggr), \qquad k \to \infty, \tag{3.1}$$

is sufficient for the following equality to hold:

$$\frac{1}{n}\sum_{i=1}^{n} f(\{i\theta\}) = (1 + o(1)) \int_{1/n}^{1} f(x)\,dx, \qquad n \to \infty. \tag{3.2}$$

If

$$\delta f(\delta) = o\biggl(\int_{\delta}^{1} f(x)\,dx\biggr), \qquad \delta \to +0, \tag{3.3}$$

then condition (3.1) is also necessary for the validity of (3.2).

Proof. Set $\delta_k = 1/q_k (\delta_k \downarrow 0)$. Since $f(x)$ is nonincreasing and $\|q_k\theta\| < 1/q_{k+1} = \delta_{k+1}$ for all $k = 1, 2, \dots$, it follows from (3.1) that

$$\delta_k f(\delta_{k+1}) = \gamma_k \int_{\delta_k}^{1} f(x)\,dx, \tag{3.4}$$

where $\gamma_k \to 0$ as $k \to \infty$. Combining this with the fact that f is nonincreasing, we obtain

$$\int_{\delta_{k+1}}^{\delta_k} f(x)\,dx \le \delta_k f(\delta_{k+1}) = \gamma_k \int_{\delta_k}^{1} f(x)\,dx, \tag{3.5}$$

so that, as $k \to \infty$,

$$\int_{\delta_{k+1}}^{1} f(x)\,dx \sim \int_{\delta_k}^{1} f(x)\,dx. \tag{3.6}$$

By Lemma 4, in view of (3.1), we can write

$$\sum_{i=1}^{n} f(\{i\theta\}) \sim \sum_{k=1}^{m} b_k q_k \int_{\delta_k}^{1} f(x)\,dx, \qquad n \to \infty. \tag{3.7}$$

In view of the Abel transformation, we have

$$\sum_{k=1}^{m} b_k q_k \int_{\delta_k}^{1} f(x)\,dx = \int_{\delta_m}^{1} f(x)\,dx \cdot \sum_{k=1}^{m} b_k q_k - \sum_{k=1}^{m-1} \int_{\delta_{k+1}}^{\delta_k} f(x)\,dx \cdot \sum_{i=1}^{k} b_i q_i. \tag{3.8}$$

By the definition of the numbers b_i, we can write

$$0 \le \sum_{i=1}^{k} b_i q_i < q_{k+1}, \tag{3.9}$$

whence

$$\sum_{k=1}^{m-1} \int_{\delta_{k+1}}^{\delta_k} f(x)\,dx \cdot \sum_{i=1}^{k} b_i q_i \le \sum_{k=1}^{m-1} q_{k+1} \int_{\delta_{k+1}}^{\delta_k} f(x)\,dx. \tag{3.10}$$

Set

$$\widetilde{n} = \sum_{k=1}^{m-1} q_{k+1} = \sum_{k=2}^{m} q_k.$$

Since $q_k \le q_{k+2} - q_{k+1}$, it follows that

$$\widetilde{n} \le \sum_{k=2}^{m-2} (q_{k+2} - q_{k+1}) + q_{m-1} + q_m = -q_3 + q_m + q_{m-1} + q_m < 3q_m \le 3n. \tag{3.11}$$

Using (3.5), (3.11) and taking into account the fact that, as k increases, so does the integral $\int_{\delta_k}^{1} f(x)\,dx$, we can write

$$\sum_{k=1}^{m-1} q_{k+1} \int_{\delta_{k+1}}^{\delta_k} f(x)\,dx \le 3n \int_{\delta_m}^{1} f(x)\,dx \cdot \biggl(\frac{1}{\widetilde{n}} \sum_{k=1}^{m-1} q_{k+1}\gamma_k\biggr) = o\biggl(n \int_{\delta_m}^{1} f(x)\,dx\biggr).$$

Hence, from (3.10) and (3.8), we obtain

$$\sum_{k=1}^{m} b_k q_k \int_{\delta_k}^{1} f(x)\,dx \sim n \int_{\delta_m}^{1} f(x)\,dx, \qquad n \to \infty. \tag{3.12}$$

It remains to use the inequalities

$$\int_{\delta_m}^{1} f(x)\,dx \le \int_{1/n}^{1} f(x)\,dx \le \int_{\delta_{m+1}}^{1} f(x)\,dx, \qquad q_m \le n < q_{m+1},$$

obtaining (see (3.6))

$$\int_{\delta_m}^{1} f(x)\,dx \sim \int_{1/n}^{1} f(x)\,dx, \qquad n \to \infty. \tag{3.13}$$

Relations (3.12) and (3.13) imply the equality (3.2).

Let us prove the concluding part of the theorem. First, note that the fact that the function f is decreasing and condition (3.3) imply

$$\int_{\delta}^{2\delta} f(x)\,dx \le \delta f(\delta) = o\biggl(\int_{\delta}^{1} f(x)\,dx\biggr). \tag{3.14}$$

Again, using the fact that the function f is decreasing and the property of the distribution of the sequence $\{i\theta\}$, $i = 1, \dots, q_k$, for $n = 1/q_k$, we can write

$$\frac{1}{n}\sum_{i=1}^{n} f(\{i\theta\}) = \frac{1}{n} f(\{n\theta\}) + \frac{1}{n}\sum_{i=1}^{n-1} f(\{i\theta\}) \ge \frac{1}{n} f(\{n\theta\}) + \int_{2/n}^{1} f(x)\,dx$$

$$= \frac{1}{n} f(\{n\theta\}) + \int_{1/n}^{1} f(x)\,dx - \int_{1/n}^{2/n} f(x)\,dx.$$

Hence, from (3.2) and (3.14), we obtain (3.1). The theorem is proved. □

Remark. It follows from Khintchin's theorem [5, Sec. 14, Theorem 32] that relation (3.1) holds for almost all θ if and only if the integral $\int_0^1 f(x)\,dx$ converges (see also [3]) and holds only for the numbers θ of measure zero when the integral diverges.

In the last case, using the theorem proved above, we can obtain asymptotically sharp estimates for averaged sums in terms of the universally accepted classification of irrational numbers. Let us do this for the sums of the inverses (and more general sums) of the fractional parts $\{i\theta\}$, following the definition given in [4, Chap. III, Sec. 2].

Let $g = g(t)$ be a function increasing on $[1, +\infty)$, $g(t) \geq 1$, and let B_0 be a natural number, $B_0 \geq 10$. We shall say that θ is a number of *main subtype* $\leq g$ for all numbers $\geq B_0$ if, for a given number $B \geq B_0$, there exists a convergent p_i/q_i to θ, such that $B < q_i \leq Bg(B)$. In particular, if p_k/q_k and p_{k+1}/q_{k+1} are two successive convergents to θ and $q_k \geq B_0$, then

$$q_{k+1} \leq q_k g(q_k).$$

Note that if θ is a number of main subtype $\leq g$, then

$$\|q_k\theta\| > \frac{1}{2q_{k+1}} \geq \frac{1}{2q_k g(q_k)}.$$

If θ is a number of main subtype $\leq \alpha g$, where $\alpha = \alpha(t) = o(1)$ as $t \to \infty$, then, in addition to the notation used in [4], we shall stipulate that θ is a number of *main subtype* $\leq o(g)$. As usual, the expression $a \sim b$ and $\alpha/\beta \to 1$ are equivalent.

Corollary 1. *If θ is a number of main subtype $\leq o(\log t)$, then, for any $p \geq 0$,*

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{\{i\theta\}}\left(\log\frac{1}{\{i\theta\}}\right)^{p} \sim \frac{1}{p+1}(\log n)^{p+1}, \qquad n \to \infty. \tag{3.15}$$

Conversely, if for any $p \geq 0$, relation (3.15) *holds, then θ is a number of main subtype $\leq o(\log t)$.*

In particular, the equality

$$\lim_{n\to\infty} \frac{1}{n\log n}\sum_{i=1}^{n}\frac{1}{\{i\theta\}} = 1$$

holds if and only if θ is a number of main subtype $\leq o(\log t)$.

Turning to the history of this question, we formulate the following result from [4, Chap. III, Sec. 2, Theorem 2].

Theorem A (S. Lang). *Let θ be a number of main subtype $\leq g$ for all numbers $\geq B_0$. Then, for all integers $n \geq B_0$, we have*

$$\sum_{i=1}^{n}\frac{1}{\{i\theta\}} \leq 2n\log n + 20ng(n) + K_0, \qquad \textit{where} \quad K_0 \leq \sum_{i=1}^{B_0 g(B_0)} \frac{1}{\{i\theta\}}.$$

It was also noted in [4] (see Chap. III, Sec. 2, Remark 1) that, for numbers θ of constant type, i.e., such that, for all natural numbers q,

$$\|q\theta\| > \frac{a}{q}, \qquad a = \text{const} > 0,$$

the sum under consideration is greater than or equal to $cn\log n$ for some constant $c > 0$ and all sufficiently large n.

Proof of Corollary 1. For

$$f(x) = \frac{1}{x}\left(\log\frac{1}{x}\right)^p,$$

relation (3.2) takes the form (3.15) relation (3.3) also holds.

Condition (3.1) takes the form

$$A(\log A)^p = B, \qquad \text{where} \quad A = \frac{1}{\|q_k\theta\|}, \quad B = o(q_k(\log q_k)^{p+1}).$$

Since $A \sim B/(\log B)^p$ as $A \to \infty$, it follows that

$$\frac{1}{\|q_k\theta\|} = o(q_k \log q_k), \qquad k \to \infty.$$

But since $q_{k+1} < 1/\|q_k\theta\| < 2q_{k+1}$, we have

$$q_{k+1} = o(q_k \log q_k), \qquad k \to \infty.$$

Thus, θ is a number of main subtype $\le o(\log t)$; in view of the given arguments (for the function $f(x)$), this is equivalent to condition (3.1) and, in view of the theorem, to equality (3.15). The corollary is proved. □

Similarly, we can establish the following corollary.

Corollary 2. *If θ is a number of main subtype $\le o((\log\log t)\log t)$, then*

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\{i\theta\}\log(e/\{i\theta\})} \sim \log\log n, \qquad n \to \infty. \tag{3.16}$$

Conversely, if the asymptotics (3.16) *is valid, then the number θ is a number of main subtype* $\le o((\log\log t)\log t)$.

Results for functions of class A (see the introduction) are illustrated by the following statement.

Corollary 3. *If θ is a number of main subtype $\le o(\log t)$, then*

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{|\sin(\pi i\theta)|} \sim \frac{2}{\pi}\log n, \qquad n \to \infty. \tag{3.17}$$

Conversely, if (3.17) *is valid, then θ is a number of main subtype* $\le o(\log t)$.

Note that the following theorem was established in [4] (see Chap. III, Sec. 2, Theorem 3).

Theorem B (S. Lang). *Under the assumptions of Theorem A and of condition $n \ge B_0$, the following inequality holds*:

$$\sum_{i=1}^{n}\frac{1}{|\sin(\pi i\theta)|} \le 4n\log n + 40ng(n) + 2K_0.$$

In conclusion, it should be noted that if condition (3.3) is weakened to the condition

$$\delta f(\delta) = O\left(\int_\delta^1 f(x)\,dx\right), \qquad \delta \to 0,$$

we can obtain order-sharp upper and lower bounds for the averaged sums of the fractional parts $\{i\theta\}$ of irrational numbers θ of constant type involving functions of the form $f(x) = 1/x^p$, $p > 1$.

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